BRS TRANSFORMATION IN LATTICE GAUGE THEORIES

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Using the results of Sharpe on the gauge fixing in lattice gauge theories we construct a lattice version of the BRS transformation. As an application we state the Ward–Takahashi identities on a lattice.

1. As is well known, to give a precise meaning to continuous gauge theories on the level of perturbation theory, one has to break the gauge invariance of the lagrangian. This gives rise to a theory whose lagrangian contains, aside from the gauge invariant part, the gauge fixing term and the term involving ghost fields. It is a remarkable fact, discovered in ref. [1], that the new lagrangian still possesses a local symmetry. The corresponding symmetry transformation involves Grassmann variables and is called the BRS transformation.

The lattice formulation of gauge theories is consistent without breaking of the gauge invariance [2]. However, to obtain the correct continuum limit of a lattice theory it seems to be indispensable to break the gauge invariance ("to fix a gauge") of the lattice action. A lattice version of the Lorentz gauge fixing was recently discussed in ref. [3], where among other things, the lattice Faddeev–Popov determinant is defined, and it is shown that the problem of Gribov ambiguities is irrelevant (on a lattice). In this note we apply the results of ref. [3] to introduce the lattice ghost fields, and prove the BRS invariance of the action. We use the BRS transformation in a standard fashion [4] to obtain the lattice Ward–Takahashi identities. For the sake of simplicity we consider the case of pure gauge theory only; the generalization to theories involving matter fields is immediate.

2. Let G be a compact semisimple Lie group, $\mathscr{g}$ its Lie algebra, and $g \to \mathrm{U}(g)$ a unitary representation of G in a linear space V. The representation (V, U) defines the corresponding representation of $\mathscr{g}$ in V (we denote it by the same letter U). For $X, Y \in \mathrm{U}(\mathscr{g})$ we define $(X, Y) = \mathrm{Tr}(X * Y)$. Let $\{T_\alpha\}_{\alpha=1}^{a}$ be an orthonormal [with respect to (,)] basis in $\mathrm{U}(\mathscr{g})$, and $f_{\alpha\beta\gamma}$ the corresponding structural constants. $f_{\alpha\beta\gamma}$ is antisymmetric with respect to all the indices α, β, γ.

Let B be a Grassmann algebra generated by $1, \eta_1, \dots, \eta_a, \bar\eta_1, \dots, \bar\eta_a$. We define a mapping $I: \mathrm{B} \to \mathrm{B}$ by setting $I(\eta_\alpha) = \bar\eta_\alpha, \alpha = 1, \dots, a$, and extending it (uniquely) to the whole B in such a way that (i) $I(\alpha\Theta + \beta\Psi) = \bar\alpha I(\Theta) + \bar\beta I(\Psi)$, $\alpha, \beta \in \mathbf{C}$; $\Theta, \Psi \in \mathrm{B}$, (ii) $I(\Theta\Psi) = I(\Psi)I(\Theta)$, (iii) $I^2 = \mathrm{id}$ (an I satisfying (i)–(iii) is called an involution of B). Later on we shall write $I(\Theta) \equiv \bar\Theta$. Define $\mathrm{U^B}(\mathscr{g}) = \mathrm{B} \otimes \mathrm{U}(\mathscr{g})$. For $\Theta, \Psi \in \mathrm{U^B}(\mathscr{g})$ we set $(\Theta, \Psi) = \mathrm{Tr}(\bar\Theta\Psi)$, i.e. $(\Theta, \Psi) = \sum_{\alpha=1}^{a} \bar\Theta_\alpha \Psi_\alpha$, if $\Theta = \sum_{\alpha=1}^{a} \Theta_\alpha \mathrm{T}_\alpha$, $\Psi = \sum_{\alpha=1}^{a} \Psi_\alpha \mathrm{T}_\alpha$; $\Theta_\alpha, \Psi_\alpha \in \mathrm{B}$. Let $\mathrm{J^B}(\mathscr{g})$ be the algebra with unity (over B) generated by $\mathrm{U^B}(\mathscr{g})$ ("the enveloping algebra of $\mathrm{U^B}(\mathscr{g})$"). There is a natural definition of trace on $\mathrm{J^B}(\mathscr{g})$. We define $\mathrm{Tr}: \mathrm{J^B}(\mathscr{g}) \to \mathrm{B}$ to be the function $\mathrm{Tr}(X) = \sum_j \Theta_j \mathrm{Tr}(X_j)$ for each $X = \sum_j \Theta_j X_j \in \mathrm{J^B}(\mathscr{g})$, where the Θ_j are the basis elements of B, and the X_j are matrices. It is also clear how to define the products $\mathrm{U}(g)X$ and $X\mathrm{U}(g)$.

3. We consider a finite subset $\Lambda \subset \mathbf{Z}^d_{1/2} \equiv (1/2, \dots, 1/2) + \mathbf{Z}^d$, e.g. a finite cube. Let Λ^* be the set of all bonds $\langle x, y\rangle$ in Λ, $\{g_{\langle x,y\rangle}\}_{\langle x,y\rangle \in \Lambda^*}$, $g_{\langle x,y\rangle} \in \mathrm{G}$, a gauge field configuration, and $A^{\mathrm{inv}}_\Lambda(g)$ the Wilson action of $\{g_{\langle x,y\rangle}\}$ corresponding to the representation U [2]. As the lattice Lorentz condition we take [3]

$$F_x(g) \equiv \prod_{\mu=1}^{d} g_{\langle x, x-e_\mu\rangle} g_{\langle x, x+e_\mu\rangle} = e, \tag{1}$$

where $x \in \Lambda^0 \equiv \Lambda \setminus \partial\Lambda$. We impose no gauge condition for $x \in \partial\Lambda$, since this would be too restrictive [3]. To get the continuum limit of (1) we write $\mathrm{U}(g_{\langle x,x+e_\mu\rangle}) = \exp[\mathrm{i}A_\mu(x)]$ and interpret $g_{\langle x,x-e_\mu\rangle}$ as $g^{-1}_{\langle x-e_\mu, x\rangle}$. (1) is, of course, not the only lattice condition which gives the correct continuum limit. The corresponding Gibbs measure is given by

$$Z_\Lambda^{-1} \det M_\Lambda(g) \exp\left(A^{\mathrm{inv}}_\Lambda(g) - \frac{1}{2\alpha} \sum_{x\in\Lambda^0} \mathrm{Tr}\{F_x[\mathrm{U}(g)] - 1\}^2 \right) \prod_{\langle x,y\rangle \in \Lambda^*} \mathrm{d}g_{\langle x,y\rangle}. \tag{2}$$

Here Z_Λ is the normalization constant, $\mathrm{d}g$ the Haar measure, and $M_\Lambda(g)$ an operator acting in $\oplus_{x\in\Lambda^0}\mathrm{U}(\mathscr{g})$ which is defined by

$$\begin{aligned}[M_\Lambda(g)\Theta]_x &= (1 - \mathrm{ad}\{F_x[\mathrm{U}(g)]\})\Theta_X \\ &+ \sum_{\mu=1}^{d} \mathrm{ad}\left(\prod_{\nu=1}^{\mu-1} \mathrm{U}(g_{\langle x,x-e_\nu\rangle} g_{\langle x,x+e_\nu\rangle}) \mathrm{U}(g_{\langle x,x-e_\mu\rangle}) \right) (\Theta_X - \Theta_{X-e_\mu}) \\ &+ \sum_{\mu=1}^{d} \mathrm{ad}\left(\prod_{\nu=1}^{\mu} \mathrm{U}(g_{\langle x,x-e_\nu\rangle} g_{\langle x,x+e_\nu\rangle}) \right) (\Theta_X - \Theta_{X+e_\mu}),\end{aligned} \tag{3}$$

where $\mathrm{ad}[\mathrm{U}(g)]\Theta = \mathrm{U}(g)\Theta\mathrm{U}(g)^{-1}$. Using the discussion of section 2 we can lift $M_\Lambda(g)$ to an operator acting in $\mathrm{U}^{\mathrm{B}}_\Lambda(\mathscr{g}) \equiv \oplus_{x\in\Lambda^0}\mathrm{B} \otimes \mathrm{U}(\mathscr{g})$, which we denote again by $M_\Lambda(g)$. Representing $\det M_\Lambda(g)$ as the Berezin integral

$$⨍ \exp[-(\eta, M_\Lambda(g)\eta)_\Lambda] \prod_{x\in\Lambda^0} \mathrm{d}\bar\eta_x \mathrm{d}\eta_x,$$

where $(\eta, M_\Lambda(g)\eta)_\Lambda = \Sigma_{x\in\Lambda^0}(\eta_x, [M_\Lambda(g)\eta]_x)$, we can rewrite (2) as

$$Z_\Lambda^{-1} \exp[A_\Lambda(g, \bar\eta, \eta)] \prod_{\langle x,y\rangle\in\Lambda^*} \mathrm{d}g_{\langle x,y\rangle} \prod_{x\in\Lambda^0} \mathrm{d}\bar\eta_x \mathrm{d}\eta_x,$$

with

$$A_\Lambda(g, \bar\eta, \eta) = A^{\mathrm{inv}}_\Lambda(g) - (\eta, M_\Lambda(g)\eta)_\Lambda - \frac{1}{2\alpha} \sum_{x\in\Lambda^0} \mathrm{Tr}\{F_x[\mathrm{U}(g)] - 1\}^2. \tag{4}$$

(4) is the lattice Faddeev–Popov action.

4. Let $\xi \in \oplus_{x\in\Lambda^0}$ B be a constant such that $\xi\eta_x = -\eta_x\xi$, and $\xi^2 = 0$. We define the following transformation of $\mathrm{U}(g_{\langle x,y\rangle})$, η_x and $\bar{\eta}_x$:

$$s\mathrm{U}(g_{\langle x,y\rangle}) = (1 + \mathrm{i}\eta_x\xi)\mathrm{U}(g_{\langle x,y\rangle})(1 - \mathrm{i}\eta_y\xi), \tag{5}$$

$$s\eta_x = \eta_x - \tfrac{1}{2}\mathrm{i}\{\eta_x, \eta_x\}\xi, \tag{6}$$

$$s\bar{\eta}_x = \bar{\eta}_x - \tfrac{1}{2}\mathrm{i}\{[M_\Lambda(g)\eta]_x, \bar{\eta}_x\}\xi + (\mathrm{i}/\alpha)\{F_x[\mathrm{U}(g)]^2 + F_x[\mathrm{U}(g)]\}\xi. \tag{7}$$

We replace in (4) the usual trace by the trace defined in section 2. It is also clear that we can replace in (3) $\mathrm{U}(g_{\langle x,y\rangle})$ by $s\mathrm{U}(g_{\langle x,y\rangle})$ and the so obtained operator is well defined as an operator in $\mathrm{U}_\Lambda^{\mathrm{B}}(\mathscr{g})$.

We claim that $A_\Lambda(g)$ is invariant with respect to s. A simple calculation shows that $A_\Lambda^{\mathrm{inv}}(g)$ is invariant under (5). Let $\delta \equiv s - \mathrm{id}$. We shall show that

$$[\delta M_\Lambda(g)]\eta_x = -\tfrac{1}{2}\mathrm{i}\{M_\Lambda(g)\eta_x, M_\Lambda(g)\eta_x\}\xi + \tfrac{1}{2}\mathrm{i}M_\Lambda(g)\{\eta_x, \eta_x\}\xi. \tag{8}$$

Let $(\nabla_\mu\Theta)_x = \Theta_x - \mathrm{ad}[\mathrm{U}(g_{\langle x,x-e_\mu\rangle})]\Theta_{x-e_\mu}$ be the "covariant derivative" and $(\nabla_\mu^*\Theta)_x = \Theta_x - \mathrm{ad}[\mathrm{U}(g_{\langle x,x+e_\mu\rangle})]\Theta_{x+e_\mu}$ the conjugate of ∇_μ. Set

$$M_\mu = \nabla_\mu + \mathrm{ad}\left[\mathrm{U}\left(g_{\langle x,x-e_\mu\rangle}\right)\right]\nabla_\mu^*.$$

We shall show first that

$$(\delta M_\mu)\eta_x = -\tfrac{1}{2}\mathrm{i}\{M_\mu\eta_x, M_\mu\eta_x\}\xi + \tfrac{1}{2}\mathrm{i}M_\mu\{\eta_x, \eta_x\}\xi. \tag{9}$$

Observe that

$$\delta M_\mu = \delta\nabla_\mu + \left\{\delta\mathrm{ad}\left[\mathrm{U}\left(g_{\langle x,x-e_\mu\rangle}\right)\right]\right\}\nabla_\mu^* + \mathrm{ad}\left[\mathrm{U}\left(g_{\langle x,x-e_\mu\rangle}\right)\right]\delta\nabla_\mu^*.$$

One verifies easily that for Θ_x such that $\Theta_x\xi = -\xi\Theta_x$ we have

$$\left\{\delta\mathrm{ad}\left[\mathrm{U}\left(g_{\langle x,x-e_\mu\rangle}\right)\right]\right\}\Theta_x = -\mathrm{i}\left\{\nabla_\mu\eta_x, \mathrm{ad}\left[\mathrm{U}\left(g_{\langle x,x-e_\mu\rangle}\right)\right]\Theta_x\right\}\xi,$$

and similarly for $\delta\mathrm{ad}[\mathrm{U}(g_{\langle x,x+e_\mu\rangle})]$ with ∇_μ replaced by ∇_μ^*. Using these identities we find

$$\left(\delta\nabla_\mu^{(*)}\right)\eta_x = \mathrm{i}\left\{\nabla_\mu^{(*)}\eta_x, \eta_x\right\}\xi - \mathrm{i}\left\{\nabla_\mu^{(*)}\eta_x, \nabla_\mu^{(*)}\eta_x\right\}\xi.$$

Since

$$\nabla_\mu^{(*)}(\eta_x\eta_x) = \left\{\nabla_\mu^{(*)}\eta_x, \eta_x\right\} - \tfrac{1}{2}\left\{\nabla_\mu^{(*)}\eta_x, \nabla_\mu^{(*)}\eta_x\right\},$$

we find

$$\begin{aligned}(\delta M_\mu)\eta_x = &-\tfrac{1}{2}\mathrm{i}\{\nabla_\mu\eta_x, \nabla_\mu\eta_x\}\xi - \tfrac{1}{2}\mathrm{i}\left\{\mathrm{ad}\left[\mathrm{U}\left(g_{\langle x,x-e_\mu\rangle}\right)\right]\nabla_\mu^*\eta_x, \mathrm{ad}\left[\mathrm{U}\left(g_{\langle x,x-e_\mu\rangle}\right)\right]\nabla_\mu^*\eta_x\right\}\xi \\ &-\mathrm{i}\left\{\nabla_\mu\eta_x, \mathrm{ad}\left[\mathrm{U}\left(g_{\langle x,x-e_\mu\rangle}\right)\right]\nabla_\mu^*\eta_x\right\}\xi + \tfrac{1}{2}\mathrm{i}\nabla_\mu\{\eta_x, \eta_x\}\xi + \tfrac{1}{2}\mathrm{i}\,\mathrm{ad}\left[\mathrm{U}\left(g_{\langle x,x-e_\mu\rangle}\right)\right]\nabla_\mu^*\{\eta_x, \eta_x\}\xi \\ &= -\tfrac{1}{2}\mathrm{i}\{M_\mu\eta_x, M_\mu\eta_x\}\xi + \tfrac{1}{2}\mathrm{i}M_\mu\{\eta_x, \eta_x\}\xi,\end{aligned}$$

which is precisely (9). To prove (8) observe that

$$M_\Lambda(g)=\sum_{\mu=1}^{d}\mathrm{ad}(\mu-1)M_\mu,$$

where

$$\mathrm{ad}(\mu)\equiv\mathrm{ad}\left(\prod_{\nu=1}^{\mu}\mathrm{U}\left(g_{\langle x,x-e_\nu\rangle}g_{\langle x,x+e_\nu\rangle}\right)\right),\quad \mathrm{ad}(0)\equiv\mathrm{id}.$$

We have:

$$[\delta\mathrm{ad}(\mu)]\Theta_x=-\mathrm{i}\left\{\sum_{\nu=1}^{\mu}\mathrm{ad}(\nu-1)M_\nu\eta_x,\mathrm{ad}(\mu)\Theta_x\right\}\xi.$$

Hence, with the help of (9) we find that

$$\begin{aligned}[\delta M_\Lambda(g)]\eta_x&=-\mathrm{i}\sum_{\mu=1}^{d}\left(\sum_{\nu=1}^{\mu-1}\mathrm{ad}(\nu-1)M_\nu\eta_x,\mathrm{ad}(\mu-1)M_\mu\eta_x\right)\xi\\&-\frac{\mathrm{i}}{2}\sum_{\mu=1}^{d}\mathrm{ad}(\mu-1)\{M_\mu\eta_x,M_\mu\eta_x\}\xi+\frac{\mathrm{i}}{2}\sum_{\mu=1}^{d}\mathrm{ad}(\mu-1)M_\mu\{\eta_x,\eta_x\}\xi\\&=-\tfrac{1}{2}\mathrm{i}\{M_\Lambda(g)\eta_x,M_\Lambda(g)\eta_x\}\xi+\tfrac{1}{2}\mathrm{i}M_\Lambda(g)\{\eta_x,\eta_x\}\xi.\end{aligned}$$

This proves formula (8).

A direct calculation using (5)–(8) shows that

$$\delta[\eta,M_\Lambda(g)\eta]=-\frac{\mathrm{i}}{\alpha}\sum_{x\in\Lambda^0}\mathrm{Tr}\left(\{E_x[\mathrm{U}(g)]^2-F_x[\mathrm{U}(g)]\}M_\Lambda(g)\eta_x\right)\xi.$$

However

$$\begin{aligned}\delta\left(\frac{1}{2\alpha}\sum_{x\in\Lambda^0}\mathrm{Tr}\{F_x[\mathrm{U}(g)]-1\}^2\right)&=\frac{\mathrm{i}}{2\alpha}\sum_{x\in\Lambda^0}\mathrm{Tr}(\{F_x[\mathrm{U}(g)]-1,M_\Lambda(g)\eta_xF_x[\mathrm{U}(g)]\})\xi\\&=\frac{\mathrm{i}}{\alpha}\sum_{x\in\Lambda^0}\mathrm{Tr}\left(\{F_x[\mathrm{U}(g)]^2-F_x[\mathrm{U}(g)]\}M_\Lambda(g)\eta_x\right)\xi,\end{aligned}$$

i.e. $\delta(A_\Lambda-A_\Lambda^{\mathrm{inv}})=0$. This proves our assertion. Finally, let us observe, that (i) δ is nilpotent: $\delta^2=1$, and (ii) $\delta I\neq I\delta$.

5. To each bond $\langle x,y\rangle\in\Lambda^*$ we assign an element $K_{\langle x,y\rangle}=\Sigma_{\alpha=1}^{a}K_{\langle x,y\rangle}^{\alpha}T_\alpha\in\mathrm{U}^{\mathrm{B}}(\mathscr{g})$, to each lattice point $x\in\Lambda^0$ an element $L_x=\Sigma_{\alpha=1}^{a}L_x^{\alpha}T_\alpha\in\mathrm{U}^{\mathrm{B}}(\mathscr{g})$. Let D be defined by

$$D\mathrm{U}(g_{\langle x,y\rangle})=\mathrm{i}\left[\eta_x\mathrm{U}(g_{\langle x,y\rangle})-\mathrm{U}(g_{\langle x,y\rangle})\eta_y\right],\quad D\eta_x=-\tfrac{1}{2}\mathrm{i}\{\eta_x,\eta_x\}.$$

Set

$$\Xi_\Lambda = A_\Lambda + \sum_{\langle x,y\rangle \in \Lambda^*} \mathrm{Tr}\left[K_{\langle x,y\rangle} D\mathrm{U}(g_{\langle x,y\rangle})\right] + \sum_{x\in\Lambda^0} \mathrm{Tr}(L_x D\eta_x).$$

One readily verifies that $\delta\Xi_\Lambda = 0$ (we set $\delta K = \delta L = 0$). Let $Z_\Lambda(\mathrm{J}, \beta, \bar\beta, K, L)$ be the following generating functional

$$Z_\Lambda(\mathrm{J}, \beta, \bar\beta, K, L) = \int dg\, d\bar\eta\, d\eta \exp\left[\Xi_\Lambda + \mathrm{i}(\eta, \beta) + \mathrm{i}(\beta, \eta) + (\mathrm{J}, U(g))\right],$$

where $\mathrm{J} = \{J_{\langle x,y\rangle}\}_{\langle x,y\rangle\in\Lambda^*}$, $J_{\langle x,y\rangle} = \Sigma_{\alpha=1}^{a} J^\alpha_{\langle x,y\rangle} T_\alpha$, and $\beta = \{\beta_x\}_{x\in\Lambda^0}$, $\beta_x \in \mathrm{U}^{\mathrm{B}}(\mathscr{g})$. By $\Gamma_\Lambda(\Phi, \bar\lambda, \lambda, K, L)$ we denote the corresponding generating functional for proper Green's functions defined by

$$\Gamma_\Lambda(\Phi, \bar\lambda, \lambda, K, L) = W_\Lambda(\mathrm{J}, \beta, \bar\beta, K, L) - (\mathrm{J}, \Phi) - (\lambda, \beta) - (\beta, \lambda),$$

where

$$W_\Lambda = -\mathrm{i} \ln Z_\Lambda, \quad \Phi_{\langle x,y\rangle} = \delta W_\Lambda / \delta J_{\langle x,y\rangle}, \quad \lambda_x = \delta W_\Lambda / \delta\bar\beta_x, \quad \bar\lambda_x = -\delta W_\Lambda / \delta\beta_x$$

with

$$\frac{\delta}{\delta J_{\langle x,y\rangle}} = \sum_{\alpha=1}^{a} T_\alpha \frac{\delta}{\delta J^\alpha_{\langle x,y\rangle}},$$

etc. Proceeding in the standard fashion [4] we find that $\Gamma_\Lambda(\Phi, \bar\lambda, \lambda, K, L)$ satisfies the following equation (the Ward–Takahashi identity):

$$\sum_{\langle x,y\rangle\in\Lambda^*} \mathrm{Tr}\left(\frac{\delta\Gamma_\Lambda}{\delta\Phi_{\langle x,y\rangle}} \frac{\delta\Gamma_\Lambda}{\delta K_{\langle x,y\rangle}}\right) + \mathrm{i} \sum_{x\in\Lambda^0} \mathrm{Tr}\left(\frac{\delta\Gamma_\Lambda}{\delta\lambda_x} \frac{\delta\Gamma_\Lambda}{\delta L_x}\right) + \mathrm{i} \sum_{x\in\Lambda^0} \mathrm{Tr}\left(\frac{\delta\Gamma_\Lambda}{\delta\bar\lambda_x} A_x\right) = 0,$$

where

$$A_x = -\tfrac{1}{2}\mathrm{i}\{[M_\Lambda(g)\eta]_x, \bar\eta_x\} + (\mathrm{i}/\alpha)\{F_x[\mathrm{U}(g)]^2 - F_x[\mathrm{U}(g)]\}.$$

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