INTEREST RATES AND FX MODELS

1. LIBOR and OIS

Andrew Lesniewski
Courant Institute of Mathematical Sciences
New York University
New York

January 30, 2013

Contents

1 Introduction

Debt and debt related markets, also known as fixed income markets, account for the lion share of the world's financial markets. Instruments trading in these markets fall into the categories of *cash instruments*, or *bonds*, and derivative instruments, such as interest rate swaps, bond futures and credit default swaps. Bonds are debt instruments issued by various entities (such as sovereigns, corporations, municipalities, US Government Sponsored Enterprizes, etc), with the purposes of raising funds in the capital markets. Derivatives are synthetic instruments which extract specific features of commonly traded cash instruments in order to make risk management and speculation easier. In this course we will focus largely on a class of derivatives, known as interest rate derivatives.

Depending on their purpose, fixed income instruments may exhibit very complex risk profiles. The value of a fixed income instrument may depend on the level and term structure of interest rates, credit characteristics of the underlying entity, foreign exchange levels, or prepayment propensity of the collateral pool of loans. Understanding, modeling, and managing each of these (and other) risks poses unique challenges. These lectures cover some aspects of the interest rate risk only.

Like all other financial markets, fixed income markets fluctuate, and the main driver of this variability is the current perception of future interest rates. Market participants have adopted the convention that a *rally* refers to falling interest rates, while a *sell off* refers to rising rates. This somewhat counterintuitive convention is best rationalized by the fact that interest rate sensitive instruments, such as bonds, typically appreciate in value when rates go down and depreciate in value when rates go up. By the same token, a fixed income portfolio is *long the market*, if its value goes up with falling rates, and is *short the market* if the opposite is true.

Each fixed income instrument is a stream of known or contingent cash flows. Such cash flows are typically periodic, and are computed as a fixed or floating *coupon*, applied to a principal (or a notional principal). The market has developed various conventions to apply the coupon to the relevant accrual period. Examples of such conventions in the US dollar market are the *money market convention* act/360 (actual number of days over a 360 day year), and the *bond market conventions* 30/360 (based on a 30 day month over a 360 day year).

2 LIBOR and LIBOR based instruments

2.1 LIBOR rates

Much of the activity in the world capital markets is tied to the LIBOR rates. They are widely used as benchmarks for short term (overnight to 1 year) interest rates. A LIBOR (= London Interbank Offered Rate) rate is the interest rate at which banks offer (at least in principle) unsecured deposits to each other. Daily *fixings* of LIBOR are published by the British Banking Association on each London business day at 11 a.m. London time. These fixings are calculated from the quotes provided by a panel of participating banks. LIBOR rates are not a risk free, but the participating banks have high credit ratings. The details on the composition of the panels and how the fixings are calculated can found on the web site www.bbalibor.com of the British Banking Association.

LIBOR is quoted in ten major currencies: US Dollar (USD), British Sterling (GBP), Euro (EUR), Japanese Yen (JPY), Swiss Franc (CHF), Canadian Dollar (CAD), Australian Dollar (AUD), Danish Krone (DKK), Swedish Krona (SEK), and New Zealand Dollar (NZD). For each currency, LIBOR rates of 15 different maturities are quoted. The LIBOR rates and the market practices surrounding them vary somewhat from currency to currency. For example, the prevalent maturity in the USD is 3 months, and when we refer to the LIBOR rate we mean the 3 month rate. On the other hand, the most popular benchmark in the EUR is the 6 month rate[1]. In order to keep things simple, throughout this course, we shall focus on the the USD market, and designate the term LIBOR to mean the 3 month LIBOR rate.

In the USD market, LIBOR applies to deposits that settle two business days from the current date (this date is called the *spot date*), and whose maturity is on an anniversary date (say, 3 months) of that settlement date. Determining the anniversary date follows two rules:

(a) If the anniversary date is not a business day, move forward to the next business day, except if this takes you over a calendar month end, in which case you move back to the last business day. This rule is known as *modified following business day convention*.

(b) If the settlement date is the last business day of a calendar month, all anniversary dates are last business days of their calendar months.

In addition to spot transactions, there are a variety of *vanilla* LIBOR based instruments actively trading both on exchanges and over the counter: LIBOR futures, forward rate agreements, and interest rate swaps. The markets for LIBOR based instruments are among the most active derivatives markets. The significance of these instruments is that:

(a) They allow portfolio managers and other financial professionals effectively hedge their interest rate exposure. We will discuss it in detail later in this course.

(b) One can use them to synthetically create desired future cash flows and thus effectively manage assets versus liabilities.

(c) They allow market participants easily express their views on the future level and shape of the term structure of interest rates, and thus are convenient vehicles of speculation.

[1] The prevalent index rate in the Eurozone is actually the EURIBOR, which is different from the euro LIBOR.

2.2 Forward rate agreements

Forward rate agreements (FRAs) are *over the counter* (OTC) transactions. This means that they are arranged between the counterparties without an involvement of an exchange or a clearing house. The trades are typically arranged through a salesman, a voice broker or an electronic platform. In a FRA transaction, one of the counterparties (A) agrees to pay the other counterparty (B) LIBOR settling t years from now applied to a certain notional amount (say, $\$100$ mm). In exchange, counterparty B pays counterparty A a pre-agreed interest rate (say, 3.05%) applied to the same notional. The contract matures on an anniversary T (say, 3 months) of the settlement date, and interest is computed on an act/360 day count basis. Anniversary dates generally follow the same modified following business day convention as the LIBOR. FRAs are quoted in terms of the annualized forward interest rate applied to the accrual period of the transaction.

Market participants use FRAs for both hedging and speculation. Because of their granular structure, they allow one to hedge the exposure or express the view on rates move on specific days. For example, an *FOMC FRA* is a FRA which fixes the day after a Federal Open Markets Committee meeting. An *FOMC switch* involves taking opposite positions in a FRA fixing before and after an FOMC meeting. Likewise, a *turn switch* allow to hedge or express the view on the rate spikes idiosyncratic to the month-, quarter- or year-end.

Econometric studies of historical rates data show that forward rates are poor predictors of future interest rates. Rather, they reflect the evolving current consensus market sentiment about the future levels of rates. Their true economic significance lies in the fact that a variety of instruments whose values derive from the levels of forward rates (such as swaps) can be liquidly traded and used to hedge against adverse future levels of rates.

2.3 LIBOR futures

LIBOR futures, known also as the *Eurodollar futures*, are *exchange traded* futures contracts on the 3 month LIBOR rate. They trade on the Chicago Mercantile Exchange, often referred to as "the Merc", which also clears and settles the trades. The parties involved in futures trading maintain a cash position in their margin accounts, which serve as a security deposit against default. The Merc makes daily "mark to market" adjustments to the margin account in order to reflect the changes in values of each of the individual contracts.

In many ways, Eurodollar futures are similar to FRAs, except that their terms, such as contract sizes and settlement dates are standardized. Each of the contracts assumes a notional principal of \$1,000,000. Interest on these contracts is computed on an act/360 day count basis assuming 90 day accrual period. That means that for a 1 basis point movement in the underlying LIBOR forward rate, the daily mark to market adjustment of the contract is

$$0.0001 \times \frac{90}{360} \times \$1,000,000 = \$25.$$

The value of the margin account goes up (money is credited to the account) if the rate drops, and it goes down (money is debited from the account) if the rate goes up. In order to make a Eurodollar future resemble a bond, the market has adopted the convention according to which the forward rate R underlying the contract is quoted in terms of the "price" defined as

$$100 \times (1 - R).$$

For example, if $R = 2.32\%$, the quoted price of the contract is 97.68. Unlike a FRA, the Eurodollar future quoted price is linear in the underlying rate.

At any time, 44 Eurodollar contracts are listed on the Merc. The *quarterly* 40 contracts expire on the third Wednesday of the months of March, June, September, and December over the next 10 years. These dates are referred to as the IMM[2] dates. As it happens, the third Wednesdays of a month have the fortunate characteristic that they are (almost) always good New York and London business days, and are thus convenient for settlement and roll overs. The Monday preceding an IMM date is the last trading day of the contract, as the underlying LIBOR rate fixes on that day. Of the 40 quarterly contracts, only the first 20 are liquid, the open interest in the remaining 20 being minimal.

For the ease of execution, traders group these contracts into *packs* of four, and designate a color code to each of the packs. The first four contracts form the White (or front) pack, followed by the Red, Green, Blue, and Gold packs. The first two packs form the *2 year bundle*, the first three packs form the *3 year bundle*, etc.

In addition, 4 *serial contracts* maturing on the third Wednesday of the nearest four months not covered by the above quarterly contracts are listed. For example, on December 12, 2011, the following serial contracts are listed: January, February, April, and May. Of these 4 contracts, typically the first two are somewhat liquid. Serial contracts are not part of the front pack.

[2] IMM stands for the International Monetary Market

Table 1 contains a snapshot of the Eurodollar market. We have included only the first 20 quarterly contracts and the serial contracts. In addition to the quoted price, we have also included the open interest, i.e. the total number of outstanding contracts, as well as the daily volume, i.e. the number of contracts traded on the day. Each of the four quarterly contracts is designated a letter code: H stands for March, M stands for June, U stands for September, and Z stands for December. The contract's ticker is a combination of the letter code and the year of expiration. Thus, EDU13 denotes the contract expiring in September 2013. The letter codes for the serial contracts are F (January), G (February), J (April), K (May), N (July), Q (August), V (October), and X (November).

Ticker	Price	Open Int	Volume
EDZ11	99.355	992506	120246
EDF12	99.415	32443	10324
EDG12	99.385	7623	2849
EDH12	99.355	933497	184855
EDJ12	99.340	426	0
EDK12	99.325	255	0
EDM12	99.310	1063728	158419
EDU12	99.295	801723	151663
EDZ12	99.290	796494	146512
EDH13	99.300	753118	94834
EDM13	99.290	593587	77793
EDU13	99.260	478959	86254
EDZ13	99.190	544016	69813
EDH14	99.085	374708	58436
EDM14	98.930	307804	47007
EDU14	98.760	217849	43230
EDZ15	98.580	182137	36593
EDH16	98.415	137627	21723
EDM16	98.250	111012	17310
EDU16	98.085	76774	18580
EDZ16	97.905	55982	18628
EDH17	97.760	47385	14877
EDM17	97.620	37211	9331
EDU17	97.490	38411	9577

Table 1: Snapshot of the Eurodollar futures market

2.4 Swaps

A fixed for floating *interest rate swap* (or simply: a swap) is an OTC transaction in which two counterparties agree to exchange periodic interest payments on a pre-specified *notional* amount. One counterparty (the fixed *payer*) agrees to pay periodically the other counterparty (the fixed *receiver*) a fixed coupon (say, 3.35% per annum) in exchange for receiving periodic LIBOR applied to the same notional.

Spot starting swaps based on LIBOR begin on a start date 2 business days from the current date and mature and pay interest on anniversary dates that use the same modified following business day conventions as the LIBOR index. Interest is usually computed on an act/360 day basis on the floating side of the swap and on 30/360 day basis in the fixed side of the pay. Typically, fixed payment dates ("coupon dates") are semiannual (every 6 months), and floating payment dates are quarterly (every 3 months) to correspond to a 3 month LIBOR. In addition to spot starting swaps, *forward starting* swaps are routinely traded. In a forward starting swap, the first accrual period can be any business day beyond spot. Swaps (spot and forward starting) are quoted in terms of the fixed coupon.

Swaps are often created on the back of bond issuance. Consider the situation depicted in Figure 1. An issuer, it could be for instance a corporation or a government sponsored agency, issues a bond which is purchased by an investor. The issuer pays a periodic coupon and repays the principal at the bond's maturity. Because of their high quality credit the issuer can fund their liabilities at a favorable spread to LIBOR rate, and thus enters into a swap with the interest rate derivatives desk of a bank offsetting their coupon payments.

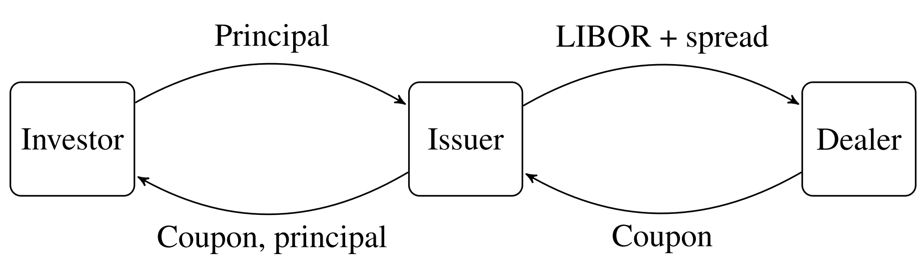

Figure 1: Swapping a bond

Table 2 contains a snapshot of the swap rate market. Each rate is the break even rate on the swap of indicated tenor paying coupon semiannually on the 30/360 basis, versus receiving 3 month LIBOR.

Tenor	Rate (%)
2Y	0.690%
3Y	0.798%
4Y	1.008%
5Y	1.248%
7Y	1.690%
10Y	2.106%
12Y	2.298%
15Y	2.478%
20Y	2.599%
25Y	2.660%
30Y	2.694%

Table 2: Snapshot of the swap market taken on 12/13/2011

3 OIS and basis swaps

In the wake of the 2008 credit crunch, LIBOR's credibility as a funding rate was put to question. Part of the issue is the *de facto* absence of the interbank unsecured lending market which raises doubts over the validity of the quotes submitted by the participating banks to the BBA. As a result, rates referred to as the OIS rates, linked to the overnight rate controlled by the local (to the currency) central bank became increasingly important as benchmark funding rates. OIS is the acronym for *overnight indexed swap*. An overnight indexed swap is a fixed for floating interest rate swap where the floating rate is based on a short term (overnight), daily compounded (rather quarterly, as in a LIBOR swap) interest rate. By default, both the fixed and floating legs accrue based on the act/360 basis. OIS swaps tend to be of short maturity, ranging from a few days to five years.

In the USD market, OIS rates are calculated by reference to daily *fed funds effective rate*. The economic significance of that rate is as follows. Under the US law, banks are required to maintain certain minimum levels of reserves, either in the form of reserves with the Fed or as vault cash. The level of reserves depends on the outstanding assets and liabilities of a bank, and is verified at the end of each business day. If the banks's reserve falls s below the mandated minimum, it must borrow funds in order to remain compliant with the regulations. The bank can borrow from another bank that happens to have an excess reserve, or directly from the Fed itself (at the "discount window"). The interest rate that the borrowing bank pays to the lending bank is negotiated between them, and the weighted average of the interest rates on all such transactions[3] on a given day is the fed funds effective rate for that day. This "fixing" is published the following day between 7 and 8 am. Unlike LIBOR, which is the result of a poll taken among participating banks, the fed funds effective rate is a trading rate, and is thus regarded a more reliable barometer of the current cost of funds.

The fed funds effective rate is to a large degree controlled by the Federal Reserve Bank which periodically sets a target for its level. In the periods of monetary *easing*, the target fed funds rate is lowered, while in the periods of *tightening*, it is raised. Typically, the changes in the target rate are announced after the monthly FOMC meetings, although, historically, unscheduled changes have also been announced. The Fed enforces its targets through open market operations at the Domestic Trading Desk at the Federal Reserve Bank of New York.

There is an active futures market operated by the Merc, similar to the Eurodollar futures market, where the underlying rate is the fed funds effective rate rather than LIBOR. There are 36 monthly futures contracts listed, three years out, and they cash settle at the end of the corresponding calendar month. Each contract assumes the notional principal of $5,000,000. Because the daily fixings of the fed funds effective rate tend to be more volatile than those of LIBOR[4], the settlement value of each contract is the monthly arithmetic average of the daily fixings (with the Friday fixings counting for the following weekend values), rather than the fixing of the rate on the settlement day. The first 12 of the the contracts are the most liquid, and Table 3 is a snapshot of the fed funds futures taken on December 13, 2011.

OIS rates in other currencies are referenced by the corresponding overnight rates. In the euro zone, the relevant short rate is EONIA (which stands for *euro overnight index average*). The banks contributing to EONIA are the same as the panel banks contributing to EURIBOR. SONIA is the acronym for *sterling overnight index average*, the reference rate for overnight unsecured transactions in the sterling market. SARON (the acronym for *Swiss average rate overnight*) is an overnight interest rates average referencing the Swiss franc interbank repo market.

The *LIBOR / OIS spread*, defined as the difference between the 3 month LIBOR and 3 month OIS rates, is regarded an important indicator of stress in the

[3]The (non-negotiated) rate at which banks borrow directly from the Fed is not included in this calculation. This rate is referred to as the discount rate, and it is typically higher than the fed funds rate.

[4]This has definitely not been the case recently, especially after the Fed pledged in August 2011 to keep rates low for two years and reinforced that pledge yesterday.

Ticker	Price	Open Int	Volume
FFZ11	99.9225	68323	1825
FFF12	99.9100	68486	1785
FFG12	99.9050	55908	2390
FFH12	99.8950	34924	1759
FFJ12	99.8850	38840	1430
FFK12	99.8750	45100	196
FFM12	99.8750	28284	556
FFN12	99.8700	35982	1155
FFQ12	99.8650	40812	1720
FFU12	99.8650	16302	208
FFV12	99.8650	37476	309
FFX12	99.8650	14755	134

Table 3: Snapshot of the fed funds futures market

capital markets. A wider spread is an indication of a decreased willingness to lend by major banks, while a tighter spread indicates easier availability of credit. The LIBOR / OIS spread is thus a barometer of market participants' view of the credit worthiness of other financial institutions and the general availability of funds for lending purposes.

The LIBOR / OIS spread has been historically hovering around 10 basis points. However, at times of elevated credit stress, the basis between LIBOR and OIS can be quite volatile. In the midst of the financial crisis that started in 2007, the spread between LIBOR and OIS was wildly volatile and peeked at an all-time high of 364 basis points in October 2008, indicating a severe credit crunch. Since that time the spread has declined substantially, dropping below 100 basis points in January 2009 and returning to the usual 10-15 basis points by September 2009. During the ongoing euro crisis, the LIBOR / OIS spread has widened again, reaching in December 2011 the levels of 40-60 basis points.

Market participants can implement their views on the future LIBOR / OIS spread through the *LIBOR / OIS basis swap* market. A LIBOR / OIS basis swap is a floating for floating swap, in which one counterparty pays the current fixing of 3 month LIBOR, while the other counterparty pays the 3 month OIS rate plus a fixed spread (say, 20 basis points). Both legs accrue on the act/360 basis, and the interest payments are netted. Table 4 contains a snapshot of the LIBOR / OIS basis market. Note that the term structure of the LIBOR / OIS basis is downward sloping, as is typically the case.

It should be noted that alternatives to LIBOR, other than the fed funds effective rate, have been proposed. An example is the *DTCC GCF repo index* which is

Term	Spread (bp)
3M	45.88
6M	49.25
9M	51.13
1Y	52.50
18M	53.50
2Y	53.00
3Y	51.63
4Y	49.25
5Y	46.25
7Y	41.38
10Y	34.50
12Y	31.75
15Y	28.88
20Y	25.50
25Y	23.38
30Y	21.88

Table 4: Snapshot of the LIBOR / OIS basis swap market

published daily and is calculated as the volume weighted rate on the overnight repo transactions backed by the US Government Treasuries. The advantage of the GCF repo index is that, unlike LIBOR, it reflects a very liquid, secured short term lending market and is virtually impossible to manipulate. In July of 2012, NYSE Liffe launched a series of futures contracts based on the GCF repo index.

4 Valuation of swaps

In this section we are concerned with valuation and risk management of non-contingent (but not necessarily known) future cash flows. The building blocks required are:

(a) *Discount factors*, which allow one to calculate present value of cash received in the future.

(b) *Forward rates*, which allow one to make assumptions as to the future levels of interest rates.

Until 2008, it was common practice to use LIBOR as both the discount rate, i.e. the interest rate used for calculating the discount factors, as well as the index rate, i.e. the rate used as the forward rate. Since then, in the wake of the financial crisis, the industry has been steadily moving away from this practice, and adopted the *multi-curve* paradigm to swap valuation: Since OIS is a better indicator of the costs of funding, it is used for discounting, while LIBOR is the index rate. It remains to be seen whether the fed funds effective rate will retain this role in the USD market.

4.1 Zero coupon bonds

A *zero coupon bond* (or *discount bond*) is a cash instrument which pays a predefined principal amount, say $\$1$, at a specified future date. More precisely, a zero coupon bond is characterized by two dates, the settlement date S which marks the start of the accrual period, and the maturity date $T > S$ on which the payment is made. Its value at settlement is thus the present value (abbreviated PV) of $\$1$ guaranteed to be paid at time T.

In practice we are interested in the value $P(t, S, T)$ of the forward zero coupon bond for any *valuation date* $t \leq S$. It is thus the time t value of zero coupon bond (whose face value is $\$1$) which settles on the date S years from now and matures in T years. The forward zero coupon bond $P\left(t, S, T\right)$ is also called the (forward) *discount factor*.

We note the following two important facts about zero coupon bonds. For all valuation dates t:

(a) The value of a dollar in the future is less than the its value now, i.e.

$$P(t, S, T) < 1. \tag{1}$$

In other words, we assume a world in which one collects (rather than pay) interest on a deposit.

(b) As the maturity of the contract increases, its value decreases,

$$\frac{\partial P(t, S, T)}{\partial T} < 0. \tag{2}$$

Throughout this course we will be assuming that the zero coupon bond prices satisfy these properties. It is worth noting, however, that in distressed market conditions, these properties are known to have been violated. For example, during the month of August of 2011, the 1 month and 3 month US Treasury bills traded above par.

Throughout this course we will adopt the following convention. If the valuation date is today, $t = 0$, then we denote the price of the zero coupon bond by $P_0(S, T)$, i.e.

$$P_0(S,T) \equiv P(0,S,T). \tag{3}$$

In particular, today's value of the discount factor for time T is denoted by $P_0(0, T)$.

There is a useful no arbitrage relationship involving $P(t, S, T)$. In order to explain it, consider a *forward contract* on a zero coupon bond: at some future time $S < T$, we deliver to the counterparty $\$1$ of a zero coupon bond of final maturity T. What is the fair price of this transaction at time t? We calculate it using the following argument which provides a risk-free replication of the forward trade in terms of spot trades.

1. At time t we buy $\$1$ of a zero coupon bond of maturity T today for the price of $P\left(t, t, T\right)$.

2. We finance this purchase by short selling a zero coupon bond of maturity S and notional $P\left(t, t, T\right) / P\left(t, t, S\right)$ for overall zero initial cost.

3. In order to make the trade self-financing, we need to charge this amount at delivery. Thus, we have the relation:

$$P\left(t,S,T\right) = \frac{P\left(t,t,T\right)}{P\left(t,t,S\right)}\ . \tag{4}$$

We will be making repeat use of (4) throughout these lectures. In particular, with $t = 0$,

$$P_0\left(S,T\right) = \frac{P_0\left(0,T\right)}{P_0\left(0,S\right)}\ . \tag{5}$$

4.2 Forward rates

Discount factors can be expressed in terms of interest rates. A convenient, albeit purely theoretical concept is that of the continuously compounded *instantaneous forward rate* $f(t, s)$. For all practical purposes, we can think about $f(t, s)$ as the forward overnight OIS rate. As discussed earlier in this lecture, in the USD market, this is the fed funds effective rate. In terms of $f(t, s)$,

$$P\left(t,S,T\right) = \exp\left(-\int_S^T f(t,s)ds\right). \tag{6}$$

This equation is merely the definition of $f(t, s)$, and expresses the discount factor as the result of continuous discounting of the value of a dollar between the value and maturity dates. Consistently with our convention, we shall use the notation:

$$f_0(s) \equiv f(0, s), \tag{7}$$

and consequently

$$P_0\,(S, T) = \exp\Big(-\int_S^T f_0(s)ds\Big). \tag{8}$$

Conversely, the instantaneous forward rate can be computed from the discount factor:

$$\begin{aligned} f(t, s) &= -\frac{1}{P(t, S, T)}\,\frac{\partial P(t, S, T)}{\partial T}\Big|_{T=s} \\ &= -\frac{\partial}{\partial T}\,\log P(t, S, T)\Big|_{T=s}. \end{aligned} \tag{9}$$

In particular,

$$\begin{aligned} f_0(s) &= -\frac{1}{P_0(S, T)}\,\frac{\partial P_0(S, T)}{\partial T}\Big|_{T=s} \\ &= -\frac{\partial}{\partial T}\,\log P_0(S, T)\Big|_{T=s}. \end{aligned} \tag{10}$$

The OIS forward rate $F\,(t, S, T)$ for start S and maturity T, as observed at time t, is defined as

$$\begin{aligned} F(t, S, T) &= \frac{1}{\delta}\,\Big(\frac{1}{P(t, S, T)} - 1\Big) \\ &= \frac{1}{\delta}\,\Big(\exp\int_S^T f(t, s)ds - 1\Big), \end{aligned} \tag{11}$$

where δ denotes the day count factor for the period $[S, T]$. The discount factor $P(t, S, T)$ can be expressed in terms of $F(t, S, T)$ by means of the formula:

$$P(t, S, T) = \frac{1}{1 + \delta F\,(t, S, T)}\,.$$

It is convenient to represent the LIBOR forward rate $L\,(t, S, T)$ for start S and maturity T, as observed at time t, in terms of the instantaneous forward rate $l(t, s)$ by means of a similar relation:

$$L(t, S, T) = \frac{1}{\delta}\,\Big(\exp\int_S^T l(t, s)ds - 1\Big). \tag{12}$$

We emphasize that the instantaneous rate $l(t, s)$ is specific to the tenor $T - S$ of LIBOR, as different LIBOR tenors trade at a basis to each other. The rate $l(t, s)$ is not used for discounting future cash flows; it merely serves as an index rate on a class of fixed income instruments. The LIBOR / OIS spread $B(t, S, T)$ is given by

$$B(t, S, T) = L(t, S, T) - F(t, S, T). \tag{13}$$

4.3 Swap rates and valuation of swaps

We consider a swap which settles at $T_0 \geq 0$ and matures at T. If T_0 is the spot date, the swap is referred to as a spot starting swap, otherwise it is called a forward starting swap. We assume that the notional principal is one currency unit.

Let $T_1^{\mathrm{c}} < \ldots < T_{n_{\mathrm{c}}}^{\mathrm{c}} = T$ denote the coupon dates of the swap, and let $0 \leq t \leq T_0$ denote the valuation date. The PV of the interest payments on the fixed leg of a swap is calculated by adding up the PVs of all future cash flows:

$$P_0^{\mathrm{fix}}(t) = \sum_{j=1}^{n_{\mathrm{c}}} \alpha_j C P_0(t, T_j^{\mathrm{c}}), \tag{14}$$

where C is the coupon rate, $P_0(t, T_j^{\mathrm{c}})$ are the discount factors to the valuation date, and α_j are the day count fractions on the fixed leg. For example, on a standard USD swap paying semi-annual coupon, the α's correspond to the modified following $30/360$ business day convention. It is useful to write this formula as

$$P_0^{\mathrm{fix}}(t) = C A_0(t), \tag{15}$$

where

$$A_0(t) = \sum_{1 \leq j \leq n_{\mathrm{c}}} \alpha_j P_0(t, T_j^{\mathrm{c}}), \tag{16}$$

is called the *annuity function* (or the *level function*) of the swap.

Likewise, let $T_1^{\mathrm{f}} < \ldots < T_{n_{\mathrm{f}}}^{\mathrm{f}} = T$ denote the LIBOR payment dates of the swap. The valuation formula for the swap's floating leg reads then:

$$P_0^{\mathrm{float}}(t) = \sum_{1 \leq j \leq n_{\mathrm{f}}} \delta_j L_j P_0(t, T_j^{\mathrm{f}}), \tag{17}$$

where

$$L_j = L_0(T_{j-1}^{\mathrm{f}}, T_j^{\mathrm{f}}) \tag{18}$$

is the LIBOR forward rate for settlement at T^{f}_{j-1}, $P_0(t, T^{\mathrm{f}}_j)$ is the discount factor, and δ_j is the day count fraction applying to the floating leg. In the USD, the payments are quarterly, and the δ's correspond to the modified following act/360 business day convention.

The PV of a swap is the difference between the PVs of the fixed and floating legs (in this order!):

$$P_0(t) = P_0^{\mathrm{fix}}(t) - P_0^{\mathrm{float}}(t).$$

A break-even (or mid-market) swap has zero PV:

$$P_0^{\mathrm{fix}}(t) = P_0^{\mathrm{float}}(t).$$

That uniquely determines the coupon on a mid-market swap:

$$S_0(T_0, T) = \frac{P_0^{\mathrm{float}}(t)}{A_0(t)}\,, \tag{19}$$

called the *break-even swap rate*. Note that the value of the swap rate is independent of the valuation date (even though the PV's of the individual legs of the swap are clearly not). Indeed, according to (5), the numerator and the denominator in the above expression both contain the factor $P_0(t, T_0)$, and one can always choose to discount to the settlement date.

4.4 Valuation of LIBOR / OIS basis swaps

Consider a spot or forward starting LIBOR / OIS basis swap of maturity T (say, 10 years), with payment dates on $T_1, \ldots, T_n = T$. The PV of the LIBOR leg is

$$P_0^{\mathrm{LIBOR}}(t) = \sum_{1 \le j \le n} \delta_j L_j P_0(t, T_j), \tag{20}$$

while the PV of the OIS leg is

$$P_0^{\mathrm{OIS}}(t) = \sum_{1 \le j \le n} \delta_j (F_j + B) P_0(t, T_j), \tag{21}$$

where B denotes the fixed basis spread for the maturity T. The break-even basis spread is thus given by

$$B_0(T_0, T) = \frac{\sum_j \delta_j (L_j - F_j) P_0(t, T_j)}{\sum_j \delta_j P_0(t, T_j)}\ . \tag{22}$$

5 Building the OIS and LIBOR curves

So far we have been assuming that the discount factors $P_0(S,T)$ and the LIBOR forward rates $L_0(S,T)$ are known for all $S < T$. In practice, interest rate markets are not liquid enough to quote the levels for all these quantities at all times. Instead, a relatively small number of liquid benchmark instruments are traded (or, at least, quoted) sufficiently frequently and in sufficient volume in order to form a basis for meaningful inference. It is now our goal to describe an interpolation technique that allows one to calculate the discount factors and LIBOR forward rates from the available market information.

The process of curve construction out of liquid market inputs is often referred to as *curve stripping*. In reality, we will be simultaneously stripping two interest rate curves: the OIS curve and the LIBOR curve. The result can be presented in various forms:

(a) As a function $S \to L_0(S,T)$ with fixed tenor $T - S$ (say, $T - S = 3$ months). The graph of this function is called the LIBOR *forward curve*.

(b) As a function $T \to P_0(0,T)$. This is called the *discount curve* (or *zero coupon curve*).

(c) As a collection of spot starting swap rates for all tenors. This is called the *par swap curve*.

The curve construction should be based on the prices of liquidly traded benchmark securities. As this set of securities is incomplete, we need a robust and efficient method involving interpolation and, if necessary, extrapolation. These benchmark instruments include deposit rates, Eurodollar futures, a number of benchmark swaps, and a number of liquidly traded LIBOR / OIS basis swaps. For example, one could use the following set of instruments:

(a) Fed funds effective rate, LIBOR fixing.

(b) The first 8 quarterly Eurodollar contracts.

(c) Spot starting swaps with maturities 2, 3, 4, 5, 7, 10, 12, 15, 20, 25, and 30 years.

(d) Spot starting LIBOR / OIS basis swaps with maturities, 2, 6, 9, 12, and 18 months, and 2, 3, 4, 5, 7, 10, 12, 15, 20, 25, and 30 years.

If more granularity in the short end is desired, one could include some spot starting OIS swaps. It is, however, not appropriate to include rates such as 1 week or 1 month LIBOR because of the basis to 3 month LIBOR present in these rates.

Various approaches to curve stripping have been developed, and we refer the interested reader to the literature. An excellent survey of these techniques is presented in [2]. Traditionally, these methodologies assume the single curve paradigm, but without much effort they can be extended to the multi-curve setup.

5.1 Splines and B-splines

The approach to curve stripping that we take in this course is based on interpolation by means of *B-splines*. We collect here a number of basic facts about B-splines. For a complete presentation, we refer the reader to [1].

Definition 5.1 *A spline of degree d is a function $f(t)$ on $\mathbb{R}$ such that:*

(a) $f(t)$ is piecewise polynomial of degree d. That means that one can partition the real line into non-overlapping intervals such that, on each of these intervals, $f(t)$ is given by a polynomial of degree d.

(b) $f(t)$ has everywhere $d-1$ continuous derivatives. That means that the constituent polynomials are glued together in a maximally smooth way, i.e. so that their derivatives to order $d-1$ match at the ends of partitioning inetrvals.

Splines of low degree provide a convenient and robust framework for data interpolation. The choice $d = 3$ (in which case they are called *cubic splines*) is particularly important, as it provides a good compromise between flexibility and computational efficiency.

A particular type of splines are B-splines. A B-spline of degree $d \geq 0$ is a function $f(t)$ of the form

$$f(t) = \sum_{-\infty<k<\infty} f_k B_k^{(d)}(t), \tag{23}$$

where the *basis functions* $B_k^{(d)}(t)$, $k = \ldots, -1, 0, 1, 2, \ldots$, is a family of degree d splines defined as follows. We choose a sequence of *knot points*:

$$\ldots < t_{-1} < t_0 < t_1 < \ldots < t_k < \ldots, \tag{24}$$

and set

$$B_k^{(0)}(t) = \begin{cases} 1, & \text{if } t_k \le t < t_{k+1}. \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

In other words, the degree 0 basis functions are simply indicator functions of the intervals between the knot points. We then define recursively:

$$B_k^{(d)}(t) = \frac{t - t_k}{t_{k+d} - t_k}\, B_k^{(d-1)}(t) + \frac{t_{k+d+1} - t}{t_{k+d+1} - t_{k+1}}\, B_{k+1}^{(d-1)}(t). \tag{26}$$

It is easy to see that a basis function $B_k^{(1)}(t)$ is tent shaped over the interval $[t_k, t_{k+2}]$, $B_k^{(2)}(t)$ is bell shaped over the interval $[t_k, t_{k+3}]$, etc. Clearly, each $B_k^{(d)}(t)$ is a spline of degree d.

Here are some key properties of the basis functions:

$$B_k^{(d)}(t) \ge 0, \tag{27}$$

and

$$B_k^{(d)}(t) = 0, \tag{28}$$

if t lies outside of the interval $[t_k,\ t_{k+d+1}]$. Furthermore,

$$\sum_{-\infty < k < \infty} B_k^{(d)}(t) = 1. \tag{29}$$

One summarizes these three properties by saying that the family of basis functions $\{B_k^{(d)}(t)\}$ forms a *partition of unity*.

Remarkably and conveniently, differentiating and integrating of the basis functions can be carried out in a recursive way as well. For the derivative we have the following recursion:

$$\frac{d}{dt}\, B_k^{(d)}(t) = \frac{d}{t_{k+d} - t_k}\, B_k^{(d-1)}(t) - \frac{d}{t_{k+d+1} - t_{k+1}}\, B_{k+1}^{(d-1)}(t). \tag{30}$$

The integral from $-\infty$ to t can be expressed in terms of a sum as follows:

$$\int_{-\infty}^{t} B_k^{(d)}(\tau)\, d\tau = \sum_{i=k}^{\infty} \frac{t_{k+d+1} - t_k}{d+1}\, B_i^{(d+1)}(t). \tag{31}$$

Despite its appearance, for each given t, this sum involves finitely many terms only. Specifically, assume that n is the smallest integer such that $t_n \le t$. Then, the sum extends over those i for which $k \le i \le t_{n+d+1}$. We require at least $d+2$ knot points greater than or equal to t. A consequence of the last equality is

$$\int_a^b B_k^{(d)}(\tau)\, d\tau = \int_{-\infty}^b B_k^{(d)}(\tau)\, d\tau - \int_{-\infty}^a B_k^{(d)}(\tau)\, d\tau. \tag{32}$$

Owing to these recursive properties, B-splines can be easily and robustly implemented in computer code.

The final bit on cubic B-splines is the issue of calculating an integral of the form

$$\int_a^b B_k''(t) B_l''(t) dt. \tag{33}$$

Let us assume that there are m knot points $w_1, \ldots, w_m$ (i.e. $w_j = t_{j+r}$, $j = 1, \ldots, m$, for some r) between a and b, and let us denote $w_0 = a$ and $w_{m+1} = b$. Integrating by parts and noting that the third derivative of the basis function $B_l(t)$ is piecewise constant (because $B_l(t)$ is piecewise cubic) yields the explicit formula

$$\begin{aligned} \int_a^b B_k''(t) B_l''(t) dt = {} & B_k'(b)\, B_l''(b) - B_k'(a)\, B_l''(a) \\ & - \sum_{j=1}^{m+1} B_l'''(w_{j-1}) \left(B_k(w_j) - B_k(w_{j-1}) \right). \end{aligned} \tag{34}$$

This formula is easy to implement in computer code.

5.2 Curve construction by B-splines fitting

Our objective is to construct the instantaneous OIS forward rate $f_0(t)$, and the instantaneous 3 month LIBOR forward rate $l_0(t)$. The "instantaneous 3 month LIBOR forward rate" sounds a bit like an oxymoron, but we want to draw a clear distinction between the actual short term (overnight) LIBOR rate and the ficticious de-compounded 3 month LIBOR rate $l_0(t)$, as they carry different credits, and cannot be identified with each other.

We represent each of these rates as a cubic B-spline, as explained in Section 5.1 above. We assume that the curve starts at $T_0 = 0$ and ends at $T_{\max}$ (say, 30 years), and choose K knot points $t_{-3}, \ldots, t_{N+4}$, with

$$t_{-3} < \ldots < t_0 = 0 < t_1 < \ldots < t_{N-1} < T_{\max} < t_N < \ldots < t_{N+4}.$$

We let $B_k(t) \equiv B_k^{(3)}(t)$, $k = -3, -2, \ldots$, be the k-th basis function corresponding to these knot points. The nodes to the left and right of the time interval $[0,\, T_{\max}]$ are auxiliary nodes required to ensure that the partition of unity property (29) of the basis functions holds inside that interval. We represent $f_0(t)$ and $l_0(t)$, for $t \in [0,\, T_{\max}]$, as linear combinations of the basis functions:

$$f_0(t) = \sum_{k=-3}^{N} f_k B_k(t), \tag{35}$$

and

$$l_0(t) = \sum_{k=-3}^{N} l_k B_k(t), \tag{36}$$

respectively. It would be somewhat pedantic, but not impossible, to assume that the basis functions corresponding to the two forward rates are based on different sets of knot points. In practice, it turns out that choosing the same knot points for both curves is a good idea.

Note that, in this representation, the discount factors $P_0\,(S, T)$, $0 \leq t \leq T \leq T_{\max}$, are simple functions of the f_k's:

$$P_0(S,T) = \exp\Big(- \sum_{k=-3}^{N} \gamma_k\,(S,T)\, f_k \Big), \tag{37}$$

where the coefficients

$$\gamma_k(S,T) = \int_S^T B_k(s)ds \tag{38}$$

can be easily computed using the algorithm presented in Section 5.1. Note that, for computational efficiency, we should use the fact that $B_k(t)$ is supported in the interval $[t_k,\, t_{k+4}]$, and thus the integral in (38) actually runs from $\max(S, t_k)$ to $\min(T, t_{k+4})$.

Likewise, from (12) we infer that the LIBOR forward rates are simple and explicit functions of the l_k's, namely

$$L_0(S,T) = \frac{1}{\delta}\,\Big(\exp\Big(\sum_{k=-3}^{N} \gamma_k\,(S,T)\, l_k \Big) - 1 \Big). \tag{39}$$

Given (37) and (39), and using formulas (11), (13), and (19), we can express all LIBOR / OIS basis spreads and all swap rates in terms of the f_k's and l_k's.

Our goal is to choose the coefficients f_k and l_k in (35) and (36) consistently with the market data. This will be achieved by minimizing a suitable objective function. Suppose now that we are given a number of benchmark rates R_j, $j = 1, \ldots, m$, such as those discussed in Section 5: fed funds effective rate, LIBOR, Eurodollar implied LIBOR forward rates, swap rates, and LIBOR / OIS bases, etc. Let $\overline{R}_j$ denote the current market value of these rates. Consider the following objective function:

$$Q(f, l) = \frac{1}{2} \sum_{j=1}^{m} (R_j - \overline{R}_j)^2 + \frac{1}{2}\, \lambda \int_{T_0}^{T_{max}} \big(f''(t)^2 + l''(t)^2 \big) dt, \tag{40}$$

where $f = (f_{-3}, \ldots, f_N)$, $l = (l_{-3}, \ldots, l_N)$, and where λ is a non-negative constant.

The second term on the right hand side of (40) is a *Tikhonov regularizer*, and its purpose is to penalize the "wiggliness" of $f(t)$ and $l(t)$ at the expense of the accuracy of the fit. Its magnitude is determined by the magnitude of λ: the bigger the value of λ, the smoother the instantaneous forward rate at the expense of the fit. One may choose to refine the Tikhonov regularizer by replacing it with

$$\int_{T_0}^{T_{max}} \lambda(t) \big(f''(t)^2 + l''(t)^2 \big) dt,$$

where $\lambda(t)$ is a non-negative (usually, piecewise constant) function. Experience shows that it is a good idea to choose $\lambda(t)$ smaller in the short end and larger in the back end of the curve.

The minimum of (40) can be found by means of standard Newton-type optimization algorithms such as the Levenberg-Marquardt algorithm. The Levenberg-Marquardt algorithm applies to an objective function which can be written as a sum of squares of "residuals". This algorithm requires explicit formulas for the partial derivatives of the residuals with respect to the parameters of the problem. In our case, these derivatives can be readily computed.

Figure 2 contains the plot of the instantaneous 3 month LIBOR forward curve based on the market snapshot taken on on December 13, 2011. We used weekly time increments in order to detail the shape of the short end of the curve.

Figure 3 presents the plot, based on the same market snapshot, of the spread between the instantaneous 3 month LIBOR forward curve and the instantaneous OIS curve. Note the downward sloping shape of the spread curve.

The same forward curve building methodology can be applied to other rates such as the 1 month or 6 month LIBOR curves. For example, having constructed the OIS and 3 month LIBOR curves, one can build the 1 month LIBOR curve. To this end, we invoke the 3M LIBOR / 1M LIBOR basis swap market[5], and calibrate the instantaneous forward rate corresponding to 1 month LIBOR.

References

[1] de Boor, C.: *A Practical Guide to Splines*, Springer Verlag (2001).

[2] Hagan, P., and West, G.: Interpolation methods for curve construction, *Appl. Math. Finanace*, 89 - 129 (2006).

[3] Tuckman, B., and A. Serrat: *Fixed Income Securities: Tools for Today's Markets*, Wiley (2011).

[5]The existence of the 3M LIBOR / 1M LIBOR (positive) basis is explained by the banks' preference to lend for 1 month rather than 3 months; thus the compounded 1 month rate is lower than the 3 month rate.

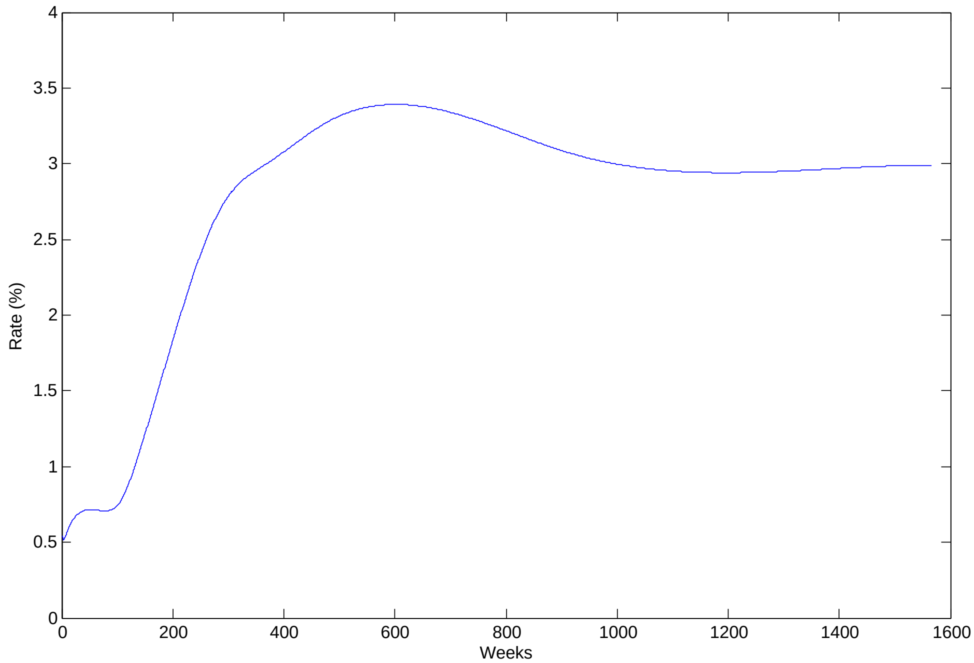

Figure 2: Instantaneous 3M LIBOR forward curve

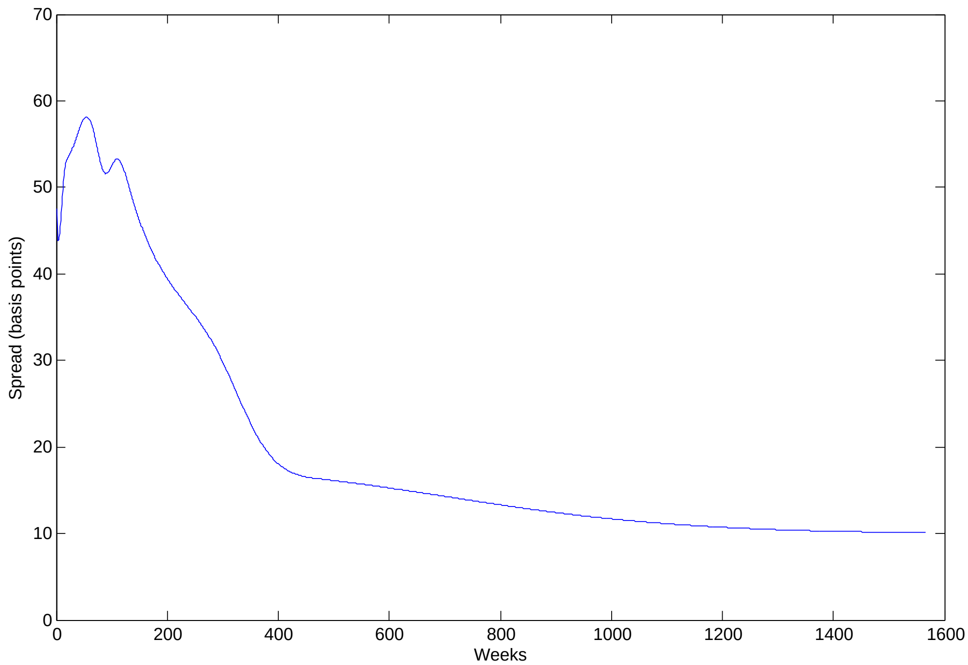

Figure 3: LIBOR / OIS spread curve