

BRS TRANSFORMATION IN LATTICE GAUGE THEORIES

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Using the results of Sharpe on the gauge fixing in lattice gauge theories we construct a lattice version of the BRS transformation. As an application we state the Ward–Takahashi identities on a lattice.

1. As is well known, to give a precise meaning to continuous gauge theories on the level of perturbation theory, one has to break the gauge invariance of the lagrangian. This gives rise to a theory whose lagrangian contains, aside from the gauge invariant part, the gauge fixing term and the term involving ghost fields. It is a remarkable fact, discovered in ref. [1], that the new lagrangian still possesses a local symmetry. The corresponding symmetry transformation involves Grassmann variables and is called the BRS transformation.

The lattice formulation of gauge theories is consistent without breaking of the gauge invariance [2]. However, to obtain the correct continuum limit of a lattice theory it seems to be indispensable to break the gauge invariance (“to fix a gauge”) of the lattice action. A lattice version of the Lorentz gauge fixing was recently discussed in ref. [3], where among other things, the lattice Faddeev–Popov determinant is defined, and it is shown that the problem of Gribov ambiguities is irrelevant (on a lattice). In this note we apply the results of ref. [3] to introduce the lattice ghost fields, and prove the BRS invariance of the action. We use the BRS transformation in a standard fashion [4] to obtain the lattice Ward–Takahashi identities. For the sake of simplicity we consider the case of pure gauge theory only; the generalization to theories involving matter fields is immediate.

2. Let G be a compact semisimple Lie group, \mathcal{g} its Lie algebra, and $g \rightarrow U(g)$ a unitary representation of G in a linear space V . The representation (V, U) defines the corresponding representation of \mathcal{g} in V (we denote it by the same letter U). For $X, Y \in U(\mathcal{g})$ we define $(X, Y) = \text{Tr}(X * Y)$. Let $\{T_\alpha\}_{\alpha=1}^a$ be an orthonormal [with respect to $(,)$] basis in $U(\mathcal{g})$, and $f_{\alpha\beta\gamma}$ the corresponding structural constants. $f_{\alpha\beta\gamma}$ is antisymmetric with respect to all the indices α, β, γ .

Let B be a Grassmann algebra generated by $1, \eta_1, \dots, \eta_a, \bar{\eta}_1, \dots, \bar{\eta}_a$. We define a mapping $I: B \rightarrow B$ by setting $I(\eta_\alpha) = \bar{\eta}_\alpha, \alpha = 1, \dots, a$, and extending it (uniquely) to the whole B in such a way that (i) $I(\alpha\Theta + \beta\Psi) = \bar{\alpha}I(\Theta) + \bar{\beta}I(\Psi), \alpha, \beta \in \mathbf{C}; \Theta, \Psi \in B$, (ii) $I(\Theta\Psi) = I(\Psi)I(\Theta)$, (iii) $I^2 = \text{id}$ (an I satisfying (i)–(iii) is called an involution of B). Later on we shall write $I(\Theta) \equiv \bar{\Theta}$. Define $U^B(\mathcal{g}) = B \otimes U(\mathcal{g})$. For $\Theta, \Psi \in U^B(\mathcal{g})$ we set $(\Theta, \Psi) = \text{Tr}(\bar{\Theta}\Psi)$, i.e. $(\Theta, \Psi) = \sum_{\alpha=1}^a \bar{\Theta}_\alpha \Psi_\alpha$, if $\Theta = \sum_{\alpha=1}^a \Theta_\alpha T_\alpha, \Psi = \sum_{\alpha=1}^a \Psi_\alpha T_\alpha; \Theta_\alpha, \Psi_\alpha \in B$. Let $J^B(\mathcal{g})$ be the algebra with unity (over B) generated by $U^B(\mathcal{g})$ (“the enveloping algebra of $U^B(\mathcal{g})$ ”). There is a natural definition of trace on $J^B(\mathcal{g})$. We define $\text{Tr}: J^B(\mathcal{g}) \rightarrow B$ to be the function $\text{Tr}(X) = \sum_j \Theta_j \text{Tr}(X_j)$ for each $X = \sum_j \Theta_j X_j \in J^B(\mathcal{g})$, where the Θ_j are the basis elements of B , and the X_j are matrices. It is also clear how to define the products $U(g)X$ and $XU(g)$.

3. We consider a finite subset $\Lambda \subset \mathbf{Z}_{1/2}^d \equiv (1/2, \dots, 1/2) + \mathbf{Z}^d$, e.g. a finite cube. Let Λ^* be the set of all bonds $\langle x, y \rangle$ in Λ , $\{g_{\langle x, y \rangle}\}_{\langle x, y \rangle \in \Lambda^*}$, $g_{\langle x, y \rangle} \in \mathbf{G}$, a gauge field configuration, and $A_\Lambda^{\text{inv}}(g)$ the Wilson action of $\{g_{\langle x, y \rangle}\}$ corresponding to the representation \mathbf{U} [2]. As the lattice Lorentz condition we take [3]

$$F_x(g) \equiv \prod_{\mu=1}^d g_{\langle x, x-e_\mu \rangle} g_{\langle x, x+e_\mu \rangle} = e, \quad (1)$$

where $x \in \Lambda^0 \equiv \Lambda \setminus \partial\Lambda$. We impose no gauge condition for $x \in \partial\Lambda$, since this would be too restrictive [3]. To get the continuum limit of (1) we write $U(g_{\langle x, x+e_\mu \rangle}) = \exp[iA_\mu(x)]$ and interpret $g_{\langle x, x-e_\mu \rangle}$ as $g_{\langle x-e_\mu, x \rangle}^{-1}$. (1) is, of course, not the only lattice condition which gives the correct continuum limit. The corresponding Gibbs measure is given by

$$Z_\Lambda^{-1} \det M_\Lambda(g) \exp \left(A_\Lambda^{\text{inv}}(g) - \frac{1}{2\alpha} \sum_{x \in \Lambda^0} \text{Tr} \{ F_x[U(g)] - 1 \}^2 \right) \prod_{\langle x, y \rangle \in \Lambda^*} dg_{\langle x, y \rangle}. \quad (2)$$

Here Z_Λ is the normalization constant, dg the Haar measure, and $M_\Lambda(g)$ an operator acting in $\Theta_{x \in \Lambda^0} \mathbf{U}(\mathcal{G})$ which is defined by

$$\begin{aligned} [M_\Lambda(g)\Theta]_x &= (1 - \text{ad} \{ F_x[U(g)] \}) \Theta_x \\ &+ \sum_{\mu=1}^d \text{ad} \left(\prod_{\nu=1}^{\mu-1} U(g_{\langle x, x-e_\nu \rangle} g_{\langle x, x+e_\nu \rangle}) U(g_{\langle x, x-e_\mu \rangle}) \right) (\Theta_x - \Theta_{x-e_\mu}) \\ &+ \sum_{\mu=1}^d \text{ad} \left(\prod_{\nu=1}^{\mu} U(g_{\langle x, x-e_\nu \rangle} g_{\langle x, x+e_\nu \rangle}) \right) (\Theta_x - \Theta_{x+e_\mu}), \end{aligned} \quad (3)$$

where $\text{ad}[U(g)]\Theta = U(g)\Theta U(g)^{-1}$. Using the discussion of section 2 we can lift $M_\Lambda(g)$ to an operator acting in $\mathbf{U}_\Lambda^{\text{B}}(\mathcal{G}) \equiv \Theta_{x \in \Lambda^0} \mathbf{B} \otimes \mathbf{U}(\mathcal{G})$, which we denote again by $M_\Lambda(g)$. Representing $\det M_\Lambda(g)$ as the Berezin integral

$$\int \exp \left[-(\eta, M_\Lambda(g)\eta)_\Lambda \right] \prod_{x \in \Lambda^0} d\bar{\eta}_x d\eta_x,$$

where $(\eta, M_\Lambda(g)\eta)_\Lambda = \sum_{x \in \Lambda^0} (\eta_x, [M_\Lambda(g)\eta]_x)$, we can rewrite (2) as

$$Z_\Lambda^{-1} \exp [A_\Lambda(g, \bar{\eta}, \eta)] \prod_{\langle x, y \rangle \in \Lambda^*} dg_{\langle x, y \rangle} \prod_{x \in \Lambda^0} d\bar{\eta}_x d\eta_x,$$

with

$$A_\Lambda(g, \bar{\eta}, \eta) = A_\Lambda^{\text{inv}}(g) - (\eta, M_\Lambda(g)\eta)_\Lambda - \frac{1}{2\alpha} \sum_{x \in \Lambda^0} \text{Tr} \{ F_x[U(g)] - 1 \}^2. \quad (4)$$

(4) is the lattice Faddeev–Popov action.

4. Let $\xi \in \Theta_{x \in \Lambda^0} \mathbf{B}$ be a constant such that $\xi \eta_x = -\eta_x \xi$, and $\xi^2 = 0$. We define the following transformation of $U(g_{\langle x, y \rangle})$, η_x and $\bar{\eta}_x$:

$$sU(g_{\langle x, y \rangle}) = (1 + i\eta_x \xi)U(g_{\langle x, y \rangle})(1 - i\eta_y \xi), \quad (5)$$

$$s\eta_x = \eta_x - \frac{1}{2}i\{\eta_x, \eta_x\}\xi, \quad (6)$$

$$s\bar{\eta}_x = \bar{\eta}_x - \frac{1}{2}i\{[M_\Lambda(g)\eta]_x, \bar{\eta}_x\}\xi + (i/\alpha)\{F_x[U(g)]^2 + F_x[U(g)]\}\xi. \quad (7)$$

We replace in (4) the usual trace by the trace defined in section 2. It is also clear that we can replace in (3) $U(g_{\langle x, y \rangle})$ by $sU(g_{\langle x, y \rangle})$ and the so obtained operator is well defined as an operator in $U_\Lambda^{\mathbf{B}}(\mathcal{P})$.

We claim that $A_\Lambda(g)$ is invariant with respect to s . A simple calculation shows that $A_\Lambda^{\text{inv}}(g)$ is invariant under (5). Let $\delta \equiv s - \text{id}$. We shall show that

$$[\delta M_\Lambda(g)]\eta_x = -\frac{1}{2}i\{M_\Lambda(g)\eta_x, M_\Lambda(g)\eta_x\}\xi + \frac{1}{2}iM_\Lambda(g)\{\eta_x, \eta_x\}\xi. \quad (8)$$

Let $(\nabla_\mu \Theta)_x = \Theta_x - \text{ad}[U(g_{\langle x, x-e_\mu \rangle})]\Theta_{x-e_\mu}$ be the "covariant derivative" and $(\nabla_\mu^* \Theta)_x = \Theta_x - \text{ad}[U(g_{\langle x, x+e_\mu \rangle})]\Theta_{x+e_\mu}$ the conjugate of ∇_μ . Set

$$M_\mu = \nabla_\mu + \text{ad}\left[U(g_{\langle x, x-e_\mu \rangle})\right]\nabla_\mu^*.$$

We shall show first that

$$(\delta M_\mu)\eta_x = -\frac{1}{2}i\{M_\mu\eta_x, M_\mu\eta_x\}\xi + \frac{1}{2}iM_\mu\{\eta_x, \eta_x\}\xi. \quad (9)$$

Observe that

$$\delta M_\mu = \delta \nabla_\mu + \left\{ \delta \text{ad}\left[U(g_{\langle x, x-e_\mu \rangle})\right] \right\} \nabla_\mu^* + \text{ad}\left[U(g_{\langle x, x-e_\mu \rangle})\right] \delta \nabla_\mu^*.$$

One verifies easily that for Θ_x such that $\Theta_x \xi = -\xi \Theta_x$ we have

$$\left\{ \delta \text{ad}\left[U(g_{\langle x, x-e_\mu \rangle})\right] \right\} \Theta_x = -i\left\{ \nabla_\mu \eta_x, \text{ad}\left[U(g_{\langle x, x-e_\mu \rangle})\right] \Theta_x \right\} \xi,$$

and similarly for $\delta \text{ad}[U(g_{\langle x, x+e_\mu \rangle})]$ with ∇_μ replaced by ∇_μ^* . Using these identities we find

$$(\delta \nabla_\mu^*)\eta_x = i\left\{ \nabla_\mu^* \eta_x, \eta_x \right\} \xi - i\left\{ \nabla_\mu^* \eta_x, \nabla_\mu^* \eta_x \right\} \xi.$$

Since

$$\nabla_\mu^*(\eta_x \eta_x) = \left\{ \nabla_\mu^* \eta_x, \eta_x \right\} - \frac{1}{2} \left\{ \nabla_\mu^* \eta_x, \nabla_\mu^* \eta_x \right\},$$

we find

$$\begin{aligned} (\delta M_\mu)\eta_x &= -\frac{1}{2}i\left\{ \nabla_\mu \eta_x, \nabla_\mu \eta_x \right\} \xi - \frac{1}{2}i\left\{ \text{ad}\left[U(g_{\langle x, x-e_\mu \rangle})\right] \nabla_\mu^* \eta_x, \text{ad}\left[U(g_{\langle x, x-e_\mu \rangle})\right] \nabla_\mu^* \eta_x \right\} \xi \\ &\quad - i\left\{ \nabla_\mu \eta_x, \text{ad}\left[U(g_{\langle x, x-e_\mu \rangle})\right] \nabla_\mu^* \eta_x \right\} \xi + \frac{1}{2}i \nabla_\mu \{\eta_x, \eta_x\} \xi + \frac{1}{2}i \text{ad}\left[U(g_{\langle x, x-e_\mu \rangle})\right] \nabla_\mu^* \{\eta_x, \eta_x\} \xi \\ &= -\frac{1}{2}i\left\{ M_\mu \eta_x, M_\mu \eta_x \right\} \xi + \frac{1}{2}i M_\mu \{\eta_x, \eta_x\} \xi, \end{aligned}$$

which is precisely (9). To prove (8) observe that

$$M_\Lambda(g) = \sum_{\mu=1}^d \text{ad}(\mu-1) M_\mu,$$

where

$$\text{ad}(\mu) \equiv \text{ad} \left(\prod_{\nu=1}^{\mu} U(g_{\langle x, x-e_\nu \rangle} g_{\langle x, x+e_\nu \rangle}) \right), \quad \text{ad}(0) \equiv \text{id}.$$

We have:

$$[\delta \text{ad}(\mu)] \Theta_x = -i \left\{ \sum_{\nu=1}^{\mu} \text{ad}(\nu-1) M_\nu \eta_x, \text{ad}(\mu) \Theta_x \right\} \xi.$$

Hence, with the help of (9) we find that

$$\begin{aligned} [\delta M_\Lambda(g)] \eta_x &= -i \sum_{\mu=1}^d \left(\sum_{\nu=1}^{\mu-1} \text{ad}(\nu-1) M_\nu \eta_x, \text{ad}(\mu-1) M_\mu \eta_x \right) \xi \\ &\quad - \frac{i}{2} \sum_{\mu=1}^d \text{ad}(\mu-1) \{ M_\mu \eta_x, M_\mu \eta_x \} \xi + \frac{i}{2} \sum_{\mu=1}^d \text{ad}(\mu-1) M_\mu \{ \eta_x, \eta_x \} \xi \\ &= -\frac{1}{2} i \{ M_\Lambda(g) \eta_x, M_\Lambda(g) \eta_x \} \xi + \frac{1}{2} i M_\Lambda(g) \{ \eta_x, \eta_x \} \xi. \end{aligned}$$

This proves formula (8).

A direct calculation using (5)–(8) shows that

$$\delta[\eta, M_\Lambda(g)\eta] = -\frac{i}{\alpha} \sum_{x \in \Lambda^0} \text{Tr}(\{ E_x[U(g)]^2 - F_x[U(g)] \} M_\Lambda(g) \eta_x) \xi.$$

However

$$\begin{aligned} \delta \left(\frac{1}{2\alpha} \sum_{x \in \Lambda^0} \text{Tr} \{ F_x[U(g)] - 1 \}^2 \right) &= \frac{i}{2\alpha} \sum_{x \in \Lambda^0} \text{Tr}(\{ F_x[U(g)] - 1, M_\Lambda(g) \eta_x F_x[U(g)] \}) \xi \\ &= \frac{i}{\alpha} \sum_{x \in \Lambda^0} \text{Tr}(\{ F_x[U(g)]^2 - F_x[U(g)] \} M_\Lambda(g) \eta_x) \xi, \end{aligned}$$

i.e. $\delta(A_\Lambda - A_\Lambda^{\text{inv}}) = 0$. This proves our assertion. Finally, let us observe, that (i) δ is nilpotent: $\delta^2 = 1$, and (ii) $\delta I \neq I \delta$.

5. To each bond $\langle x, y \rangle \in \Lambda^*$ we assign an element $K_{\langle x, y \rangle} = \sum_{\alpha=1}^a K_{\langle x, y \rangle}^\alpha T_\alpha \in U^B(\mathcal{G})$, to each lattice point $x \in \Lambda^0$ an element $L_x = \sum_{\alpha=1}^a L_x^\alpha T_\alpha \in U^B(\mathcal{G})$. Let D be defined by

$$DU(g_{\langle x, y \rangle}) = i[\eta_x U(g_{\langle x, y \rangle}) - U(g_{\langle x, y \rangle}) \eta_y], \quad D\eta_x = -\frac{1}{2} i \{ \eta_x, \eta_x \}.$$

Set

$$\Xi_{\Lambda} = A_{\Lambda} + \sum_{\langle x, y \rangle \in \Lambda^*} \text{Tr} [K_{\langle x, y \rangle} DU(g_{\langle x, y \rangle})] + \sum_{x \in \Lambda^0} \text{Tr} (L_x D\eta_x).$$

One readily verifies that $\delta\Xi_{\Lambda} = 0$ (we set $\delta K = \delta L = 0$). Let $Z_{\Lambda}(J, \beta, \bar{\beta}, K, L)$ be the following generating functional

$$Z_{\Lambda}(J, \beta, \bar{\beta}, K, L) = \int dg d\bar{\eta} d\eta \exp [\Xi_{\Lambda} + i(\eta, \beta) + i(\beta, \eta) + (J, U(g))],$$

where $J = \{J_{\langle x, y \rangle}\}_{\langle x, y \rangle \in \Lambda^*}$, $J_{\langle x, y \rangle} = \sum_{\alpha=1}^a J_{\langle x, y \rangle}^{\alpha} T_{\alpha}$, and $\beta = \{\beta_x\}_{x \in \Lambda^0}$, $\beta_x \in U^B(\mathcal{G})$. By $\Gamma_{\Lambda}(\Phi, \bar{\lambda}, \lambda, K, L)$ we denote the corresponding generating functional for proper Green's functions defined by

$$\Gamma_{\Lambda}(\Phi, \bar{\lambda}, \lambda, K, L) = W_{\Lambda}(J, \beta, \bar{\beta}, K, L) - (J, \Phi) - (\lambda, \beta) - (\beta, \lambda),$$

where

$$W_{\Lambda} = -i \ln Z_{\Lambda}, \quad \Phi_{\langle x, y \rangle} = \delta W_{\Lambda} / \delta J_{\langle x, y \rangle}, \quad \lambda_x = \delta W_{\Lambda} / \delta \bar{\beta}_x, \quad \bar{\lambda}_x = -\delta W_{\Lambda} / \delta \beta_x$$

with

$$\frac{\delta}{\delta J_{\langle x, y \rangle}} = \sum_{\alpha=1}^a T_{\alpha} \frac{\delta}{\delta J_{\langle x, y \rangle}^{\alpha}},$$

etc. Proceeding in the standard fashion [4] we find that $\Gamma_{\Lambda}(\Phi, \bar{\lambda}, \lambda, K, L)$ satisfies the following equation (the Ward–Takahashi identity):

$$\sum_{\langle x, y \rangle \in \Lambda^*} \text{Tr} \left(\frac{\delta \Gamma_{\Lambda}}{\delta \Phi_{\langle x, y \rangle}} \frac{\delta \Gamma_{\Lambda}}{\delta K_{\langle x, y \rangle}} \right) + i \sum_{x \in \Lambda^0} \text{Tr} \left(\frac{\delta \Gamma_{\Lambda}}{\delta \lambda_x} \frac{\delta \Gamma_{\Lambda}}{\delta L_x} \right) + i \sum_{x \in \Lambda^0} \text{Tr} \left(\frac{\delta \Gamma_{\Lambda}}{\delta \bar{\lambda}_x} A_x \right) = 0,$$

where

$$A_x = -\frac{1}{2} i \{ [M_{\Lambda}(g)\eta]_x, \bar{\eta}_x \} + (i/\alpha) \{ F_x[U(g)]^2 - F_x[U(g)] \}.$$

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References

- [1] C. Becchi, A. Rouet and R. Stora, *Ann. Phys.* 98 (1976) 287.
- [2] K. Osterwalder and E. Seiler, *Ann. Phys.* 110 (1978) 440;
D. Brydges, J. Fröhlich and E. Seiler, *Ann. Phys.* 121 (1979) 227.
- [3] B. Sharpe, Thesis, University of British Columbia (1984).
- [4] C. Itzykson and J.-B. Zuber, *Quantum field theory* (McGraw-Hill, New York, 1980).