

Chern Character in Equivariant Entire Cyclic Cohomology*

SLAWOMIR KLIMEK and ANDRZEJ LESNIEWSKI

Department of Mathematics, Harvard University, Cambridge MA 02138, U.S.A.

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Abstract. We construct the Chern character in the equivariant entire cyclic cohomology. We prove a general index theorem for the G -invariant Dirac operator.

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1. Introduction

In this paper, we continue our study of equivariant entire cyclic cohomology [6]. Entire cyclic cohomology [1] is a cohomological framework designed to deal with global properties of ‘infinite-dimensional noncommutative space’. While our first paper [6] was cohomological in character, here we describe the ‘differential geometric’ aspects of the theory (in the spirit of A. Connes’ noncommutative differential geometry).

The paper is organized as follows. In Section 2, we define a G -quantum algebra. It consists of a \mathbb{C}^* -dynamical system (\mathcal{A}, G, ρ) , with G finite, a \mathbb{Z}_2 -grading γ on \mathcal{A} , and a superderivation d on \mathcal{A} . Roughly speaking, d is determined by a ‘Dirac operator’. A G -quantum algebra generalizes the notion of a quantum algebra introduced in [2, 3].

In Section 3, we construct an equivariant entire cyclic cocycle, the Chern character. Central to the construction is the notion of an equivariant super-KMS form on \mathcal{A} . Super-KMS forms have been studied before in [5, 3, 4], where their relation to ordinary entire cyclic cohomology and their stability properties were established. Since the proofs of [5, 3, 4] easily carry over to the equivariant context, our arguments in this section are very sketchy.

In Section 4 we prove a general index theorem for the Dirac operator Q of a G -quantum algebra. We express the character-valued index of Q in terms of the Chern character constructed in Section 3 and an equivariant K -theory class. The theorem generalizes Connes’ index theorem [1]. We believe that a special case of the (putative) $U(1)$ version of this theorem is Witten’s elliptic genera computation [8].

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2. Quantum Algebras

Let G be a finite group. By $\mathcal{F}(G)$ we denote the space of functions $f: G \rightarrow \mathbb{C}$ and by $R(G) \subset \mathcal{F}(G)$ we denote the space of central function on G . Let \hat{G} denote the set of equivalence classes of irreducible unitary representations of G .

A G -quantum algebra is a \mathbb{C}^* -dynamical system [7] (\mathcal{A}, G, ρ) with the following structures:

- (i) \mathcal{A} is \mathbb{Z}_2 -graded, i.e., there is a homomorphism $\Gamma: \mathbb{Z}_2 \rightarrow \text{Aut}(\mathcal{A})$, where $\mathbb{Z}_2 = \{0, 1\}$. We denote $\gamma := \Gamma(1)$ and let $\mathcal{A}_\pm := \{a \in \mathcal{A} : \gamma(a) = \pm a\}$. Then $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$. It is convenient to write $a^\gamma := \gamma(a)$.
- (ii) There is a norm-continuous, one-parameter group of automorphism $\mathbb{R} \ni t \rightarrow \alpha_t \in \text{Aut}(\mathcal{A})$ such that

$$[\gamma, \alpha_t] = 0, \quad t \in \mathbb{R}. \tag{2.1}$$

By D we denote the generator of α_t ,

$$D = -i \frac{d}{dt} \alpha_t \Big|_{t=0}. \tag{2.2}$$

- (iii) There is a densely defined closed superderivation $d: \mathcal{A} \rightarrow \mathcal{A}$, i.e.,

$$d(ab) = da \cdot b + a^\gamma db, \quad a, b \in D(d), \tag{2.3}$$

such that

$$\{d, \gamma\} = 0, \tag{2.4}$$

and

$$d^2 = D \quad \text{on } D(d). \tag{2.5}$$

Observe that $D(d)$ is a Banach algebra with the norm

$$\|a\|_* := \|a\| + \|da\|. \tag{2.6}$$

- (iv) For any $g \in G$,

$$[\gamma, \rho_g] = [d, \rho_g] = 0. \tag{2.7}$$

As a consequence of (2.7) and (2.5),

$$[\alpha_t, \rho_g] = 0. \tag{2.8}$$

It is straightforward to define functorial operations in the category of G -quantum algebras. For example, if $(\mathcal{A}, G, \rho, \gamma, \alpha, d)$ and $(\mathcal{A}', G, \rho', \gamma', \alpha', d')$ are G -quantum algebras then their *direct sum* is the G -quantum algebra $(\mathcal{A} \oplus \mathcal{A}', G, \rho \oplus \rho', \gamma \oplus \gamma', \alpha \oplus \alpha', d \oplus d')$. Their *tensor product* is $(\mathcal{A} \hat{\otimes} \mathcal{A}', G, \rho \otimes \rho', \gamma \otimes \gamma', \alpha \otimes \alpha', d \otimes \text{id} + \gamma \otimes d')$, where $\mathcal{A} \hat{\otimes} \mathcal{A}'$ is a suitably topologized \mathbb{Z}_2 -graded tensor product of \mathcal{A} and \mathcal{A}' , and where $(\rho \otimes \rho')_g := \rho_g \otimes \rho'_g$.

The above abstract structure does not involve any concrete representation of \mathcal{A} on a Hilbert space. In applications, one encounters often a concrete \mathbb{C}^* -algebra of operators on a Hilbert space with d implemented by a self-adjoint operator, whose square has a trace class heat kernel. We refer to such a structure as a Θ -summable G -quantum algebra. More precisely, a Θ -summable G -quantum algebra is a \mathbb{C}^* -algebra \mathcal{A} of operators acting on \mathbb{Z}_2 -graded Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Let $\gamma = \pm \mathbf{1}$ on \mathcal{H}_\pm . We set $a^\gamma := \gamma a \gamma$ for $a \in \mathcal{A}$; then $\mathcal{A}_\pm = \{a \in \mathcal{A} : a^\gamma = \pm a\}$. Let Q be a self-adjoint operator on \mathcal{H} such that $\{Q, \gamma\} = 0$ and let

$$da := Qa - a^\gamma Q. \tag{2.9}$$

We assume that $D(d)$ is dense in \mathcal{A} . We set

$$H := Q^2, \tag{2.10}$$

and

$$\alpha_t := \exp\{it \operatorname{Ad}(H)\}. \tag{2.11}$$

We assume that there is a unitary representation λ of G on \mathcal{H} such that

$$[\gamma, \lambda_g] = [Q, \lambda_g] = 0. \tag{2.12}$$

We set

$$\rho_g(a) := \lambda_g a \lambda_{g^{-1}}. \tag{2.13}$$

Finally, we impose the following trace classness condition. Let

$$\mathcal{H} = \bigoplus_{\sigma \in \hat{G}} \mathcal{H}_\sigma \tag{2.14}$$

be the decomposition of \mathcal{H} into G -invariant subspaces \mathcal{H}_σ . Each \mathcal{H}_σ is the direct sum of all subspaces of \mathcal{H} which carry the irreducible representation σ . Let P_σ denote the projection onto \mathcal{H}_σ . We require that

$$\operatorname{tr}(P_\sigma e^{-tH}) < \infty, \quad \text{for } t > 0. \tag{2.15}$$

Remark. Since \hat{G} is finite, condition (2.15) is equivalent to

$$\operatorname{tr}(e^{-tH}) < \infty. \tag{2.16}$$

This is, however, not the case if the assumption of finiteness of G is dropped.

3. sKMS-Forms and Chern Characters

Let \mathcal{A}_α be a norm dense subalgebra of \mathcal{A} such that $\mathbb{R} \ni t \rightarrow \alpha_t(a)$, $a \in \mathcal{A}_\alpha$, extends to an entire \mathcal{A} -valued function.

DEFINITION 3.1. A linear mapping $\omega : \mathcal{A} \rightarrow \mathcal{F}(G)$ is called an equivariant sKMS form on \mathcal{A} if it satisfies the following conditions:

(α) ω is G -equivariant, i.e.,

$$\omega(\rho_h(a))(g) = \omega(a)(h^{-1}gh); \tag{3.1}$$

(β) for $a \in D(d)$,

$$\omega(da)(g) = 0; \tag{3.2}$$

(γ) for $a, b \in \mathcal{A}_x$,

$$\omega(ab)(g) = \omega(\rho_{g^{-1}}(b^y)\alpha_i(a))(g); \tag{3.3}$$

(δ) ω is continuous with respect to the norm $\|\cdot\|$,

$$\max_{g \in G} |\omega(a)(g)| \leq C \|a\|. \tag{3.4}$$

Observe that as a consequence of (α) and (γ), ω is \mathbb{Z}_2 -invariant.

It is easy to construct an sKMS form on a Θ -summable G -quantum algebra.

THEOREM 3.2. *Let \mathcal{A} be a Θ -summable G -quantum algebra. Then*

$$\omega(a)(g) := \text{tr}(\gamma\lambda(g)a e^{-H}) \tag{3.5}$$

is an equivariant sKMS form on \mathcal{A} .

Proof. The existence and continuity of ω is a consequence of (2.15). Properties (α)–(γ) follow by elementary calculations. □

The following theorem is of analytic nature. It is the main technical input needed in the construction of the equivariant Chern character and in the study of its stability properties. Its proof is a standard application of the Phragmen–Lindeloff principle and we do not reproduce it here (it is a simple extension of the proof of Theorem 2.2 in [4]). We set

$$D_n := \{z \in \mathbb{C}^n : 0 \leq \text{Im } z_1 \leq \dots \leq \text{Im } z_n \leq 1\}. \tag{3.6}$$

THEOREM 3.3. *Let ω be an equivariant sKMS form on \mathcal{A} . Then for $(t_1, \dots, t_n) \in \mathbb{R}^n$ and $a_1, \dots, a_n \in \mathcal{A}$, $(t_1, \dots, t_n) \rightarrow \omega(a_0\alpha_{t_1}(a_1) \cdots \alpha_{t_n}(a_n))(g)$ is the boundary value of a function denoted by $\omega(a_0\alpha_{z_1}(a_1) \cdots \alpha_{z_n}(a_n))(g)$ holomorphic inside D_n and continuous in \bar{D}_n , and such that*

$$\max_{g \in G} |\omega(a_0\alpha_{z_1}(a_1) \cdots \alpha_{z_n}(a_n))(g)| \leq KC^n \exp\left\{\sum_{j=1}^n |\text{Re } z_j|\right\} \prod_{j=0}^n \|a_j\|, \tag{3.7}$$

with some constants K and C .

DEFINITION 3.4. Let ω be an equivariant sKMS form. For $a_0, a_1, \dots, a_{2k} \in \mathcal{A}$ we define

$$\begin{aligned} &\tau_{2k}(a_0, a_1, \dots, a_{2k})(g) \\ &:= (-1)^k \int_{\sigma_{2k}} \omega(a_0 \alpha_{is_1}(da_1^y) \alpha_{is_2}(da_2) \cdots \times \\ &\quad \times \alpha_{is_{2k-1}}(da_{2k-1}^y) \alpha_{is_{2k}}(da_{2k}))(g) d^{2k}s, \end{aligned} \tag{3.8}$$

where

$$\sigma_{2k} := \{(t_1, \dots, t_{2k}) \in \mathbb{R}^{2k} : 0 \leq t_1 \leq \dots \leq t_{2k} \leq 1\}. \tag{3.9}$$

We call the sequence $\tau = (\tau_0, \tau_2, \dots)$ the Chern character associated with ω .

Now let

$$\cdots \rightarrow \mathcal{C}_G^e(\mathcal{A}) \xrightarrow{\partial} \mathcal{C}_G^o(\mathcal{A}) \xrightarrow{\partial} \mathcal{C}_G^e(\mathcal{A}) \rightarrow \cdots$$

denote the G -equivariant entire cyclic complex of the Banach algebra \mathcal{A} (see [6] for definitions). For $(j, g) \in \mathbb{Z}_2 \times G, j = 0, 1$, we set

$$\tilde{\tau}_{2k}(a_0, a_1, \dots, a_{2k})(j, g) = \begin{cases} 0, & \text{if } j = 0, \\ \tau_{2k}(a_0, a_1, \dots, a_{2k})(g), & \text{if } j = 1. \end{cases} \tag{3.10}$$

THEOREM 3.5. (i) *The sequence $\tilde{\tau}$ is an element of $\mathcal{C}_{\mathbb{Z}_2 \times G}^e(D(d))$. Moreover, there are constants K and C such that*

$$\max_{g \in G} |\tau_{2k}(a_0, a_1, \dots, a_{2k})(g)| \leq KC^{2k} \frac{1}{(2k)!} \prod_{j=0}^{2k} \|a_j\|_*. \tag{3.11}$$

(ii) *The cochain $\tilde{\tau}$ is a cocycle,*

$$\partial \tilde{\tau} = 0. \tag{3.12}$$

Proof. Part (i) is an immediate consequence of (3.8) and (3.9). Part (ii) follows by a computation analogous to that of [2] (see also [5] and [3]). \square

Observe that if \mathcal{A} is a Θ -summable G -quantum algebra, then

$$\begin{aligned} &\tau_{2k}(a_0, a_1, \dots, a_{2k})(g) \\ &= (-1)^k \int_{\sigma_{2k}} \text{tr}(\gamma \lambda(g) a_0 da_1^y(s_1) da_2(s_2) \cdots da_{2k-1}^y(s_{2k-1}) \times \\ &\quad \times da_{2k}(s_{2k}) e^{-H}) d^{2k}s, \end{aligned} \tag{3.13}$$

where

$$a(s) := e^{-sH} a e^{sH}. \tag{3.14}$$

Finally, we observe that the stability result of [4] can be extended to the equivariant context. Let $q \in \mathcal{A}_- \cap D(d)$ be G -invariant and let d_q be the superderivation defined by

$$d_q a := da + (qa - a^{\flat}q), \tag{3.15}$$

i.e., d_q is a bounded perturbation of d . The square of d_q is the derivation

$$D_q := d^2 + \text{Ad}(\Omega_q), \tag{3.16}$$

where $\Omega_q := d_q + q^2$. Let α_t^q be the group of automorphisms of \mathcal{A} generated by D_q and let γ_t^q be the group of transformations

$$\gamma_t^q(a) := \exp\{it(D + \Omega_q)\}(a) \tag{3.17}$$

(see Section III of [4] for the details).

THEOREM 3.6. (i) *If ω is an sKMS form for $(\mathcal{A}, G, \rho, \gamma, \alpha, d)$, then*

$$\omega^q(a)(g) := \omega(a\gamma_t^q(\mathbf{1}))(g) \tag{3.18}$$

is an sKMS form for $(\mathcal{A}, G, \rho, \gamma, \alpha^q, d_q)$.

(ii) *Let τ^q be the Chern character associated with ω^q . Then $\tilde{\tau}^q$ and $\tilde{\tau}$ are in the same equivariant entire cyclic cohomology class.*

The proof of this theorem is a repetition of the arguments of [4]. Observe, that if \mathcal{A} is a Θ -summable G -quantum algebra, then the situation discussed in Theorem 3.6 arises if Q is replaced by

$$Q_q := Q + q, \tag{3.19}$$

with $q \in \mathcal{A}_- \cap D(d)$ and G -invariant.

4. The Index Theorem

In this section we assume that $(\mathcal{A}, G, \rho, \gamma, \alpha, d)$ is a Θ -summable G -quantum algebra. Since Q anticommutes with γ , it can be written as

$$Q = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix}, \tag{4.1}$$

where $Q_{\pm} : \mathcal{H}_{\pm} \rightarrow \mathcal{H}_{\mp}$, $(Q_+)^* = Q_-$. We let

$$Q_{\sigma} := P_{\sigma} Q P_{\sigma}, \tag{4.2}$$

and

$$Q_{\sigma_{\pm}} := P_{\sigma} Q_{\pm} P_{\sigma}. \tag{4.3}$$

Clearly, $Q_{\sigma}^2 = P_{\sigma} Q^2 P_{\sigma} = P_{\sigma} H P_{\sigma} =: H_{\sigma}$. As a consequence of (2.15), $\exp(-tH_{\sigma})$ is trace class on \mathcal{H}_{σ} , and thus Q_{σ_+} is a Fredholm operator on \mathcal{H}_{σ} . Let $i(Q_{\sigma_+})$ be the index of this operator.

DEFINITION 4.1. The function $i(Q_+) \in R(G)$ defined by

$$i(Q_+)(g) := \sum_{\sigma \in \hat{G}} i(Q_{\sigma+})\chi_{\sigma}(g), \tag{4.4}$$

where χ_{σ} is the character of σ , is called the character valued index of Q .

The following theorem is a version of the McKean-Singer representation of the index.

THEOREM 4.2. For $t > 0$,

$$i(Q_+)(g) = \text{tr}(\gamma\lambda(g) e^{-tH}). \tag{4.5}$$

Proof. Since γ and H are G -invariant,

$$\text{tr}(\gamma\lambda(g) e^{-tH}) = \sum_{\sigma \in \hat{G}} \text{tr}_{\mathcal{H}_{\sigma}}(\gamma\lambda(g) e^{-tH_{\sigma}}), \tag{4.6}$$

in the notation of (2.14). By the standard argument, the only contribution to $\text{tr}_{\mathcal{H}_{\sigma}}(\gamma\lambda(g) \exp\{-tH_{\sigma}\})$ comes from the kernel of H_{σ} . But

$$\text{tr}_{\ker(H_{\sigma})}(\gamma\lambda(g) e^{-tH_{\sigma}}) = \text{tr}_{\ker(H_{\sigma})}(\gamma\lambda(g)) = i(Q_{\sigma+})\chi_{\sigma}(g),$$

where we have used the fact that, as a consequence of (2.12), each of the subspaces $\ker(H_{\sigma}) \cap \mathcal{H}_{\pm}$ is a direct sum of representation spaces of σ . Consequently,

$$\text{tr}(\gamma\lambda(g) e^{-tH}) = \sum_{\sigma \in \hat{G}} i(Q_{\sigma+})\chi_{\sigma}(g) = i(Q_+)(g). \quad \square$$

Let now $\tilde{\tau} \in \mathcal{C}_{\mathbb{Z}_2 \times G}^e(D(d))$ be the Chern character associated with the G -sKMS form (3.6). Let $[e] \in K_0^{Z_2 \times G}(\mathcal{A})$. Since \mathcal{A} is a \mathbb{C}^* -algebra, $[e]$ has a representative p which is an orthogonal projection, $p^* = p = p^2$ (see pp. 31–33 of [7]). Let \mathcal{V} be a finite-dimensional vector space carrying a unitary representation U of G such that $p \in \mathcal{A} \otimes \text{End}(\mathcal{V})$. Assume that $p \in D(Q \otimes I)$, where I is the identity operator on \mathcal{V} , and consider the following operator

$$Q^p := p(Q \otimes I)p. \tag{4.7}$$

Let $\langle \cdot, \cdot \rangle: K_0^G(\mathcal{A}) \times \mathcal{H}_G^e(\mathcal{A}) \rightarrow R(G)$ be the pairing defined in Section 5 of [6]. Then we have the following index theorem.

THEOREM 4.3. Assume that $[Q \otimes I, p] \in \mathcal{A} \otimes \text{End}(\mathcal{V})$. Then

$$i(Q^p_+)(g) = \langle p, \tilde{\tau} \rangle(1, g). \tag{4.8}$$

Proof. The pairing $\langle \cdot, \cdot \rangle$ is defined in terms of normalized cocycles (see [1, 6]). In the formulas below, $\tilde{\tau}$ and $\tilde{\tau}'$ denote the normalizations of the respective cocycles, as given by formulas (3.46) and (3.47) of [6].

By functoriality (see [6]) we can assume that $p \in \mathcal{A}$ and $Q^p = pQp$. We consider the operator

$$Q' := Q + [p, dp], \quad (4.9)$$

with $dp = [Q, p]$. Observe that

$$d'p := [Q', p] = 0, \quad (4.10)$$

and thus, if $\tilde{\tau}'$ is the (normalization of the) Chern character defined by Q' , then

$$\langle p, \tilde{\tau}' \rangle(1, g) = \text{tr}(\gamma\lambda(g)p e^{-(Q')^2}).$$

Using the $\mathbb{Z}_2 \times G$ -invariance of p , we obtain

$$\text{tr}(\gamma\lambda(g)p e^{-(Q')^2}) = \text{tr}_{p\mathcal{A}}(\gamma\lambda(g) e^{-p(Q')^2p}).$$

But

$$p(Q')^2p = pQpQp = (pQp)^2 = (Q^p)^2,$$

and thus by Theorem 4.2,

$$\langle p, \tilde{\tau}' \rangle(1, g) = i(Q_+^p)(g).$$

Notice that the terms with $k \geq 1$ in formula (5.2) of [6] vanish, as a consequence of (4.10) and the definition of τ_{2k} . On the other hand, as a consequence of Theorem 3.6, $\tilde{\tau}$ and $\tilde{\tau}'$ are cohomological. Hence, by the theorem of Section 5.1 of [6],

$$\langle p, \tilde{\tau}' \rangle(1, g) = \langle p, \tilde{\tau} \rangle(1, g),$$

and the theorem is proved. \square

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