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Asymptotically commuting families of operators*

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Abstract. We study families of symmetric operators $\{Q_n\}$ with domains given by the range of self-adjoint contraction semigroups $\{e^{-tH_n}\}$. Assuming the asymptotic commutativity, $\lim_n [Q_n, e^{-tH_n}] = 0$, and certain other estimates, we establish the existence and properties of a limiting self-adjoint operator $Q = \lim_n Q_n$. We apply these results to the study of an elementary supersymmetry algebra.

I. Asymptotically commuting families

One often has occasion to obtain an operator Q as a limit of approximating symmetric operators Q_n. We are interested in sufficient conditions on the convergence $Q_n \to Q$ so that Q is self-adjoint, as well as possibly having other desired properties. The method we use here is to associate with Q_n a contraction semigroup $K_n(t) = e^{-tH_n}$, generated by a positive self-adjoint operator H_n. We assume that $R(K_n(t)) \subset D(Q_n)$ for every $t > 0$. Here R and D denote the range and domain respectively. Our crucial assumption is that Q_n and H_n commute asymptotically as $n \to \infty$; more precisely, that st. $\lim_n [Q_n, K_n(t)] \to 0$. We also require other estimates on the convergence as $n \to \infty$, in order to conclude the existence of an operator Q which is self-adjoint. We remark that we have previously discussed a related question, but with stronger hypotheses than the ones used here [JLO]. We have tried to formulate the estimates here in such a way that they are conveneint to establish in applications. In particular, the convergence estimates rely on approximation properties of heat kernel regularizations. In concrete examples, it may be possible to establish such convergence using estimates on path integral or random walk representations of the heat kernels.

Let us now formulate two general theorems. We give separately the hypotheses of the first theorem, as they are somewhat technical.

(α) Let $\{H_n\}$ be a sequence of positive, self-adjoint operators on a Hilbert space $\mathscr{H}$. These operators approximate the positive self-adjoint operator H in the norm

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resolvent sense, or equivalently

$$\lim_{n\to\infty} \|e^{-tH_n} - e^{-tH}\| = 0, \quad \text{for all } t \geq 0. \tag{I.1}$$

Let

$$\mathscr{D}_n = \bigcup_{t>0} R(e^{-tH_n}), \tag{I.2}$$

and

$$\mathscr{D} = \bigcup_{t>0} R(e^{-tH}). \tag{I.3}$$

(β) Let $\{Q_n\}$ be a sequence of symmetric operators on $\mathscr{H}$ with domains $\mathscr{D}_n$ defined by (I.2). Assume that for $t > 0$ there is a function $M(t)$ independent of n, and such that

$$\|Q_n e^{-tH_n}\| \leq M(t). \tag{I.4}$$

(Note that this estimate is a consequence of a bound such as $Q_n^2 \leq H_n$ or $\pm Q_n \leq H_n$.)
(γ) Assume that for any s, $t > 0$, the Q_n converge in the sense

$$\lim_{n,m\to\infty} \|e^{-tH_n}(Q_n - Q_m) e^{-sH_m}\| = 0. \tag{I.5}$$

(δ) Assume that Q_n and H_n asymptotically commute in the sense that for each $t > 0$,

$$\text{st.}\lim_{n\to\infty} [Q_n, e^{-tH_n}] = 0. \tag{I.6}$$

THEOREM 1. *Assume* ($\alpha, \beta, \gamma, \delta$). *Then there exists an operator* Q *with domain* $\mathscr{D}$, *such that* $Q\, e^{-tH}$ *is bounded for any* $t > 0$, *and*

$$Q\, e^{-tH} = \text{st.}\lim_{n\to\infty} Q_n e^{-tH_n}. \tag{I.7}$$

Furthermore

(i) $Q : \mathscr{D} \to \mathscr{D}$.
(ii) $Q \restriction \mathscr{D}$ *is essentially self-adjoint.* (*We let* Q *also denote the closure of* $Q \restriction \mathscr{D}$.)
(iii) $Q\, e^{-tH}$ *is bounded for* $t > 0$ *by*

$$\|Q\, e^{-tH}\| \leq M(t). \tag{I.8}$$

(iv) $Q\, e^{-tH} = e^{-tH} Q$.

THEOREM 2. *Assume* $(\alpha, \beta, \gamma, \delta)$ *and for each* $t > 0$ *assume*

$$(\varepsilon) \quad \text{w.} \lim_{n \to \infty} e^{-tH_n}(Q_n^2 - H_n)\, e^{-tH_n} = 0. \tag{I.9}$$

Then $H = Q^2$, *for* Q *the self-adjoint operator of Theorem* 1.

The remainder of this section is devoted to the proof of these two statements. We first construct the heat kernel regularization of Q.

DEFINITION 3. Let $K_n(t) = \exp(-tH_n)$ be a family of strongly continuous, contraction semigroups with $\mathscr{D}_n$ defined by (I.2). If Q_n is a bilinear form on $\mathscr{D}_n \times \mathscr{D}_n$, then $K_n(t)Q_nK_n(t)$ is the weak heat kernel regularization (weak HKR) of Q_n with respect to H_n. If Q_n is an operator with domain $\mathscr{D}_n$, then $Q_nK_n(t)$ is the strong heat kernel regularization of Q_n with respect to H_n.

PROPOSITION 4. (*Limit of weak* HKR's) *Assume* (α, β, γ). *Then for every* $t > 0$, *as* $n \to \infty$

$$\text{norm} \lim_{n \to \infty} K_n(t)Q_nK_n(t) = A(t) \tag{I.10}$$

exists. Furthermore

$$A(t) = e^{-tH}Q\, e^{-tH}, \tag{I.11}$$

where Q *is a symmetric operator with domain* $\mathscr{D}$, *and bounded by* $\|Q\, e^{-tH}\| \leq M(t)$, *namely* (I.8). *Finally,* $Q_n\, e^{-tH_n}$ *converges weakly to* $Q\, e^{-tH}$ *as* $n \to \infty$.

Proof. Let $A_n(t) = K_n(t)Q_nK_n(t)$. The convergence of $A_n(t)$ follows easily. We fix t and do not write it. By (β), Q_nK_n and K_nQ_n are bounded. Hence

$$\|A_n - A_m\| \leq \|(K_n - K_m)Q_nK_n\| + \|K_m(Q_n - Q_m)K_n\| + \|K_mQ_m(K_n - K_m)\|. \tag{I.12}$$

The first and third terms on the right of (I.12) converge to zero as $n, m \to \infty$ as a consequence of (I.1) and (I.4). The remaining term converges to zero as a consequence of (I.5). Thus there exists a self-adjoint operator $A(t)$ with

$$\|A_n(t) - A(t)\| \to 0, \qquad \|A(t)\| \leq M(t). \tag{I.13}$$

We now show that there exists a symmetric bilinear form Q with $A(t) = K(t)QK(t)$.

In fact, we claim that for $t > 0$, $s \geq 0$,

$$A(t + s) = K(s)A(t)K(s). \tag{I.14}$$

This follows from

$$\begin{aligned} A(t + s) &= \text{norm} \lim_n K_n(s)A_n(t)K_n(s) \\ &= \text{norm} \lim_n \{(K_n(s) - K(s))A_n(t)K_n(s) \\ &\quad + K(s)A_n(t)(K_n(s) - K(s)) + K(s)A_n(t)K(s)\}. \end{aligned} \tag{I.15}$$

The first two terms on the right of (I.15) converge in norm to zero, as $\|A_n(t)\| \leq M(t)$, $\|K_n(s)\| \leq 1$, and $\|K_n(s) - K(s)\| \to 0$. The last term converges in norm by (I.14). But (I.13) is symmetric in s and t, so also $A(t + s) = K(t)A(s)K(t)$, as long as $s > 0$. Thus let us assume that $x \in \mathscr{D}$ and for some $s > 0$, $x = K(s)\bar{x}_s$. We have

$$\langle x, A(t)x \rangle = \langle \bar{x}_s, A(t + s)\bar{x}_s \rangle = \langle K(t)\bar{x}_s, A(s)K(t)\bar{x}_s \rangle. \tag{I.16}$$

The right side of (I.16) is continuous in t as $t \to 0$, so we define

$$\langle x, Qx \rangle = \lim_{t \to 0} (x, A(t)x \rangle = \langle \bar{x}_s, A(s)\bar{x}_s \rangle. \tag{I.17}$$

By polarization, this extends to $\mathscr{D} \times \mathscr{D}$. Furthermore $A(t)$ is bounded and self-adjoint for $t > 0$, so Q is a symmetric bilinear form.

We now show that the bilinear form Q uniquely determines a symmetric operator with domain $\mathscr{D}$. It is sufficient to prove that for $x, y \in \mathscr{D}$, the form $\langle x, Qy \rangle$ is continuous in x. Suppose that $y = K(s)\bar{y}_s$, for some $s > 0$. then

$$\begin{aligned} \langle x, Qy \rangle = \langle x, QK(s)\bar{y}_s \rangle &= \lim_{t \to 0+} \langle x, A(t)K(s)\bar{y}_s \rangle \\ &= \lim_{t \to 0+} \lim_{n \to \infty} \langle x, A_n(t)K(s)\bar{y}_s \rangle \\ &= \lim_{t \to 0+} \lim_{n \to \infty} \langle x, A_n(t)K_n(s)\bar{y}_s \rangle. \end{aligned} \tag{I.18}$$

Thus by (I.4),

$$|\langle x, Qy \rangle| = |\langle x, QK(s)\bar{y}_s \rangle| \leq M(s) \|x\| \|\bar{y}_s\|, \tag{I.19}$$

which is continuous in x as desired. Note that (I.19) also proves (I.8). Furthermore, this argument proves weak convergence of $Q_n K_n(t)$ to $QK(t)$ on $\mathscr{D} \times \mathscr{D}$. This convergence extends to $\mathscr{H} \times \mathscr{H}$ by continuity, so the proof of the proposition is complete.

PROPOSITION 5. *Assume* $(\alpha, \beta, \gamma, \delta)$. *Then* Q *defined in Proposition* 4 *has the properties* (I.7) *and* (i–iv) *of Theorem* 1.

Proof. First we prove that for any $t \geq 0$, Q commutes with $K(t) = e^{-tH}$. In fact, since $QK(t)$ is bounded for $t > 0$, so is $K(t)Q$. Hence it is sufficient to fix t and show that $[Q, K(t)] = 0$ as a bilinear form on the dense domain $\mathscr{D} \times \mathscr{D}$. By polarization, it is sufficient to restrict to the diagonal. Using the weak convergence established in Proposition 4,

$$\langle x, [Q, e^{-tH}]x\rangle = \lim_n \langle x, [Q_n, K_n(t)]x\rangle. \tag{I.20}$$

By (I.6), this limit is zero, so Q commutes with e^{-tH}. Thus

$$Q : R(e^{-tH}) \to R(e^{-sH}), \quad 0 < s < t.$$

In particular $Q : \mathscr{D} \to \mathscr{D}$.

We now show that the two spaces $(Q \pm i)\mathscr{D}$ are dense, and therefore that $Q \restriction \mathscr{D}$ is essentially self-adjoint. Let us suppose the contrary. Then there exists $x \in \mathscr{H}$ orthogonal to $(Q + i)\mathscr{D}$, or to $(Q - i)\mathscr{D}$. Assume the former; then for some $t > 0$,

$$0 = \langle x, (Q + i)\, e^{-tH}x\rangle = \langle x, e^{-tH/2}Q\, e^{-tH/2}x\rangle + i\langle x, e^{-tH}x\rangle, \tag{I.21}$$

where we use the vanishing of (I.20). But (I.21) is the sum of a real number and an imaginary number and hence can hold only if $x = 0$. Finally we show that $Q_nK_n(t)$ converges strongly to $QK(t)$ for every $t > 0$. Write

$$Q_nK_n(t) = K_n(t/2)Q_nK_n(t/2) + [Q_n, K_n(t/2)]K_n(t/2). \tag{I.22}$$

The first term on the right of (I.22) converges in norm to

$$K(t/2)QK(t/2) = QK(t).$$

This is a consequence of (I.10) and the commutativity of Q with $K(t)$. The second term in (I.22) converges strongly to zero. In fact $K_n(t/2)$ converges in norm to $K(t/2)$, so

$$\begin{aligned}[Q_n, K_n(t/2)]K_n(t/2)x &= [Q_n, K_n(t/2)]K(t/2)x + O(1) \\ &= O(1),\end{aligned}$$

using (I.6). This completes the proof of Proposition 5 and the proof of Theorem 1.

Proof of Theorem 2. We need to prove that $H = Q^2$. Since $\mathscr{D}$ is a core for H and for Q, it is sufficient to prove that $H = Q^2$ on $\mathscr{D} \times \mathscr{D}$. By polarization we may consider the diagonal. Thus we consider for $x \in D$,

$$\begin{aligned}\langle x, Q^2 x\rangle &= \langle Qx, Qx\rangle = \langle Q\, e^{-tH}\bar{x}_t, Q\, e^{-tH}\bar{x}_t\rangle \\ &= \lim_{n\to\infty} \langle Q_n\, e^{-tH_n}\bar{x}_t, Q_n\, e^{-tH_n}\bar{x}_t\rangle \\ &= \lim_{n\to\infty} \langle \bar{x}_t, e^{-tH_n}Q_n^2\, e^{-tH_n}\bar{x}_t\rangle \\ &= \lim_{n\to\infty} \langle \bar{x}_t, e^{-tH_n}H_n\, e^{-tH_n}\bar{x}_t\rangle = \langle x, Hx\rangle.\end{aligned}$$

Here we use (ε) and the fact that $\|H_n e^{-tH_n} - He^{-tH}\| \to 0$. This completes the proof.

II. Application to a supersymmetry algebra and the spectral condition

In this section we give estimates which are sufficient to construct three operators Q_1, Q_2 and P on the domain $\mathscr{D}$, which leave $\mathscr{D}$ invariant, and which satisfy the supersymmetry algebra

$$Q_1^2 = H + P, \qquad Q_2^2 = H - P, \qquad Q_1 Q_2 + Q_2 Q_1 = 0. \tag{II.1}$$

Here Q_1, Q_2, and P are each essentially self-adjoint on $\mathscr{D}$, and P commutes with Q_1, Q_2, and H. As a consequence of (II.1),

$$Q = \frac{1}{\sqrt{2}}(Q_1 + Q_2) \tag{II.2}$$

satisfies $Q^2 = H$. Also $\pm P \le H$. (This last inequality is called the spectral condition.)

As in Section I, we assume that H is a positive self-adjoint operator which can be approximated in the norm resolvent sense by a self-adjoint family $\{H_n\}$. We formulate hypotheses on two families $\{Q_{1,n}\}$ and $\{Q_{2,n}\}$ of approximating operators such that we can construct limits Q_1 and Q_2 with the desired properties: essential self-adjointness and satisfying the algebra (II.1). We assume the following:

(α') The operators $Q_{1,n}$, $Q_{2,n}$ are symmetric on the domain $\mathscr{D}_n$ of (I.2), and the operator P is essentially self-adjoint on $\mathscr{D}_n$ and P commutes with $Q_{1,n}$, with $Q_{2,n}$, and with H_n on $\mathscr{D} \times \mathscr{D}$. We also assume the approximate algebra

$$Q_{1,n}^2 = Q_n^2 + P, \qquad Q_{2,n}^2 = Q_n^2 - P, \qquad Q_{1,n}Q_{2n} + Q_{1,n} = 0. \tag{II.3}$$

(β') In (II.3), the operators Q_n are

$$Q_n = \frac{1}{\sqrt{2}}(Q_{1,n} + Q_{2,n}),$$

and satisfy the hypothesis (β) of Section I. As a consequence

$$\|Q_{\alpha,n}\, e^{-tH_n}\| \leq M(t), \quad \alpha = 1, 2. \tag{II.4}$$

(γ') For every $s, t > 0$, the $Q_{\alpha,n}$ converge as $n \to \infty$ in the sense that

$$\lim_{n,m \to \infty} \|e^{-tH_n}(Q_{\alpha,n} - Q_{\alpha,m})\, e^{-sH_m}\| = 0, \quad \alpha = 1, 2. \tag{II.5}$$

(δ') The $Q_{\alpha,n}$ and H_n approximately commute in the sense that

$$\text{st.} \lim_{n \to \infty} [Q_{\alpha,n}, e^{-tH_n}] = 0, \quad \alpha = 1, 2. \tag{II.6}$$

THEOREM 6. *Assume* ($\alpha, \beta, \gamma, \delta, \epsilon$) *of Section* I *and* ($\alpha', \beta', \gamma', \delta'$). *Then there eixst symmetric operators* Q_1, Q_2 *with domain* $\mathscr{D}$, *such that for* $t > 0$,

$$Q_\alpha\, e^{-tH} = \text{norm} \lim_{n \to \infty} Q_{\alpha,n}\, e^{-tH_n}, \quad \alpha = 1, 2. \tag{II.7}$$

Furthermore $\|Q_\alpha\, e^{-tH}\| \leq M(t)$, *and*

(i) $Q_\alpha : \mathscr{D} \to \mathscr{D}$.
(ii) Q_α *is essentially self-adjoint on* $\mathscr{D}$ *for* $\alpha = 1, 2$. (*Let* Q_α *denote the closure* $Q_\alpha \restriction \mathscr{D}$.)
(iii) *The algebra* (II.1) *holds.*
(iv) *The spectral condition holds,*

$$\pm P \leq H. \tag{II.8}$$

Proof. We use Theorem 1 to establish the existence and self-adjointness of Q_1 and Q_2 as limits of $Q_{1,n}$ and $Q_{2,n}$. We now remark on some properties of $P = \frac{1}{2}(Q_{1,n}^2 - Q_{2,n}^2)$. We infer from ($\alpha', \beta$) that $\pm P \leq Q_n^2$, so

$$\|e^{-tH_n} P\, e^{-tH_n}\| \leq M(t)^2.$$

Since P and H_n commute, we obtain

$$\|P\, e^{-tH_n}\| \leq M(t/2)^2. \tag{II.9}$$

Define $B_n = P\,e^{-tH_n}$ as a sequence of bounded operators. We claim that

$$\operatorname{norm}\lim_{n\to\infty} B_n = B(t) = P\,e^{-tH}. \tag{II.10}$$

In fact writing $K = e^{-tH/2}$, etc., we have

$$\begin{aligned}\|B_n - B_m\| &= \|K_nPK_n - K_mPK_m\|\\ &= \|(K_n - K_m)PK_n + K_mP(K_n - K_m)\|\\ &\le 2M(t/2)^2\|K_n - K_m\| \to 0, \quad \text{as } n, m \to \infty.\end{aligned}$$

Thus $B(t) = \lim_n B_n$ exists. The operators P and H_n satisfy the hypotheses of Theorem 1 with P replacing Q_n. Consequently P is essentially self-adjoint on $\mathscr{D}$ and $B(t) = P\,e^{-tH} = e^{-tH}P$.

Let us now return to the proof of (II.1). First we work on the domain $\mathscr{D} \times \mathscr{D}$. From (II.3) we have

$$e^{-tH_n}Q_{1,n}^2\,e^{-tH_n} = e^{-tH_n}Q_n^2\,e^{-tH_n} + e^{-tH_n}P\,e^{-tH_n},$$

and this converges as $n \to \infty$ to

$$e^{-tH}Q_1^2\,e^{-tH} = e^{-tH}(H + P)\,e^{-tH}.$$

On $\mathscr{D} \times \mathscr{D}$ we let $t \to 0$ to obtain $Q_1^2 = H + P$. Likewise we conclude $Q_2^2 = H - P$. Finally we have on $\mathscr{D} \times \mathscr{D}$,

$$\begin{aligned}\langle x, Q_1Q_2x\rangle &= \lim_n \langle Q_{1,n}\,e^{-tH_n}\bar{x}_t, Q_{2,n}\,e^{-tH_n}\bar{x}_t\rangle\\ &= -\lim_n \langle Q_{2,n}\,e^{-tH_n}\bar{x}_t, Q_{1,n}\,e^{-tH_n}\bar{x}_t\rangle = -\langle x, Q_2Q_1x\rangle.\end{aligned}$$

A similar argument shows that P commutes with the self-adjoint operators Q, Q_1, and Q_2.

This completes the verification of (II.1) on the domain $\mathscr{D} \times \mathscr{D}$. Since $\mathscr{D}$ is a core for Q, Q_1, and Q_2, and since $\mathscr{D}$ is invariant under each of these operators, the algebra (II.1) extends to the self-adjoint limits.

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