

# Interest Rate Volatility

## VI. Managing interest rate volatility risk

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# Outline

- 1 Portfolio delta risk
- 2 Bartlett delta
- 3 Portfolio vega risk

# Risk metrics

- One of the most important tasks faced by a portfolio manager, a trader, or a risk manager is to quantify and monitor the interest rate exposure of a portfolio of fixed income securities such as government bonds, corporate bonds, mortgage backed securities, structured interest rate products, etc.
- This interest rate risk may manifest itself in various ways:
  - (i) risk to the level of rates (“duration risk”),
  - (ii) risk to the convexity of instruments (“convexity risk”), and
  - (iii) risk to the volatility of rates (“vega risk”).

# Risk metrics

- Traditional risk measures of options are the *greeks*: delta, gamma, vega, theta, etc.<sup>1</sup>, see for example [4]. Recall, for example, that the delta of an option is the derivative of the premium with respect the underlying.
- This poses a bit of a problem in the world of interest rate derivatives, as the interest rates play a dual role in the option valuation formulas:
  - (i) as the underlyings, and
  - (ii) as the discount rates.
- One thus has to perturb both the underlying *and* the discount factor when calculating the delta of a swaption.

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<sup>1</sup> Rho, vanna, volga,...

# Risk metrics

- The key issue is to quantify the portfolio exposure and, if required, offset aspects of it by entering appropriate positions in liquid vanilla instruments such as Eurodollar futures, swaps, swaptions, caps/floors, etc.
- In addition to the various facets of interest rate risk, fixed income portfolios carry other kinds of risk:
  - (i) government bonds carry foreign exchange risk and sovereign credit risk,
  - (ii) corporate bonds are exposed to credit and liquidity risk,
  - (iii) mortgage backed securities have prepayment and credit risk,
  - (iv) ...
- These other types of risk are the defining characteristics of the relevant instruments and are, in fact, their *raison d'être*.
- Here we focus on interest rate risk only.

# Measuring delta

- We begin with the dominant portion of the interest rate risk, namely the delta risk. Traditionally, this risk has been designated to as the *duration risk*.
- We let  $\Pi$  denote this portfolio, whose detailed composition is not important for our discussion. We will discuss two commonly used approaches to measure the interest rate risk of  $\Pi$ :
  - (i) Sensitivity to the inputs
  - (ii) Sensitivity to the forward curve

Two methods of computing the delta are commonly used in the industry.

# Input perturbation deltas

- In this approach we compute the sensitivities of the portfolio to the benchmark instruments used in the curve construction, and replicate the risk of the portfolio by means of a portfolio consisting of the suitably weighted benchmark instruments:
  - (i) Compute the *partial DVO1s* of the portfolio  $\Pi$  to each of the benchmark instruments  $B_j$ : We shift each of the benchmark rates down 1 bp and calculate the corresponding changes  $\delta_j \Pi$  in the present value of the portfolio.
  - (ii) Compute the DVO1s  $\delta_j B_j$  of the present values of the benchmark instruments under these shifts.
  - (iii) The hedge ratios  $\Delta_j$  of the portfolio to the benchmarks are given by:

$$\Delta_j = \frac{\delta_j \Pi}{\delta_j B_j} .$$

- This way of computing portfolio risk has the disadvantage that the shifts of each of the individual inputs (while keeping the others fixed) into the (multi-)curve construction propagate erratically throughout the entire curve and produce curves with nonintuitive shapes.

# Regression based deltas

- An alternative and more robust approach consists in computing the sensitivities of the portfolio to a number of virtual “micro scenarios”, and expressing these sensitivities in terms of the sensitivities of a suitably selected hedging portfolio. We proceed as follows.
- First, we select a hedging portfolio and the rates scenarios. The hedging portfolio consists of vanilla instruments such as spot or forward starting swaps, Eurodollar futures, and forward rate agreements.
- The choice of instruments in the hedging portfolio should be made judiciously, based on understanding of the nature of the portfolio and liquidity of the instruments intended as hedges. Typically, a fixed income portfolio shows a great deal of sensitivity to the short end of the curve, and it is a good idea to include the first two years worth of Eurodollar futures.
- We let

$$\Pi_{\text{hedge}} = \{B_1, \dots, B_n\}.$$

denote this hedging portfolio.



# Regression based deltas

- We now let  $C_0$  denote the current snapshot of the LIBOR / OIS multi-curve to which we refer as the base scenario. A micro scenario is a perturbation of the base scenario in which a segment  $a \leq t < b$  of both the instantaneous LIBOR and OIS rates are shifted in parallel by a prescribed amount.
- For example, a micro scenario could result from  $C_0$  by shifting the first 3 month segment down by 1 basis point. Choose a *complete* set of non-overlapping micro scenarios

$$C_1, \dots, C_p.$$

- What we mean by this is that
  - (i) the shifted segments  $a_i \leq t < b_i$  and  $a_j \leq t < b_j$  of distinct  $C_i$  and  $C_j$  do not overlap, and
  - (ii) the union of all  $a_i \leq t < b_i, i = 1, \dots, p$  is  $(0, T_{\max})$ . There are of course, infinitely many ways of choosing a complete set of non-overlapping micro scenarios.

# Regression based deltas

- Ideally, we would select a large number of scenarios corresponding to narrowly spaced shifted segments but this may be impractical because of computational budget constraints.
- A reasonable alternative is a choice in which the short end of the curve is covered with narrow shifted segments which become sparser as we move toward the back end of the curve.
- We then compute the sensitivities of the portfolio and the hedging portfolio under these curve shifts. The vector  $\delta\Pi$  of portfolio's *sensitivities* under these micro scenarios is

$$\delta_i\Pi = \Pi(C_i) - \Pi(C_0), \quad i = 1, \dots, p, \quad (1)$$

where by  $\Pi(C_i)$  we denote the value of the portfolio given the shifted forward curve  $C_i$ .

- The matrix  $\delta B$  of sensitivities of the hedging instruments to these scenarios is

$$\delta_i B_j = B_j(C_i) - B_j(C_0). \quad (2)$$

In order to avoid accidental co-linearities between its rows or columns, we should always use more micro scenario than hedging instruments.

# Regression based deltas

- Finally, we translate the risk of the portfolio to the vector of hedge ratios with respect to the instruments in the hedging portfolio. We do this by means of *ridge regression*.
- The vector  $\Delta$  of *hedge ratios* is calculated by minimizing the following objective function:

$$\mathcal{L}(\Delta) = \frac{1}{2} \|\delta B \Delta - \delta \Pi\|^2 + \frac{1}{2} \lambda \|Q \Delta\|^2. \quad (3)$$

- Here,  $\lambda$  is an appropriately chosen small smoothness parameter (similar to the Tikhonov regularizer!), and  $Q$  is the smoothing operator (say, the identity matrix). Explicitly,

$$\Delta = ((\delta B)^\top \delta B + \lambda Q^\top Q)^{-1} (\delta B)^\top \delta \Pi,$$

where the superscript  $^\top$  denotes matrix transposition.

# Regression based deltas

- One can think of the component  $\Delta_j$  as the sensitivity of the portfolio to the hedging instrument  $B_j$ .
- This method of calculating portfolio sensitivities is called the *ridge regression* method. It is very robust, and allows one to view the portfolio risk in a flexible way.
- In addition, one should quantify the exposure of the portfolio to the LIBOR / OIS basis by performing suitable sensitivity analysis of the portfolio under perturbing the spread curve.
- Regression based deltas are ideally suited for working with SABR-LMM, as its dynamics traces the evolution of the entire forward curve. Perturbing a segment of the forward curve is essentially equivalent to perturbing a set of consecutive LIBOR forwards.
- Specifically, we proceed as follows.

# Regression based deltas

- We use the current forward curve  $C_0$  as the initial condition for the Monte Carlo simulations based on SABR-LMM. Using these paths, we calculate the values of the portfolio  $\Pi$  as well as each of the hedging instruments  $B_j$  (the latter may not require using simulations).
- This way we calculate the values  $\Pi(C_0)$  and  $B_j(C_0)$  introduced above. Next, for each of the micro scenarios  $C_i, i = 1, \dots, p$ , we generate the same number of Monte Carlo paths using  $C_i$  as the initial condition.
- It is important that the paths in each scenario are generated using the same seed for the random number generator (or the same Sobol numbers); otherwise additional sampling noise will be introduced into the process.
- We use them to compute the perturbed values  $\Pi(C_i)$  and, if need be,  $B_j(C_i)$ .

# Delta risk under SABR

- The discussion above is general in the sense that no specific dynamic model has been assumed: the methods explained work for any generic interest rate model using the forward curve as an input.
- We now discuss the aspects of delta risk which are inherent to the SABR and LMM-SABR models.
- As explained above, the key measures of the delta risk of a fixed income portfolio are the sensitivities to selected segments of the curve. They can be calculated either by perturbing the inputs to the curve construction or by perturbing a segment of the OIS / forward curve, and calculating the impact of this perturbation on the value of the portfolio.
- Likewise, the vega risk is the sensitivity of the portfolio to volatility and is traditionally measured as the derivative of the option price with respect to the implied volatility.
- The choice of volatility model impacts not only the prices of (out of the money) options but also, at least equally significantly, their risk sensitivities.

# Delta risk under SABR

- One has to take into account the following issues:
  - (i) How is the vega risk defined: as the sensitivity to the lognormal volatility, normal volatility, or another volatility parameter?
  - (ii) How is the delta risk defined: which volatility parameter should be kept constant when taking the derivative with respect to the underlying?
- Since we assume the SABR model specification, and thus the relevant volatility parameter is the beta vol  $\sigma_0$ .

# Delta risk under SABR

- The delta risk of an option is calculated by shifting the current value of the underlying while keeping the current value of implied volatility  $\sigma_0$  fixed.
- In the case of a caplet / floorlet or a swaption, this amounts to shifting the relevant forward rate without changing the implied volatility:

$$\begin{aligned} F_0 &\rightarrow F_0 + \Delta F_0, \\ \sigma_0 &\rightarrow \sigma_0, \end{aligned} \tag{4}$$

where  $\Delta F_0$  is, say,  $-1$  bp.

- Assuming the normal model for valuation, this scenario leads to the option delta:

$$\Delta = \frac{\partial V}{\partial F_0} + \frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial F_0}. \tag{5}$$



# Delta risk under SABR

- The first term on the right hand side in the formula above is the original Black model delta, and the second arises from the systematic change in the implied (normal) volatility as the underlying changes.
- This formula shows that, in stochastic volatility models, there is an interaction between classic Black-Scholes style greeks. In the case at hand, the classic delta and vega contribute both to the smile adjusted delta.
- Notice that this way of calculating the delta risk is practical for a single option only. For a portfolio of caps / floors and swaptions (of various expirations, strikes and underlyings), we should follow one of the approaches discussed above.
- For example, in the regression based approach, we subject the portfolio to a number of forward rate shocks and replicate the resulting risk profile with the risk profile of a portfolio of liquid swaps, FRAs, etc. This simply means replacing the first of the shifts (4) by the corresponding partial shift of the OIS / forward curve.
- In the following discussion we will implicitly mean these partial shifts while (for the sake of conceptual simplicity) we talk about shifting a single forward rate.

# Gamma

- The gamma of a portfolio is a measure of the non-constancy of its delta under the evolving market.
- In the case of an individual European option, the gamma is defined as the second derivative of the option price with respect to the underlying.
- Such a definition is rather useless for a portfolio of complex fixed income securities, as it would amount to calculating a noisy, high dimensional matrix of second partial derivatives.

# Gamma

- A more practical way to look at the gamma risk is to view it as the change in the portfolio delta under specified macro scenarios:

$$\Xi_0, \Xi_1, \dots, \Xi_r, \quad (6)$$

with  $\Xi_0$  base scenario (no change in rates).

- These could be, for instance, the scenarios produced by several principal components of the curve covariance matrix, or by specified hypothetical market moves.
- For example, we could take:

$$\begin{aligned} \Xi_{+50} &: \text{all rates up 50 basis points,} \\ \Xi_{+25} &: \text{all rates up 25 basis points,} \\ \Xi_{-25} &: \text{all rates down 25 basis points,} \\ \Xi_{-50} &: \text{all rates down 50 basis points.} \end{aligned} \quad (7)$$

# Gamma

- For each of the macro scenarios, we calculate the deltas

$$\Delta_1, \dots, \Delta_r, \quad (8)$$

as explained in the previous section.

- The quantities:

$$\begin{aligned} \Gamma_1 &= \Delta_1 - \Delta_0, \\ &\vdots \\ \Gamma_r &= \Delta_r - \Delta_0, \end{aligned} \quad (9)$$

are the portfolio gammas under the corresponding scenarios.

- For intermediate market moves, the portfolio gamma can be calculated by linearly interpolating gammas corresponding to the specified macro scenarios.

# Delta risk under SABR

- The issue with the scenario (4) is, however, that it is incompatible with the stochastic volatility dynamics of SABR.
- Namely, decomposing the Brownian motion  $Z$  into a component proportional to  $W$  and a component  $W^\perp$  independent of  $W$ , we can write the dynamics in the form:

$$\begin{aligned}dF(t) &= \sigma(t) C(F(t)) dW(t), \\d\sigma(t) &= \alpha\sigma(t) (\rho dW(t) + \sqrt{1 - \rho^2} dW^\perp(t)).\end{aligned}\tag{10}$$

- This shows that

$$d\sigma(t) = \frac{\rho\alpha}{C(F(t))} dF(t) + \text{independent noise}.\tag{11}$$

- As a result, for non-zero correlation  $\rho$ , a move in the forward tends to move the volatility parameter by an amount proportional to the change in the forward.

# Delta risk under SABR

- This leads us to the conclusion that a scenario consistent with the dynamics is of the form:

$$\begin{aligned}F_0 &\rightarrow F_0 + \Delta F_0, \\ \sigma_0 &\rightarrow \sigma_0 + \delta_F \sigma_0.\end{aligned}\tag{12}$$

- Here

$$\delta_F \sigma_0 = \frac{\rho \alpha}{F_0^\beta} \Delta F_0\tag{13}$$

is the expected change in  $\sigma_0$  caused by the change in the underlying forward.

- The new delta risk [3] is given by

$$\Delta = \frac{\partial V}{\partial F_0} + \frac{\partial V}{\partial \sigma_n} \left( \frac{\partial \sigma_n}{\partial F_0} + \frac{\partial \sigma_n}{\partial \sigma_0} \frac{\rho \alpha}{F_0^\beta} \right).\tag{14}$$

This delta risk, the “Bartlett delta”, respects the dynamics of SABR.

# Delta risk under SABR

- Figure 1 shows the classic SABR delta corresponding to three different calibrations:  $\beta = 0$  (black line),  $\beta = 0.5$  (red line), and  $\beta = 1$  (green line):

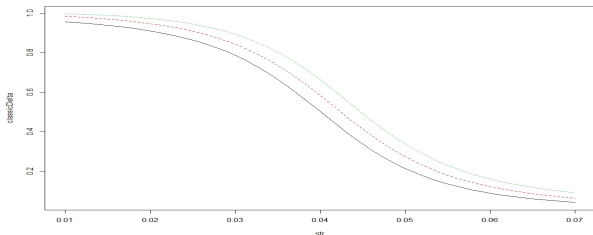


Figure: 1. Classic SABR delta for different values of  $\beta$ .

- Even though all three sets of parameters closely fit the market smile, they lead to different conventional hedges, even near the money.
- Choosing the incorrect beta can lead to good fits of the smile, but relatively poor delta hedges.

# Delta risk under SABR

- Figure 2 shows Bartlett's deltas for the same three sets of parameters:

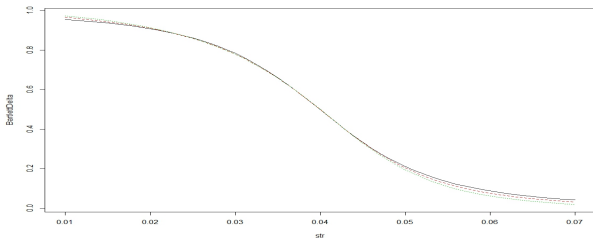


Figure: 2. Bartlett's SABR delta for different values of  $\beta$ .

- Bartlett's delta is nearly independent of  $\beta$ . It depends mainly on the actual market skew / smile, and not on how the smile is parameterized.
- Bartlett's deltas tends to provide more robust hedges.



# Delta risk under SABR

- Figure 3 below (taken from [1]) presents empirical data illustrating the presence of the Bartlett correction in the 1Y into 10Y swaption deltas

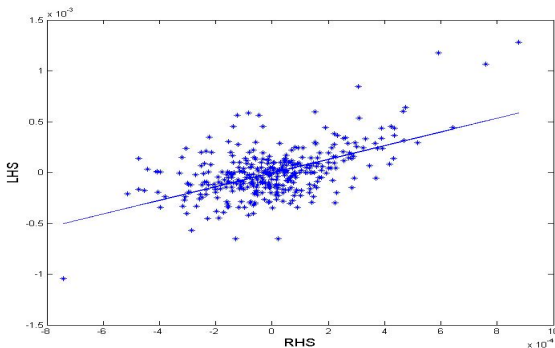


Figure: 3. Regression of  $\delta_F \sigma_0$  against  $\rho \alpha / F^\beta$  for the 1Y into 10Y swaption  $\beta = 0.5$ .

# Delta risk under SABR

- Figure 4 below (taken from [1]) presents empirical data illustrating the presence of the Bartlett correction in the 5Y into 5Y swaption deltas.

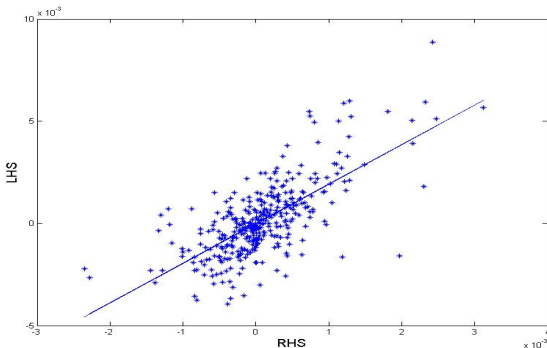


Figure: 4. Regression of  $\delta_F \sigma_0$  against  $\rho \alpha / F^\beta$  for the 5Y into 5Y swaption  $\beta = 0.75$ .

# Delta risk under SABR-LMM

- The arguments above can be extended to the SABR-LMM model. Consider a LIBOR forward  $L_j$ , and write the corresponding in the form:

$$\begin{aligned}dL_j(t) &= \sigma_j(t) L_j(t)^{\beta_j} (\{\dots\} dt + dW_j(t)), \\d\sigma_j(t) &= \alpha_j(t) \sigma_j(t) (\{\dots\} dt + r_{jj} dW_j + \sqrt{1 - r_{jj}^2} dW_j^\perp(t)),\end{aligned}\tag{15}$$

where  $\{\dots\}$  stands for expressions whose explicit form is of no relevance for this calculation.

- This shows that

$$d\sigma_j(t) = \frac{r_{jj}\alpha_j(t)}{L_j(t)^{\beta_j}} dL_j(t) + \text{independent noise} + \{\dots\} dt.\tag{16}$$

- The drift term  $\{\dots\} dt$  above is small relative to the first term on the right hand side of the equation above, and we will neglect it.

# Delta risk under SABR-LMM

- This shows that it is natural to follow a shift in the LIBOR forward rate:

$$L_{j0} \rightarrow L_{j0} + \Delta L_{j0} \quad (17)$$

by the following shift in the corresponding volatility parameter:

$$\sigma_{j0} \rightarrow \sigma_{j0} + \frac{r_{jj}\alpha_j}{L_{j0}^{\beta_j}} \Delta L_{j0}, \quad (18)$$

where  $\alpha_j = \sqrt{\frac{1}{T_m} \int_0^{T_m} \alpha_j(t)^2 dt}$  is the average instantaneous vol of vol.

- This results in the following “Bartlett correction” to the portfolio sensitivity to the shift (17) of the LIBOR forward:

$$\frac{\partial \Pi}{\partial \sigma_{j0}} \frac{r_{jj}\alpha_j}{L_{j0}^{\beta_j}} \Delta L_{j0}. \quad (19)$$

# Vega risk under SABR

- Similarly, the vega risk is calculated from

$$\begin{aligned} F_0 &\rightarrow F_0, \\ \sigma_0 &\rightarrow \sigma_0 + \Delta\sigma_0, \end{aligned} \tag{20}$$

and is given by

$$\Lambda = \frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial \sigma_0}. \tag{21}$$

These formulas are the classic SABR greeks.

- Modified SABR greeks below attempt to make a better use of the model dynamics. Since the processes for  $\sigma$  and  $F$  are correlated, whenever  $F$  changes, on average  $\sigma$  changes as well.
- This change is proportional to the correlation coefficient  $\rho$  between the Brownian motions driving  $F$  and  $\sigma$ .

# Bartlett vega risk

- The vega risk should be calculated from the scenario:

$$\begin{aligned} F_0 &\rightarrow F_0 + \delta_\sigma F_0, \\ \sigma_0 &\rightarrow \sigma_0 + \Delta\sigma_0, \end{aligned} \tag{22}$$

where

$$\delta_\sigma F_0 = \frac{\rho F_0^\beta}{\alpha} \Delta\sigma_0 \tag{23}$$

is the average change in  $F_0$  caused by the change in the beta vol.

- This leads to the modified vega risk

$$\Lambda = \frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial \sigma_0} + \left( \frac{\partial V}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial F_0} + \frac{\partial V}{\partial F_0} \right) \frac{\rho F_0^\beta}{\alpha}. \tag{24}$$

The first term on the right hand side of the formula above is the classic SABR vega, while the second term accounts for the change in volatility caused by the move in the underlying forward rate.

# Measuring portfolio vega risk

- In order to quantify the vega risk of a portfolio, we have to first design appropriate volatility scenarios.
- In Presentation IV we discussed how SABR-LMM stores internally the volatility data in the volatility matrix  $\mathcal{V}$ . We construct volatility micro scenarios by accessing  $\mathcal{V}$  and shifting selected non-overlapping segments.
- Let us call these scenarios

$$\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_q, \quad (25)$$

with  $\mathcal{V}_0 = \mathcal{V}$  being the base scenario.





- Next, we choose a hedging portfolio  $\Pi_{\text{hedge}}$  which may consist of liquid instruments such as swaptions, caps and floors, Eurodollar options, or other instruments.

# Vega

- Now we *verbatim* follow the steps described in the case of delta risk:
  - (i) We calculate the sensitivities of the portfolio to the volatility scenarios (25).
  - (ii) Next, we calculate the sensitivities of the hedging portfolio to the volatility scenarios.
  - (iii) Finally, we use ridge regression to find the hedge ratios.
- This method of managing the vega risk works allows us, in particular, to separate the exposure to swaptions from the exposure to caps / floors.



# References

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