

# Interest Rate Volatility

## IV. The SABR-LMM model

**Andrew Lesniewski**

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# Outline

- 1 The LMM methodology
- 2 Dynamics of the SABR-LMM model
- 3 Covariance structure of SABR-LMM

# The LMM methodology

- The main shortcoming of short rate models is that they do not allow for close calibration to the entire volatility cube.
- This is not a huge concern on a trading desk, where locally calibrated term structure models allow for accurate pricing and executing trades.
- It is, however, a concern for managers of large portfolios of fixed income securities (such as mortgage backed securities) which have exposures to various segments of the curve and various areas of the vol cube.
- It is also relevant in enterprise level risk management in large financial institutions where consistent risk aggregation across businesses and asset classes is important.
- A methodology that satisfies these requirements is the LIBOR market model (LMM) methodology, and in particular its stochastic volatility extensions.
- That comes at a price: LMM is less tractable than some of the popular short rate models. It also tends to require more resources than those models.
- We will discuss a natural extension of stochastic volatility LMM, namely the SABR-LMM model.

# Classic LMM

- Let us begin with a very brief review of the classic LMM model, see eg. [1].
- We consider a sequence of approximately equally spaced dates  $0 = T_0 < T_1 < \dots < T_N$  which will be termed the *standard tenors*. A standard LIBOR forward rate  $L_j$ ,  $j = 0, 1, \dots, N - 1$  is associated with a FRA which starts on  $T_j$  and matures on  $T_{j+1}$ .
- Usually, it is assumed that  $N = 120$  and the  $L_j$ 's are 3 month LIBOR forward rates. Note that these dates refer to the actual start and end dates of the contracts rather than the LIBOR "fixing dates", i.e. the dates on which the LIBOR rates settle. To simplify the notation, we shall disregard the difference between the contract's start date and the corresponding forward rate's fixing date.
- Each LIBOR forward  $L_j$  is modeled as a continuous time stochastic process  $L_j(t)$ . Clearly, this process has the property that it gets killed at  $t = T_j$ . The dynamics of the forward process is driven by an  $N - 1$ -dimensional, correlated Wiener process  $W_1(t), \dots, W_{N-1}(t)$ .
- We let  $\rho_{jk}$  denote the correlation coefficient between  $W_j(t)$  and  $W_k(t)$ :

$$dW_j(t) dW_k(t) = \rho_{jk} dt. \quad (1)$$

# No arbitrage condition

- To motivate the form of the LIBOR forwards dynamics, consider first a single LIBOR forward  $L_k$ . Under the measure  $Q_k$ ,  $L_k$  is a martingale so that

$$\begin{aligned} dL_k(t) &= C(t) dW_k(t), \\ L_k(0) &= L_{k0}, \end{aligned} \tag{2}$$

where  $L_{k0}$  is the current value of the forward (as given by the curve model).

- The coefficient

$$C_k(t) = C_k(t, L_k(t)) \tag{3}$$

is the instantaneous volatility and it determines the internal volatility smile dynamics.

- For  $j \neq k$ ,  $L_j$  is *not* a martingale under  $Q_k$ , and so we must have

$$\begin{aligned} dL_j(t) &= \Delta_j^{(k)}(t) dt + C_j(t) dW_j(t), \\ L_j(0) &= L_{j0}. \end{aligned} \tag{4}$$

- We shall determine the drifts  $\Delta_j^{(k)}(t)$  by requiring absence of arbitrage.
- The *no arbitrage* requirement of asset pricing imposes a relationship between the drift term and the diffusion term, whose form depends on the choice of  $k$ .

# Change of numeraire formula

- This is accomplished by means of the change of numeraire technique: it allows us to modify the probability law (the measure) of the process so that, under the new measure, the process is driftless, i.e. it is a martingale.
- Before we proceed, let us take a minute to recall the *change of numeraire formula*. It is a consequence of Girsanov's theorem, and its derivation can be found e.g. in [1].
- Consider a financial asset whose dynamics is given in terms of the state variable  $X$ . Under the measure  $\mathbb{P}$ , associated with the numeraire  $\mathcal{N}$ , its dynamics reads:

$$dX(t) = \Delta^{\mathbb{P}}(t) dt + C(t) dW^{\mathbb{P}}(t). \quad (5)$$

- We can relate this dynamics to the dynamics of the same asset under an equivalent martingale measure  $\mathbb{Q}$ , associated with the numeraire  $\mathcal{M}$ :

$$dX(t) = C(t) dW^{\mathbb{Q}}(t) \quad (6)$$

(remember that the diffusion coefficients in these equations are unaffected by the change of measure!).

- Namely, the drift coefficient  $\Delta^{\mathbb{P}}(t)$  can be expressed in terms of  $\mathcal{N}$  and  $\mathcal{M}$  as follows:

$$\Delta^{\mathbb{P}}(t) dt = dX(t) d\left(\log \frac{\mathcal{N}(t)}{\mathcal{M}(t)}\right). \quad (7)$$

# No arbitrage condition

- We shall now apply this formula to LMM.
- Let us first assume that  $j < k$ . The numeraires for the measures  $\mathbb{Q}_j$  and  $\mathbb{Q}_k$  are the prices  $P(t, T_{j+1})$  and  $P(t, T_{k+1})$  of the zero coupon bonds expiring at  $T_{j+1}$  and  $T_{k+1}$ , respectively.
- Explicitly,

$$P(t, T_{j+1}) = P(t, T_{\gamma(t)}) \prod_{\gamma(t) \leq i \leq j} \frac{1}{1 + \delta_i F_i(t)}, \quad (8)$$

where  $F_i$  denotes the OIS forward<sup>1</sup> spanning the accrual period  $[T_i, T_{i+1})$ , and where  $\gamma : [0, T_N] \rightarrow \mathbb{Z}$  is defined by

$$\gamma(t) = m + 1, \quad \text{if } t \in [T_m, T_{m+1}).$$

- Notice that  $P(t, T_{\gamma(t)})$  is the “stub” discount factor over the incomplete accrual period  $[t, T_{\gamma(t)}]$ .

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<sup>1</sup> Recall that all discounting is done on OIS.

# No arbitrage condition

- Since the drift of  $L_j(t)$  under  $Q_j$  is zero, formula (7) yields:

$$\begin{aligned}
 \Delta_j(t) dt &= dL_j(t) d \log \frac{P(t, T_{j+1})}{P(t, T_{k+1})} \\
 &= -dL_j(t) d \log \prod_{j+1 \leq i \leq k} (1 + \delta_i F_i(t)) \\
 &= - \sum_{j+1 \leq i \leq k} dL_j(t) \frac{\delta_i dF_i(t)}{1 + \delta_i F_i(t)} \\
 &= -C_j(t) \sum_{j+1 \leq i \leq k} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt,
 \end{aligned} \tag{9}$$

where, in the third line, we have used the fact that the spread between  $L_j$  and  $F_j$  is deterministic, and thus its contribution to  $dL_j dF_i$  is zero.

- Similarly, for  $j > k$ , we find that

$$\Delta_j(t) = C_j(t) \sum_{k+1 \leq i \leq j} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)}. \tag{10}$$



# No arbitrage condition

- We can thus summarize the above discussion as follows. The dynamics of the LMM model is given by the following system of SDEs: for  $t < \min(T_k, T_j)$ ,

$$dL_j(t) = C_j(t) \times \begin{cases} -\sum_{j+1 \leq i \leq k} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt + dW_j(t), & \text{if } j < k, \\ dW_j(t), & \text{if } j = k, \\ \sum_{k+1 \leq i \leq j} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt + dW_j(t), & \text{if } j > k. \end{cases} \quad (11)$$

- In addition to the forward measures discussed above, it is convenient to use the spot measure. It is expressed in terms of the rolling banking account numeraire:

$$B(t) = \frac{P(t, T_{\gamma(t)})}{\prod_{1 \leq i \leq \gamma(t)} P(T_{i-1}, T_i)}. \quad (12)$$

- Under the spot measure, the LMM dynamics reads:

$$dL_j(t) = C_j(t) \left( \sum_{\gamma(t) \leq i \leq j} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt + dW_j(t) \right). \quad (13)$$

# Classic LMM

- Before proceeding to the stochastic volatility extension of LMM, let us make a few observations.
- LMM allows exact calibration to the current LIBOR / OIS multi-curve. Unlike the short rate models discussed earlier, this calibration is done *separately* from the calibration of the dynamic aspects of the model: the current curve is simply the initial condition for the dynamics of the model.
- LMM is intrinsically multi-factor, meaning that it is capable of capturing accurately various aspects of the curve dynamics: parallel shifts, steepenings / flattenings, butterflies, etc.
- Its specification allows for a rich variety of volatility regimes.
- LMM is not Markovian and all valuations require Monte Carlo simulations.

# LMM and smile dynamics

- The classic LMM model has a severe drawback: while it is possible to calibrate it to match at the money option prices, it generally misprices out of the money options.
- The reason for this is its specification. While the market uses stochastic volatility models in order to price out of the money vanilla options, LMM is incompatible with such models.
- In order to remedy the problem, we describe a model that combines the key features of the LMM and SABR models.
- To this end, we assume that the instantaneous volatilities  $C_j(t)$  of the forward rates  $L_j$  are of the form

$$C_j(t) = C_j(t, L_j(t), \sigma_j(t)), \quad (14)$$

with stochastic volatility parameters  $\sigma_j(t)$ .

- The SABR specification corresponds to the choice

$$C_j(t) = \sigma_j(t) L_j(t)^{\beta_j}, \quad (15)$$

with  $\beta_j \leq 1$ .

# LMM and smile dynamics

- Furthermore, we assume that, under the  $T_{k+1}$ -forward measure  $Q_k$ , the full dynamics of the forward is given by the stochastic system:

$$\begin{aligned}dL_k(t) &= C_k(t) dW_k(t), \\d\sigma_k(t) &= D_k(t) dZ_k(t).\end{aligned}\tag{16}$$

where the diffusion coefficient  $D_k(t)$  of the process  $\sigma_k(t)$  is a process of the form

$$D_j(t) = D_j(t, L_j(t), \sigma_j(t)).\tag{17}$$

- The SABR specification corresponds to the choice

$$D_k(t) = \alpha_k(t) \sigma_k(t).\tag{18}$$

- We assume that the instantaneous vol of vol  $\alpha_k(t)$  is a *deterministic function* of  $t$  rather than a constant. This extra flexibility is added in order to make sure that the model can be calibrated to market data.

# LMM and smile dynamics

- From the general principles discussed above, we expect that the  $Q_k$ -dynamics of  $j$ -th LIBOR forward is given by

$$\begin{aligned}dL_j(t) &= \Delta_j^{(k)}(t) dt + C_j(t) dW_j(t), \\d\sigma_j(t) &= \Gamma_j^{(k)}(t) dt + D_j(t) dZ_j(t),\end{aligned}\tag{19}$$

- These equations are supplemented by the initial conditions:

$$\begin{aligned}L_j(0) &= L_{j0}, \\ \sigma_j(0) &= \sigma_{j0},\end{aligned}\tag{20}$$

where  $L_{j0}$ 's and  $\sigma_{j0}$ 's are the currently observed values.

- We impose the following correlation structure among the Brownian motions:

$$\begin{aligned}dW_i(t) dZ_j(t) &= r_{ij} dt, \\ dZ_i(t) dZ_j(t) &= \eta_{ij} dt.\end{aligned}\tag{21}$$

# LMM and smile dynamics

- We note that a pair of SDEs for a fixed  $j$  describes the dynamics of a forward maturing at  $T_{j+1}$ . The parameter  $\sigma_{j0}$  is simply the SABR  $\sigma_0$ -parameter, while the correlation  $r_{jj}$  is the  $\rho$ -parameter. The SABR  $\alpha$ -parameter  $\alpha_j$  is given by

$$\alpha_j = \sqrt{\frac{1}{T_j} \int_0^{T_j} \alpha_j(t)^2 dt} . \quad (22)$$

- The model can thus be calibrated exactly to the cap / floor market.
- The matrix  $\{r_{ij}\}$  is not symmetric. Its off-diagonal elements do not affect cap / floor valuation.
- The block correlation matrix

$$\Pi = \begin{bmatrix} \rho & r \\ r^\top & \eta \end{bmatrix} \quad (23)$$

is assumed to be strictly positive definite.

# No arbitrage condition continued

- We determine the drifts  $\Delta_j^{(k)}(t)$  and  $\Gamma_j^{(k)}(t)$  by requiring absence of arbitrage.
- The calculation in the case of  $\Delta_j^{(k)}(t)$  is essentially identical to the derivation of the drift terms for the classic LMM.
- As a result, under  $Q_k$ ,

$$dL_j(t) = C_j(t) \times \begin{cases} -\sum_{j+1 \leq i \leq k} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt + dW_j(t), & \text{if } j < k, \\ dW_j(t), & \text{if } j = k, \\ \sum_{k+1 \leq i \leq j} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt + dW_j(t), & \text{if } j > k. \end{cases} \quad (24)$$

- Similarly, under the spot measure, the SABR-LMM dynamics reads:

$$dL_j(t) = C_j(t) \left( \sum_{\gamma(t) \leq i \leq j} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt + dW_j(t) \right). \quad (25)$$

## No arbitrage condition continued

- Let us now compute the drift term  $\Gamma_j^{(k)}(t)$  in the SDE for  $\sigma_j(t)$ .
- We proceed as in the case of calculating  $\Delta_j^{(k)}(t)$ . Assuming first that  $j < k$  and using formula (7):

$$\begin{aligned}
 \Gamma_j(t) dt &= d\sigma_j(t) d \log \frac{P(t, T_{j+1})}{P(t, T_{k+1})} \\
 &= -d\sigma_j(t) d \log \prod_{j+1 \leq i \leq k} (1 + \delta_i F_i(t)) \\
 &= - \sum_{j+1 \leq i \leq k} d\sigma_j(t) \frac{\delta_i dF_i(t)}{1 + \delta_i F_i(t)} \\
 &= -D_j(t) \sum_{j+1 \leq i \leq k} \frac{r_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt.
 \end{aligned}$$

- Similarly, for  $j > k$ , we find that

$$\Gamma_j(t) = D_j(t) \sum_{k+1 \leq i \leq j} \frac{r_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)}.$$



# No arbitrage condition continued

- This leads to the following stochastic dynamics. Under the measure  $Q_k$ ,

$$d\sigma_j(t) = D_j(t) \times \begin{cases} -\sum_{j+1 \leq i \leq k} \frac{r_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt + dZ_j(t), & \text{if } j < k, \\ dZ_j(t), & \text{if } j = k, \\ \sum_{k+1 \leq i \leq j} \frac{r_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)} dt + dZ_j(t), & \text{if } j > k. \end{cases} \quad (26)$$

- Similarly, under the spot measure,

$$d\sigma_j(t) = D_j(t) \left( \sum_{\gamma(t) \leq i \leq j} \frac{r_{ji} \delta_i D_i(t)}{1 + \delta_i F_i(t)} dt + dZ_j(t) \right). \quad (27)$$

# No arbitrage condition continued

- We now substitute the volatility specification (14) and (17). Under the measure  $Q_k$ , the dynamics of the full model reads:

$$\begin{aligned}
 dL_j(t) &= \sigma_j(t) L_j(t)^{\beta_j} \\
 &\times \begin{cases} -\sum_{j+1 \leq i \leq k} \frac{\rho_{ji} \delta_i \sigma_i(t) L_i(t)^{\beta_i}}{1 + \delta_i F_i(t)} dt + dW_j(t), & \text{if } j < k, \\ dW_j(t), & \text{if } j = k, \\ \sum_{k+1 \leq i \leq j} \frac{\rho_{ji} \delta_i \sigma_i(t) L_i(t)^{\beta_i}}{1 + \delta_i F_i(t)} dt + dW_j(t), & \text{if } j > k, \end{cases} \\
 d\sigma_j(t) &= \alpha_j(t) \sigma_j(t) \\
 &\times \begin{cases} -\sum_{j+1 \leq i \leq k} \frac{r_{ji} \delta_i \sigma_i(t) L_i(t)^{\beta_i}}{1 + \delta_i F_i(t)} dt + dZ_j(t), & \text{if } j < k, \\ dZ_j(t), & \text{if } j = k, \\ \sum_{k+1 \leq i \leq j} \frac{r_{ji} \delta_i \sigma_i(t) L_i(t)^{\beta_i}}{1 + \delta_i F_i(t)} dt + dZ_j(t), & \text{if } j > k. \end{cases}
 \end{aligned} \tag{28}$$

# No arbitrage condition continued

- Similarly, under the spot measure  $Q_0$ , the dynamics is given by the stochastic system:

$$\begin{aligned}
 dL_j(t) &= \sigma_j(t) L_j(t)^{\beta_j} \left( \sum_{\gamma(t) \leq i \leq j} \frac{\rho_{ji} \delta_i \sigma_i(t) L_i(t)^{\beta_i}}{1 + \delta_i F_i(t)} dt + dW_j(t) \right), \\
 d\sigma_j(t) &= \alpha_j(t) \sigma_j(t) \left( \sum_{\gamma(t) \leq i \leq j} \frac{r_{ji} \delta_i \sigma_i(t) L_i(t)^{\beta_i}}{1 + \delta_i F_i(t)} dt + dZ_j(t) \right).
 \end{aligned} \tag{29}$$

- The stochastic dynamics given by the equations above defines the SABR-LMM model. It was proposed in [3].

## Other specifications

- Other SABR-style extensions of LMM have also been developed. The model discussed in [4] assumes one common factor driving stochastic volatility for all LIBOR forwards.
- Specifically, the  $Q_k$ -dynamics is assumed to be of the form:

$$\begin{aligned}dL_j(t) &= \Delta_j^k(t) dt + \sigma_j v(t) L_j(t)^\beta dW_j(t), \\ dv(t) &= \alpha v(t) dZ(t),\end{aligned}\tag{30}$$

with

$$dW_j(t) dZ(t) = r_j dt.\tag{31}$$

- The coefficients  $\sigma_j$  and  $\alpha$  are deterministic constants.
- The no-arbitrage condition determines the form of the drift terms in (30); they are of the form similar to (9) - (10).

# Calibratable parameters

- In order to fully calibrate the SABR-LMM, the following parameters have to be determined:
  - (i) Initial values of the LIBOR forwards  $L_{j0}$ .
  - (ii) LIBOR/OIS basis curve.
  - (iii) Initial values of the volatility parameters  $\sigma_{j0}$ .
  - (iv) Instantaneous vol of vol curves  $\alpha_j(t)$ .
  - (v) CEV exponents  $\beta_j$ .
  - (vi) Correlation matrix  $\Pi$ .
- As already pointed out, calibration of items (i) and (ii) is straightforward, as the relevant numbers can be directly read off the LIBOR / OIS multi-curve.
- Calibration of the remaining parameters on the list requires cross-sectional fitting to the prices of swaptions and caps / floors.

## Choosing the $\beta$ 's

- SABR-LMM specifies the values of the CEV exponents  $\beta_j$  for each LIBOR forward  $L_j$  but it does not explicitly refer to the CEV exponents  $\beta_{mn}$  for the benchmark forward swap rates  $S_{mn}$ .
- The latter are internally implied by the model and should be tied to the values implied by the vanilla swaption markets.
- There is no simple relation between  $\beta_j$ 's and  $\beta_{mn}$ 's. An approximation which works well in practice is given by the following formulas:

$$\begin{aligned}\beta_{mn} &= \sum_{m \leq j \leq n-1} a_{mn,j} \beta_j + b_{mn} \\ &\triangleq \mathbf{a}_{mn}^T \boldsymbol{\beta} + b_{mn},\end{aligned}\tag{32}$$

where

$$\begin{aligned}a_{mn,k} &= \frac{2 \log L_{k0}}{(n-m)^2} \sum_{m \leq j \leq n-1} \frac{1}{\log L_{j0} + \log L_{k0}}, \\ b_{mn} &= \frac{1}{(n-m)^2} \sum_{m \leq j, k \leq n-1} \frac{\log \rho_{jk}}{\log L_{j0} + \log L_{k0}}.\end{aligned}\tag{33}$$

## Choosing the $\beta$ 's

- Note that

$$\sum_{m \leq j \leq n-1} a_{mn,j} = 1. \quad (34)$$

- Consequently, the CEV exponent of a swaption is a weighted average of the CEV powers of the spanning forwards plus a convexity correction. Under a perfectly flat forward curve we would have

$$a_{mn,j} = 1/(n - m), \quad (35)$$

for all  $j$ .

- The convexity correction  $b$  is rather small. On a typical market snapshot it is of the order of magnitude  $10^{-3}$ , and thus, for all practical purposes, it can be assumed zero.
- For reasons of practicality, we should parameterize the the curve  $\beta_j$ , by a small number of parameters.

# SABR-LMM volatility matrix

- Volatility calibration of SABR-LMM requires setting the values of:
  - the volatility parameters  $\sigma_{j0}$ ,
  - the deterministic instantaneous vol of vol functions  $\alpha_j(t)$ .
- Parameters  $\sigma_{j0}$  are simply the  $\sigma$ -parameters in the SABR model for the caplet on the LIBOR forward  $L_j$ .
- For the vol of vol functions, we choose

$$\alpha_j(t) = h(T_j - t), \quad (36)$$

where  $h(t)$  is the hump function discussed in the Presentation I. It is linked to the  $\alpha$ -parameter  $\alpha_j$  for the caplet on  $L_j$  via

$$\alpha_j = \sqrt{\frac{1}{T_j} \int_0^{T_j} \alpha_j(t)^2 dt}. \quad (37)$$



# SABR-LMM volatility matrix

- In order to help the intuition, we represent the instantaneous stochastic volatility as a lower triangular matrix:

	$t \in [T_0, T_1)$	$t \in [T_1, T_2)$	...	$t \in [T_{N-1}, T_N)$
$\sigma_0(t)$	0	0	...	0
$\sigma_1(t)$	$\sigma_{1,0}$	0	...	0
$\sigma_2(t)$	$\sigma_{2,0}$	$\sigma_{2,1}$	...	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sigma_{N-1}(t)$	$\sigma_{N-1,0}$	$\sigma_{N-1,1}$	...	0

Table: 1. SABR-LMM volatility matrix

- The nonzero entries in Table 1 are given by

$$\sigma_{i,j}(t) = \sigma_i(t), \text{ for } T_j \leq t < T_{j+1}. \quad (38)$$

- We refer to the lower triangular matrix in Table 1, SABR-LMM's internal representation of volatility, as the *SABR-LMM volatility matrix*. We will denote it with the symbol  $\mathcal{V}$ .

# SABR-LMM volatility matrix

- This representation allows us to map the domain of the SABR-LMM volatility matrix which affects a given instrument.
- For example, the domain affecting a 6M by 5Y cap is marked with • in the table below:

	0	1	2	...	17	18	19	...	$N - 1$
0	0	0	0	...	0	0	0	...	0
1	○	0	0	...	0	0	0	...	0
2	•	•	0	...	0	0	0	...	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
17	•	•	•	...	0	0	0	...	0
18	•	•	•	...	•	0	0	...	0
19	•	•	•	...	•	•	0	...	0
20	○	○	○	...	○	○	○	...	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$N - 1$	○	○	○	...	○	○	○	...	0

Table: 2. The domain of vol sensitivity of a cap

# SABR-LMM volatility matrix

- On the other hand, a related instrument, 6M into 4.5Y swaption has the following vol sensitivity domain on the SABR-LMM vol matrix:

	0	1	2	...	17	18	19	...	$N - 1$
0	0	0	0	...	0	0	0	...	0
1	○	0	0	...	0	0	0	...	0
2	●	●	0	...	0	0	0	...	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
17	●	●	○	...	0	0	0	...	0
18	●	●	○	...	○	0	0	...	0
19	●	●	○	...	○	○	0	...	0
20	○	○	○	...	○	○	○	...	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$N - 1$	○	○	○	...	○	○	○	...	0

Table: 3. The domain of vol sensitivity of a swaption

# The correlation matrix

- Simultaneous calibration to the cap / floor and swaption markets in a stable and consistent manner requires calibrating the high dimensional correlation matrix  $\Pi$ . The dimensionality of  $\Pi$  is  $O(N^2)$ , far exceeding the number of instruments available for calibration.
- A small subset of elements of  $\Pi$  can be directly calibrated to the cap / floor market. Namely, as already pointed out, the diagonal elements  $r_{jj}$  of the matrix  $r$  are the  $\rho$ -parameters of the SABR model for the caplet on  $L_j$ .
- The off-diagonal elements of  $r$  (which have no impact on cap / floor pricing) should be approximately constant along each row, monotone declining as the distance from the diagonal increases. A natural parametric choice is

$$r_{ij} = r_{ii} e^{-\nu |T_i - T_j|}, \quad (39)$$

where  $\nu > 0$  is a calibratable parameter.

# The correlation matrix

- There is no reliable way of calibrating the correlation matrices  $\rho$  and  $\eta$  to market implieds. In particular, using “no arbitrage” relations between caps and swaptions may lead to wrong results.
- Instead, we estimate  $\rho$  and  $\eta$  from the historical data. A relatively long observation window (at least 6 months) should be used.
- To assure the stability of the calibrated parameters, it is common to model the correlation parameters in a parametric form. A convenient and intuitive parameterized form of  $\rho_{ij}$  is given by the formula:

$$\rho_{ij} = \bar{\varrho}_{\min(i,j)} + (1 - \bar{\varrho}_{\min(i,j)}) \exp(-\lambda_{\min(i,j)} |T_i - T_j|), \quad (40)$$

where

$$\bar{\varrho}_k = \varrho \tanh(\mu T_k), \quad (41)$$

and

$$\lambda_k = \lambda T_k^{-\kappa}. \quad (42)$$






# The correlation matrix

- The meaning of the parameters is as follows:
  - (i)  $\varrho$  is the asymptotic level of correlations,
  - (ii)  $\mu$  is a measure of speed at which  $\varrho$  is approached,
  - (iii)  $\lambda$  is a the decay rate of correlations,
  - (iv)  $\kappa$  is an asymmetry parameter.
- Intuitively, positive  $\kappa$  means that two consecutive forwards with short maturities are less correlated than two such forwards with long maturities.
- The parameters in this formula are fixed by fitting to the historical data.
- A word of caution: this parameterization produces a matrix that is only approximately positive definite<sup>2</sup>.
- The matrix  $\eta$  is parameterized in a similar fashion.

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<sup>2</sup> typically it has a few tiny negative eigenvalues

# References

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