

INTEREST RATES AND FX MODELS

8. Portfolio greeks

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1 Introduction

One of the most important tasks faced by a portfolio manager, a trader, or a risk manager is to manage the interest rate exposure of a portfolio of fixed income securities such as government bonds, corporate bonds, mortgage backed securities, structured interest rate products, etc. This interest rate risk may manifest itself in various ways: (i) risk to the level of rates, (ii) risk to the convexity of instruments, and (iii) risk to the volatility of rates.

Traditional risk measures of options are the *greeks*: delta, gamma, vega, theta, etc.¹, see for example [3]. Recall, for example, that the delta of an option is the derivative of the premium with respect the underlying. This poses a bit of a problem in the world of interest rate derivatives, as the interest rates play a dual role in the option valuation formulas: (a) as the underlyings, and (b) as the discount rates. One has thus to differentiate both the underlying *and* the discount factor when calculating the delta of a swaption!

The key issue is to quantify this exposure and, if required, offset aspects it by taking positions in liquid vanilla instruments such as Eurodollar futures, swaps, swaptions, caps/floors, etc.

In addition to the various facets of interest rate risk, fixed income portfolios carry other kinds of risk. Government bonds carry foreign exchange risk and sovereign credit risk, corporate bonds are exposed to credit and liquidity risk, and mortgage backed securities have prepayment and credit risk. CAT bonds carry

¹Rho, vanna, volga,...

risks to natural catastrophes such as earth quakes and hurricanes. These other types of risk are the defining characteristics of the relevant instruments and are, in fact, their *raison d'être*. Discussion of these risks is beyond the scope of this course, and we refer the reader to specialized literature.

There are other widely used risk metrics such as the value at risk (VaR), which we do not discuss in this course. The focus by VaR is less myopic than the sensitivity measure given by the greeks; instead it is a measure of the global tail risk. Under a given level of risk tolerance, what is the maximum loss that a portfolio can sustain? VaR is a tool used by banks to set aside their economic capital, clearing houses to set the margin levels, and hedge funds to set the risk capital to their traders.

2 Delta risk management

We begin with the dominant portion of the interest rate risk, namely the delta risk. Traditionally, this risk has been designated to as the *duration risk*. We let Π denote this portfolio, whose detailed composition is not important for our discussion. We will discuss two commonly used approaches to measure the interest rate risk of Π . Two methods of computing the delta are commonly used in the industry.

2.1 Input perturbation sensitivities

In this approach we compute the sensitivities of the portfolio to the benchmark instruments used in the curve construction, and replicate the risk of the portfolio by means of a portfolio consisting of the suitably weighted benchmark instruments.

- (a) Compute the *partial DVO1s* of the portfolio Π to each of the benchmark instruments B_i : We shift each of the benchmark rates down 1 bp and calculate the corresponding changes $\delta_i\Pi$ in the present value of the portfolio.
- (b) Compute the DVO1s $\delta_i B_i$ of the present values of the benchmark instruments under these shifts.
- (c) The hedge ratios Δ_i of the portfolio to the benchmarks are given by:

$$\Delta_i = \frac{\delta_i\Pi}{\delta_i B_i} .$$

This way of computing portfolio risk has the disadvantage that the shifts of each of the individual inputs (while keeping the others fixed) into the (multi-)curve construction propagate erratically throughout the entire curve, leading thus to its unintuitive shapes.

2.2 Regression based sensitivities

An alternative and more robust approach consists in computing the sensitivities of the portfolio to a number of virtual “micro scenarios”, and expressing these sensitivities in terms of the sensitivities of a suitably selected hedging portfolio. We proceed as follows.

First, we select a hedging portfolio and the rates scenarios. The hedging portfolio consists of vanilla instruments such as spot or forward starting swaps, Eurodollar futures, and forward rate agreements. The choice of instruments in the hedging portfolio should be made judiciously, based on understanding of the nature of the portfolio and liquidity of the instruments intended as hedges. Typically, a fixed income portfolio shows a great deal of sensitivity to the short end of the curve, and it is a good idea to include the first two years worth of Eurodollar futures. We let

$$\Pi_{\text{hedge}} = \{B_1, \dots, B_n\}.$$

denote this hedging portfolio.

We now let \mathcal{C}_0 denote the current snapshot of the LIBOR / OIS multi-curve to which we refer as the base scenario. A micro scenario is a perturbation of the base scenario in which a segment $a \leq t < b$ of both the instantaneous LIBOR and OIS rates are shifted in parallel by a prescribed amount. For example, a micro scenario could result from \mathcal{C}_0 by shifting the first 3 month segment down by 1 basis point. Choose a *complete* set of non-overlapping micro scenarios

$$\mathcal{C}_1, \dots, \mathcal{C}_p.$$

What we mean by this is that (i) the shifted segments $a_i \leq t < b_i$ and $a_j \leq t < b_j$ of distinct \mathcal{C}_i and \mathcal{C}_j do not overlap, and (ii) the union of all $a_i \leq t < b_i$, $i = 1, \dots, p$ is $(0, T_{\max})$. There are of course, infinitely many ways of choosing a complete set of non-overlapping micro scenarios. Ideally, we would select a large number of scenarios corresponding to narrowly spaced shifted segments but this may be impractical because of computational budget constraints. A reasonable alternative is a choice in which the short end of the curve is covered with narrow

shifted segments which become sparser as we move toward the back end of the curve.

We then compute the sensitivities of the portfolio and the hedging portfolio under these curve shifts. The vector $\delta\Pi$ of portfolio's *sensitivities* under these micro scenarios is

$$\delta_i\Pi = \Pi(\mathcal{C}_i) - \Pi(\mathcal{C}_0), \quad i = 1, \dots, p, \quad (1)$$

where by $\Pi(\mathcal{C}_i)$ we denote the value of the portfolio given the shifted forward curve \mathcal{C}_i . The matrix δB of sensitivities of the hedging instruments to these scenarios is

$$\delta_i B_j = B_j(\mathcal{C}_i) - B_j(\mathcal{C}_0). \quad (2)$$

In order to avoid accidental co-linearities between its rows or columns, we should always use more micro scenario than hedging instruments.

Finally, we translate the risk of the portfolio to the vector of hedge ratios with respect to the instruments in the hedging portfolio. We do this by means of *ridge regression*. The vector Δ of *hedge ratios* is calculated by minimizing the following objective function:

$$\mathcal{L}(\Delta) = \frac{1}{2} \|\delta B \Delta - \delta\Pi\|^2 + \frac{1}{2} \lambda \|Q \Delta\|^2. \quad (3)$$

Here, λ is an appropriately chosen small smoothness parameter (similar to the Tikhonov regularizer!), and Q is the smoothing operator (say, the identity matrix). Explicitly,

$$\Delta = ((\delta B)^t \delta B + \lambda Q^t Q)^{-1} (\delta B)^t \delta\Pi,$$

where the superscript t denotes matrix transposition.

One can think of the component Δ_j as the sensitivity of the portfolio to the hedging instrument B_j . This method of calculating portfolio sensitivities is called the *ridge regression* method. It is very robust, and allows one to view the portfolio risk in a flexible way.

In addition, one should quantify the exposure of the portfolio to the LIBOR / OIS basis by performing suitable sensitivity analysis of the portfolio under perturbing the spread curve.

LMM is ideally suited to implement this approach, as its dynamics traces the evolution of the entire forward curve. Specifically, we proceed as follows. We use the current forward curve \mathcal{C}_0 as the initial condition for the Monte Carlo simulations based on LMM. Using these paths, we calculate the values of the portfolio

Π as well as each of the hedging instruments B_j (the latter may not require using simulations). This way we calculate the values $\Pi(\mathcal{C}_0)$ and $B_j(\mathcal{C}_0)$ introduced above. Next, for each of the micro scenarios $\mathcal{C}_i, i = 1, \dots, p$, we generate the same number of Monte Carlo paths using \mathcal{C}_i as the initial condition. It is important that the paths in each scenario are generated using the same seed for the random number generator (or the same Sobol numbers); otherwise additional sampling noise will be introduced into the process. We use them to compute the perturbed values $\Pi(\mathcal{C}_i)$ and, if need be, $B_j(\mathcal{C}_i)$.

3 Gamma risk management

The gamma of a portfolio is a measure of the non-constancy of its delta under the evolving market. In the case of an individual European option, the gamma is defined as the second derivative of the option price with respect to the underlying. Such a definition is rather useless for a portfolio of complex fixed income securities, as it would amount to calculating a noisy, high dimensional matrix of second partial derivatives.

A more practical way to look at the gamma risk is to view it as the change in the portfolio delta under specified macro scenarios:

$$\Xi_0, \Xi_1, \dots, \Xi_r, \quad (4)$$

with Ξ_0 base scenario (no change in rates). These could be, for instance, the scenarios produced by several principal components of the curve covariance matrix, or by specified hypothetical market moves. For example, we could take:

$$\begin{aligned} \Xi_{+50} &: \text{all rates up 50 basis points,} \\ \Xi_{+25} &: \text{all rates up 25 basis points,} \\ \Xi_{-25} &: \text{all rates down 25 basis points,} \\ \Xi_{-50} &: \text{all rates down 50 basis points.} \end{aligned} \quad (5)$$

For each of the macro scenarios, we calculate the deltas

$$\Delta_1, \dots, \Delta_r, \quad (6)$$

as explained in the previous section. The quantities:

$$\begin{aligned} \Gamma_1 &= \Delta_1 - \Delta_0, \\ &\vdots \\ \Gamma_r &= \Delta_r - \Delta_0, \end{aligned} \quad (7)$$

are the portfolio gammas under the corresponding scenarios. For intermediate market moves, the portfolio gamma can be calculated by linearly interpolating gammas corresponding to the specified macro scenarios.

4 Vega risk management

In order to quantify the vega risk we have to first design appropriate volatility scenarios. In Section 3 of Lecture 5, we explained how LMM stores its internal representation \mathcal{S} of the volatility surface. We construct volatility micro scenarios by accessing \mathcal{S} and shifting selected non-overlapping segments. Let us call these scenarios

$$\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_q, \quad (8)$$

with $\mathcal{S}_0 = \mathcal{S}$ being the base scenario. Next, we choose a hedging portfolio Π_{hedge} which may consist of liquid instruments such as swaptions, caps and floors, Eurodollar options, or other instruments.

The rest is a *verbatim* repeat of the delta story. We calculate the sensitivities of the portfolio to the volatility scenarios (8). We calculate the sensitivities of the hedging portfolio to the volatility scenarios. Finally, we use ridge regression to find the hedge ratios. This method of managing the vega risk works remarkably well and allows one, in particular, to separate the exposure to swaptions from the exposure to caps / floors.

The method outlined above allows us to measure and manage the at the money vega risk only. Proper smile risk management has to be based on methods explained in Lecture 3 in the context of the SABR model.

5 Risk management under SABR

The discussion of the previous section is general in the sense that no specific dynamic model has been assumed. The methods explained work for any generic interest rate model such as the Black, Hull-White, or LMM models. In this section, we discuss aspects of risk managements which are inherent to the SABR model and, by extension, to the SABR / LMM model.

As discussed in Section 2, the key measures of risk of a fixed income portfolio are sensitivities to selected segment of the curve. They can be calculated either by perturbing the inputs to the curve construction or by perturbing a segment of the OIS / forward curve, and calculating the impact of this perturbation on the

value of the portfolio. Likewise, the vega risk is the sensitivity of the portfolio to volatility and is traditionally measured as the derivative of the option price with respect to the implied volatility. The choice of volatility model impacts not only the prices of (out of the money) options but also, at least equally significantly, their risk sensitivities. One has to think about the following issues:

- (a) How is the vega risk defined: as the sensitivity to the lognormal volatility, normal volatility, or another volatility parameter?
- (b) How is the delta risk defined: which volatility parameter should be kept constant when taking the derivative with respect to the underlying?

In this section, we assume the SABR model specification, and thus the relevant volatility parameter is the beta vol σ_0 .

The delta risk of an option is calculated by shifting the current value of the underlying while keeping the current value of implied volatility σ_0 fixed. In the case of a caplet / floorlet or a swaption, this amounts to shifting the relevant forward rate without changing the implied volatility:

$$\begin{aligned} F_0 &\rightarrow F_0 + \Delta F_0, \\ \sigma_0 &\rightarrow \sigma_0, \end{aligned} \tag{9}$$

where ΔF_0 is, say, -1 bp. Assuming the normal model for valuation, this scenario leads to the option delta:

$$\Delta = \frac{\partial V}{\partial F_0} + \frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial F_0}. \tag{10}$$

The first term on the right hand side in the formula above is the original Black model delta, and the second arises from the systematic change in the implied (normal) volatility as the underlying changes. This formula shows that, in stochastic volatility models, there is an interaction between classic Black-Scholes style greeks! In the case at hand, the classic delta and vega contribute both to the smile adjusted delta.

This way of calculating the delta risk is practical for a single option only. If our task is to hedge a portfolio of caps / floors and swaptions (of various expirations, strikes and underlyings), we should follow the approach of Section 2 of this lecture. Namely, we subject the portfolio to a number of forward rate shocks and replicate the resulting risk profile with the risk profile of a portfolio of liquid swaps, FRAs, etc. This simply means replacing the first of the shifts (9) by the

corresponding partial shift of the OIS / forward curve. In the following discussion we will implicitly mean these partial shifts while (for the sake of conceptual simplicity) we talk about shifting a single forward rate.

Similarly, the vega risk is calculated from

$$\begin{aligned} F_0 &\rightarrow F_0, \\ \sigma_0 &\rightarrow \sigma_0 + \Delta\sigma_0, \end{aligned} \quad (11)$$

to be

$$\Lambda = \frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial \sigma_0}. \quad (12)$$

These formulas are the classic SABR greeks.

Modified SABR greeks below attempt to make a better use of the model dynamics. Since the processes for σ and F are correlated, whenever F changes, on average σ changes as well. This change is proportional to the correlation coefficient ρ between the Brownian motions driving F and σ . It is easy to see that a scenario consistent with the dynamics is of the form

$$\begin{aligned} F_0 &\rightarrow F_0 + \Delta F_0, \\ \sigma_0 &\rightarrow \sigma_0 + \delta_F \sigma_0. \end{aligned} \quad (13)$$

Here

$$\delta_F \sigma_0 = \frac{\rho \alpha}{F_0^\beta} \Delta F_0 \quad (14)$$

is the average change in σ_0 caused by the change in the underlying forward. The new delta risk is given by

$$\Delta = \frac{\partial V}{\partial F_0} + \frac{\partial V}{\partial \sigma_n} \left(\frac{\partial \sigma_n}{\partial F_0} + \frac{\partial \sigma_n}{\partial \sigma_0} \frac{\rho \alpha}{F_0^\beta} \right). \quad (15)$$

This risk incorporates the average change in volatility caused by the change in the underlying.

Similarly, the vega risk should be calculated from the scenario:

$$\begin{aligned} F_0 &\rightarrow F_0 + \delta_\sigma F_0, \\ \sigma_0 &\rightarrow \sigma_0 + \Delta\sigma_0, \end{aligned} \quad (16)$$

where

$$\delta_\sigma F_0 = \frac{\rho F_0^\beta}{\alpha} \Delta\sigma_0 \quad (17)$$

is the average change in F_0 caused by the change in the beta vol. This leads to the modified vega risk

$$\Lambda = \frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial \sigma_0} + \left(\frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial F_0} + \frac{\partial V}{\partial F_0} \right) \frac{\rho F_0^\beta}{\alpha}. \quad (18)$$

The first term on the right hand side of the formula above is the classic SABR vega, while the second term accounts for the change in volatility caused by the move in the underlying forward rate.

References

- [1] Andersen, L., and Piterbarg, V.: *Interest Rate Modeling*, Vol. 3, Atlantic Financial Press (2010).
- [2] Bartlett, B.: Hedging under SABR model, *Wilmott Mag.*, **July/August**, 2 - 4 (2006)..
- [3] Tuckman, B.: *Fixed Income Securities*, Wiley (2002).