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Classical limits for quantum maps on the torus

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We provide a rigorous canonical quantization for the following toral automorphisms: cat maps, generalized kicked maps, and generalized Harper maps. For each of these systems we construct a unitary propagator and prove the existence of a well-defined classical limit. We also provide an alternative derivation of Hannay and Berry results for the cat map propagator on the plane. © *1998 American Institute of Physics.* [S0022-2488(98)02404-9]

I. INTRODUCTION

The burgeoning field of ''quantum chaos'' or, more precisely, the quantum mechanics of classically chaotic dynamics has opened the door to several issues in quantization (see e.g., Ref. 1). One of these is the quantization of classical maps.[2–5]

Classical maps arise naturally in the study of chaos. In the Poincaré surface of section maps (see Ref. 6), for instance, the flow on a d-dimensional phase space is ''reduced'' to a $(d-1)$-dimensional map by considering the discrete evolution of points of intersection of the trajectory with a co-dimension of one hypersurface. Chaotic properties of the flow typically leave tell-tale signs on the surface of section. It is clear that the two problems are intimately connected, and numerical results indicate this. Another typical way in which classical flows lead to maps is via a coarse graining of Hamilton's equation,

$$p' = p + f(x)\Delta t, \quad x' = x + p',$$

where $f(x) = -\nabla_x u(x)$ is the classical force function for the Hamiltonian $H = p^2/2 + u(x)$, and where in the second equation p' is used rather than p to ensure the map is area preserving. For maps corresponding to integrable flows, the parameter Δt is a type of ''order parameter'' for the transition to chaos in the corresponding family of maps. Typically for large Δt, the system is completely chaotic, while for small Δt, the system mimics its continuous time progenitor and remains on regular trajectories for all time. (Examples abound. See, e.g., Ref. 6.)

The quantization of classically chaotic maps has recently become an interesting issue in its own right. Hannay and Berry[3] provide a quantization of the cat map

$$(x', p') = \beta(x, p) \mod 1, \quad \beta \in SL(2, \mathbb{Z}), \tag{1}$$

using a semiclassical argument. First they impose the integrality condition of Planck's constant $h = 1/N$. Then argue that on the torus the Hilbert space has dimension N. This is semiclassical in that it corresponds to the quantization condition

$$\oint p\,dx = nh. \tag{2}$$

For the unit torus, the left-hand side of (2) is unity, and the integrality condition for Planck's constant follows. In fact, this is also the integrality condition of geometric quantization which requires the symplectic form divided by Planck's constant to define a deRham cohomology class with integer coefficients.[7] With the Hannay–Berry prescription for quantization, a further restriction was necessary: only a subset of $SL(2,\mathbb{Z})$, the so-called ''checkerboard'' matrices

$$\begin{pmatrix} odd & even \\ even & odd \end{pmatrix}, \quad \begin{pmatrix} even & odd \\ odd & even \end{pmatrix} \tag{3}$$

could be quantized.

In Refs. 8, 4, and 5, a different approach toward quantizing the cat maps was introduced, in which the dynamics is described in the Heisenberg picture as the evolution of a noncommutative algebra of observables. The Hilbert space remains infinite dimensional, and no restrictions on Planck's constant or the allowed classical maps are made. The quantum torus is constructed by deforming the classical algebra of observables. We let

$$U=Q_\hbar(e^{2\pi ix}),\quad V=Q_\hbar(e^{2\pi ip}),$$

where $Q_\hbar(\sigma)$ is an operator on the Hilbert space $\mathscr{H}$ corresponding to the classical symbol σ, and the algebra of observables $\mathfrak{A}_\hbar$ is then generated by $\{U,V,U^\dagger,V^\dagger\}$. Technically, the algebra is given with an appropriate closure. For example, taking $\mathfrak{A}_\hbar=\{U,V,U^\dagger,V^\dagger\}''$, where R'' is the bicommutant of R, the algebra generated is a von Neumann algebra. The quantum dynamics is implemented as a discrete automorphism group of $\mathfrak{A}_\hbar$ generated by a unitary operator (called the propagator) F such that as $\hbar\to 0$, $F^\dagger Q_\hbar(\sigma)F\to Q_\hbar(\sigma\circ\beta)$ in a suitable topology.

A quantization scheme of the kind just described was given in Refs. 4 and 5 for the cat map, along with the connection to the original quantization proposed by Hannay and Berry. In this paper, we extend the results to three classes of maps.

We divide the remainder of the paper into three sections, corresponding to the different maps. In Sec. II, we review the relevant results of Refs. 4 and 5 for the cat maps, and also provide a canonical derivation of the propagator *on the plane* obtained by Ref. 3 using a ''discrete time path integral.'' In Sec. III, we provide a quantization scheme for ''generalized kick maps'' $\kappa(x,p)=(x',p')$, with

$$p'=p+\epsilon f(x),\quad x'=x+p'. \tag{4}$$

Here ϵ is a real constant, and f is a continuous real function (the force function) on the unit circle with Fourier coefficients f_k obeying the bound

$$\sum_{k\in\mathbb{Z}} k^2|f_k|<\infty. \tag{5}$$

In Sec. IV, we quantize ''generalized Harper maps'': $\eta(x,p)=(x',p')$, with

$$p'=p+\epsilon_1 f(x),\quad x'=x+\epsilon_2 g(p'), \tag{6}$$

where ϵ_1 and ϵ_2 are real constants, and f and g are real functions on the unit circle with Fourier coefficients satisfying the bound:

$$\sum_j j^2|f_j|<\infty,\quad \sum_k k^2|g_k|\exp\left(2\pi\epsilon_1 k\left(\sum_j |f_j|\right)\right)<\infty.$$

Note for generic f in kick maps, and generic f and g in Harper maps, the coefficients ϵ and ϵ_1, ϵ_2 are the parameters which induce a transition to chaos as they are increased in magnitude.

II. THE CAT MAP REVISITED

The cat map (1) has become a paradigm for the study of quantum chaos (see, e.g., Ref. 9). The first quantization scheme for the cat map was given by Hannay and Berry[3] in which a path integral argument from continuous time linear dynamics was used to write down the propagator for quantum evolution. In Ref. 4, a canonical quantization was proposed using Toeplitz operators in Bargmann space, and an explicit integral kernel for the unitary one-step evolution on the entire complex plane was given. Recall that Bargmann space $\mathscr{H}^2(\mathbb{C},d\mu_\hbar)\subset L^2(\mathbb{C},d\mu_\hbar)$ is the Hilbert space of entire functions on $\mathbb{C}$ which are square integrable with respect to the measure $d\mu_\hbar(z)=(\pi\hbar)^{-1}\exp(-|z|^2/\hbar)d^2z$. Formally, Toeplitz quantization $T_\hbar$ of a symbol over phase space is the operator of multiplication on $L^2(\mathbb{C},d\mu_\hbar)$ by this symbol followed by a projection onto $\mathscr{H}^2(\mathbb{C},d\mu_\hbar)$. For calculations, we must simply note that this gives an anti-Wick ordering of the corresponding quantum operator: $T_\hbar=(z^n\overline{z^m})=T_\hbar(\overline{z^m})T_\hbar(z^n)$. The quantum torus is then constructed by restricting the algebra of observables to be the Toeplitz quantization of Fourier series with classical generators $\exp(2\pi ix)$ and $\exp(2\pi ip)$. As in Refs. 4 and 5, we define the unitary operators U and V on $\mathscr{H}^2(\mathbb{C},d\mu_\hbar)$ to be the quantization of classical generators for $f\in C(\mathbb{T}^2)$, i.e.,

$$U=\exp(2\pi i\hat{x})=\exp(\pi^2\hbar)T_\hbar(\exp(2\pi ix)),$$

$$V=\exp(2\pi i\hat{p})=\exp(\pi^2\hbar)T_\hbar(\exp(2\pi ip)),$$

where $\hat{x}=T_\hbar(x)$, $\hat{p}=T_\hbar(p)$, and $z=(1/\sqrt{2})(x-ip)$. Observe also that the canonical position and momentum variables on the plane satisfy the usual Heisenberg relation $[\hat{x},\hat{p}]=i\hbar$, and so

$$UV=e^{-4\pi^2\hbar i}VU. \tag{7}$$

Bargmann space is natural in that a phase space dynamics is given a phase space quantization. However, it is clear from the isomorphism between $\mathscr{H}^2(\mathbb{C},d\mu_\hbar)$ and $L^2(\mathbb{R},dx)$ (see Ref. 10) that we should be able to reformulate the problem in $L^2(\mathbb{R})$. Following Ref. 4, to the classical cat map β defined in Eq. (1) we assign a corresponding quantum propagator C. For linear dynamics *on the torus* we have the following result.

Theorem 1: *There exists a unique (up to a phase) unitary quantum propagator C satisfying the following properties*:

(1)

$$C^\dagger\hat{x}C=a\hat{x}+b\hat{p}=\hat{x}', \quad C^\dagger\hat{p}C=c\hat{x}+d\hat{p}=\hat{p}', \tag{8}$$

where $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$, *such that C maps the harmonic oscillator ground state into the unique ground state of the transformed coordinates.*

(2) The mapping $U\rightarrow U'=C^\dagger UC$, *and* $V\rightarrow V'=C^\dagger VC$ *extends to an automorphism of the von Neumann algebra* $\mathfrak{A}_\hbar$.

(3) For $\sigma\in C(\mathbb{T}^2)$,

$$\|C^\dagger T_\hbar(\sigma)C-T_\hbar(\sigma\circ\beta)\|\rightarrow 0 \quad \text{as } \hbar\rightarrow 0.$$

(4) Acting on a state $\Phi\in L^2(\mathbb{R})$, *C can be written explicitly as*

$$C\Phi(x)=\left(\frac{1}{hb}\right)^{1/2}\int_{\mathbb{R}}e^{i(ay^2-2yx+dx^2)/2b}\Phi(y)dy. \tag{9}$$

Remark 1: $C(x,y)=(1/hb)^{1/2}e^{i(ay^2-2yx+dx^2)/2b}$ *is precisely the integral kernel proposed in Ref. 3 using a semiclassical path integral argument.*

Proof: The proofs of (1), (2), and (3) are given in Ref. 4.

We prove statement (4). Following Ref. 10, the isomorphism between $\mathscr{H}^2(\mathbb{C},d\mu_\hbar)$ and $L^2(\mathbb{R})$ is implemented by the Bargmann transform. For $x\in\mathbb{R}$, and $z\in\mathbb{C}$, the integral kernel $B:L^2(\mathbb{R})\rightarrow\mathscr{H}^2(\mathbb{C},d\mu_\hbar)$ is given by

$$B(z,x)=\left(\frac{1}{\pi\hbar}\right)^{1/4}\exp(\sqrt{2}xz/\hbar-x^2/2\hbar-z^2/2\hbar).$$

Note also that

$$B^{-1}(z,x)=B(\overline{z},x).$$

The integral kernel for the linear dynamics on the plane in Bargmann representation was found in Ref. 4 to be

$$F(z,w)=\frac{1}{\sqrt{|\alpha|}}\exp\left\{-\frac{\overline{\beta}z^2}{2\hbar\alpha}+\frac{\overline{w}z}{\hbar\alpha}+\frac{\beta\overline{w}^2}{2\hbar\alpha}\right\},$$

where

$$\alpha=\tfrac{1}{2}(a+d+i(b-c)),\quad \beta=\tfrac{1}{2}(a-d+i(b+c)).$$

Then

$$\begin{aligned}C(x,y)&=\int B(\overline{z},x)F(z,w)B(w,y)d\mu_\hbar(w)d\mu_\hbar(z)\\&=\frac{1}{\sqrt{\pi\hbar|\alpha|}}\int\ \exp(\sqrt{2}x\overline{z}/\hbar-x^2/2\hbar-\overline{z}^2/2\hbar-\overline{\beta}z^2/2\hbar\alpha+\overline{w}z/\hbar\alpha+\beta\overline{w}^2/2\hbar\alpha\\&\quad+\sqrt{2}yw/\hbar-y^2/2\hbar-w^2/2\hbar-|z|^2/\hbar-|w|^2/\hbar)d^2zd^2w.\end{aligned}$$

This integral is a straightforward but tedious Gaussian integral. It can be calculated directly with the following result. For complex numbers A, B, C, D, E such that $\mathrm{Re}\,A>0$ and $(\mathrm{Re}\,A)^2>(\mathrm{Re}\,B+\mathrm{Re}\,C)^2+(\mathrm{Im}\,B-\mathrm{Im}\,C)^2$, we find

$$\begin{aligned}&\int_C \exp(-A|z|^2+Bz^2+C\overline{z}^2+Dz+E\overline{z})d^2z\\&\quad=\frac{\pi}{\sqrt{A^2-4BC}}\exp\frac{(D+E)^2(A^2-4BC)-(A(D-E)+2(EB-CD))^2}{4(A-B-C)(A^2-4BC)}.\end{aligned}$$

Applying this formula twice to the above integral gives the desired result. ■

III. QUANTUM KICK MAPS

We provide here a procedure for quantizing another class of automorphisms of the torus described in Sec. I. Exponentiating (4), we have for the classical maps on the torus

$$e^{2\pi ix'}=e^{2\pi i(x+p')},\quad e^{2\pi ip'}=e^{2\pi i(p+\epsilon f(x))}.\tag{10}$$

In fact, we let (10) also give the quantum evolution, that is, we find a quantum propagator such that the one-step dynamics is given by (4) exactly, with x and p replaced by the Toeplitz operators $\hat{x}$ and $\hat{p}$. However, it is not immediately clear that such a map will give a well-defined automorphism of the quantum algebra $\mathfrak{A}$ on the torus. We now demonstrate this.

Lemma 2: Let f be a real valued function satisfying (5). Then the sum $\Sigma_{k\in Z}\tilde{f}_kU^k$, where

$$\tilde{f}_k=\left(\frac{1-e^{-i\lambda k}}{i\lambda k}\right)f_k\tag{11}$$

defines a bounded self-adjoint operator.

Proof: This is an immediate consequence of the estimate

$$\sum_k \|\tilde{f}_kU^k\|\leq\sum_k |\tilde{f}_k|\leq\sum_k |f_k|<\infty.$$

The second inequality follows from (19), while the last bound on the Fourier coefficients is by assumption. ■

We now define the quantum map corresponding to (10),

$$U'=e^{-i\lambda/2}V'U,\quad V'=V\ \exp\left(2\pi i\epsilon\sum_{k\in\mathbb{Z}}\tilde{f}_kU^k\right),\tag{12}$$

where

$$\lambda=4\pi^2\hbar.$$

In the following we use the standard functional calculus for self-adjoint operators to define functions of the operators $\hat{x}$ and $\hat{p}$.

Lemma 3: With the above definitions,

$$U' = \exp(2\pi i(\hat{x}+\hat{p}')), \quad V' = \exp 2\pi i(\hat{p}+\epsilon f(\hat{x})). \tag{13}$$

Proof: Using Trotter's formula (see, e.g., Ref. 11), for T an integer,

$$\exp 2\pi i(\hat{p}+\epsilon f(\hat{x})) = \lim_{T\to\infty} (\exp[2\pi i\hat{p}/T]\exp[2\pi i\epsilon f(\hat{x})/T])^T, \tag{14}$$

where the limit is meant in the strong operator topology. Using

$$e^{-2\pi i\hat{p}/T}\hat{x}e^{2\pi i\hat{p}/T} = \hat{x} - 2\pi\hbar/T,$$

we see that the right-hand side of (14) reduces to

$$\begin{aligned}
&\exp(2\pi i\hat{p}) \lim_{T\to\infty} \exp \frac{2\pi i\epsilon}{T} \sum_{j=0}^{T-1} \sum_k f_k \exp[2\pi ik(\hat{x}-2\pi j\hbar/T)] \\
&= \exp(2\pi i\hat{p}) \lim_{T\to\infty} \exp\left(\frac{2\pi i\epsilon}{T} \sum_k f_k \exp[2\pi ik\hat{x}] \frac{1-\exp(-4\pi^2\hbar ik)}{1-\exp(4\pi^2\hbar ik/T)}\right) \\
&= \exp(2\pi i\hat{p})\exp\left(2\pi i\epsilon\sum_k \tilde{f}_k \exp(2\pi ik\hat{x})\right) = V',
\end{aligned}$$

where $\tilde{f}_k$ is defined in (11). We also find

$$\exp(2\pi i(\hat{x}+\hat{p}')) = \exp(2\pi i\hat{p}')\exp(2\pi i\hat{x})\exp(-2\pi^2[\hat{x},\hat{p}']) = V'Ue^{-i\lambda/2} = U',$$

and the claim follows. ■

Theorem 4: *Let f be a real valued function satisfying (5).*

(1) There exists a unitary operator (the quantum propagator) which implements (12), i.e.,

$$U' = K^\dagger UK, \quad V' = K^\dagger VK.$$

Explicitly, this operator is given by

$$K = e^{-i\hat{p}^2/2\hbar} e^{i\epsilon u(\hat{x})/\hbar}, \tag{15}$$

where u is a real differentiable function on the unit circle defined (up to a real constant) by $du(x)/dx = f(x)$.

(2) The map $U \to U'$, $V \to V'$ defined above extends to an automorphism of the von Neumann algebra of observables on the quantized torus. We call this automorphism the quantized kick map.

(3) The quantum dynamics has a well-defined classical limit in the norm topology. For $\sigma \in C(\mathbb{T}^2)$, we have

$$\|K^{-1}T_\hbar(\sigma)K - T_\hbar(\sigma\circ\kappa)\| \to 0, \text{ as } \hbar \to 0,$$

where $\kappa(x,p) = (p',x')$ is defined in Eq. (4).

Proof: (1) This is a straightforward calculation. Observe for K as in (15),

$$\begin{aligned}
K^\dagger VK &= \exp(2\pi iK^\dagger\hat{p}K) = \exp[2\pi i\{\exp(-i\epsilon u(\hat{x})/\hbar)\hat{p} \exp(i\epsilon u(\hat{x})/\hbar)\}] \\
&= \exp\left[2\pi i\left(\hat{p}+\epsilon \frac{du}{dx}(\hat{x})\right)\right] = V',
\end{aligned}$$

where in the last step we have used Lemma 3. Likewise,

$$K^\dagger U K = \exp[2\pi i(\exp(-i\epsilon u(\hat{x})/\hbar)\exp(i\hat{p}^2/2\hbar)\hat{x}\ \exp(-i\hat{p}^2/2\hbar)\exp(i\epsilon u(\hat{x})/\hbar))]$$
$$= \exp[2\pi i(\hat{p}' + \hat{x})] = U'.$$

(2) It follows easily from part (1) that U' and V' satisfy (7). Since U and V can be expressed in terms of U' and V', namely,

$$U = e^{i\lambda} V'^\dagger U', \quad V = V' \exp\left(-2\pi i \epsilon \sum_{k\in\mathbb{Z}} \tilde{f}_k (e^{i\lambda} V'^\dagger U')^k\right),$$

this implies that U' and V' generate $\mathfrak{A}_\hbar$. The map is thus an automorphism of the algebra $\mathfrak{A}_\hbar$.

(3) As the proof of (3) is somewhat involved, we divide it into steps.

Step 1. We first calculate the terms in (3) for the case of the pure harmonic

$$\sigma_{mn}(x,p) \coloneqq \exp(2\pi i(mx + np)).$$

Applying the Toeplitz quantization to the symbol σ_{mn} and using the Baker–Hausdorff–Campbell relation in the last step, we get

$$T_\hbar(\sigma_{mn}) = T_\hbar(\exp(\sqrt{2}\pi i(m-ni)\bar{z})\exp(\sqrt{2}\pi i(m+ni)z))$$
$$= \exp(\pi i(m-ni)(\hat{x}+i\hat{p}))\exp(\pi i(m+ni)(\hat{x}-i\hat{p}))$$
$$= V^n U^m \exp(-\pi^2\hbar(n^2+m^2+2inm)). \tag{16}$$

Thus, using (14) once again, we find

$$K^\dagger T_\hbar(\sigma_{mn})K = (V')^n (U')^m \exp(-\pi^2\hbar(n^2+m^2+2inm))$$
$$= (V')^{n+m} U^m \exp(-i\lambda m/2)\exp(-i\lambda[m(m-1)]/2)\exp(-\pi^2\hbar(n^2+m^2+2inm))$$
$$= V^{n+m} U^m \exp(-\lambda(n^2+m^2+2inm+2im^2)/4)$$
$$\times \exp\left(2\pi i\epsilon \sum_k (m+n)\tilde{f}_k(m+n)U^k\right), \tag{17}$$

where, for any integer n,

$$\tilde{f}_k(n) = \left(\frac{1-\exp(-i\lambda kn)}{i\lambda kn}\right) f_k. \tag{18}$$

Expanding the right-hand side of (17), we have

$$K^{\dagger}T_{\hbar}(\sigma_{mn})K=V^{m+n}U^{m}\ \exp(-\lambda(n^{2}+m^{2}+2inm+2im^{2})/4)$$
$$\times\sum_{l=0}^{\infty}\frac{(2\pi i\epsilon)^{l}(m+n)^{l}}{l!}\sum_{k_{1},\ldots,k_{l}}\prod_{j=1}^{l}\widetilde{f}_{k_{j}}(m+n)U^{k_{j}}.$$

We want to compare this last expression with

$$T_{\hbar}(\sigma_{mn}\circ\kappa(x,p))=T_{\hbar}(\exp(2\pi i[n(p+\epsilon f(x))+m(x+p+\epsilon f(x))]))$$
$$=T_{\hbar}\left(\exp 2\pi i((m+n)p+mx)\sum_{l=0}^{\infty}\frac{[2\pi i(m+n)\epsilon(\Sigma_{k\in\mathbb{Z}}f_{k}e^{2\pi ikx})]^{l}}{l!}\right).$$

Using (16) to expand the right-hand side of the equation above, we find

$$T_{\hbar}(\sigma_{mn}\circ\kappa(x,p))=V^{m+n}U^{m}\ \exp\left(-\frac{\lambda}{4}[(m+n)^{2}+2i(m^{2}+mn)]\right)$$
$$\times\sum_{l=0}^{\infty}\frac{(2\pi i\epsilon)^{l}(m+n)^{l}}{l!}\sum_{k_{1},\ldots,k_{l}}\prod_{j=1}^{l}f_{k_{j}}U^{k_{j}}$$
$$\times\exp\left(-\frac{\lambda}{4}[(m+k_{1}+\cdots+k_{l})^{2}+2i(m+n)(k_{1}+\cdots+k_{l})]\right).$$

Step 2. Next we prove (3) for the case of a pure harmonic. Since $\widetilde{f}_{k_j}(n)$ approaches f_k as $\hbar\to 0$, it is clear that term by term (in powers of V^pU^q) the difference $K^{\dagger}T_{\hbar}(\sigma_{mn})K$ and $T_{\hbar}(\sigma_{mn}\circ\kappa)$ vanishes as $\hbar\to 0$. To show the entire sum vanishes requires a little more work.

We shall see that this indeed happens provided the force function f satisfies our assumption. To this end, we define the following norm of f:

$$\|f\|_{2}:=\sum_{k}(1+|k|)^{2}|f_{k}|<\infty.$$

Notice first of all the difference

$$\|K^{\dagger}T_{\hbar}(\sigma_{mn})K-T_{\hbar}(\sigma_{mn}\circ\kappa)\|\leq\Bigg\|\exp\left(-\frac{\lambda}{4}(n^{2}+m^{2}+2inm+2im^{2})\right)$$
$$\times\sum_{l=0}^{\infty}\frac{(2\pi i\epsilon)^{l}(m+n)^{l}}{l!}\sum_{k_{1},\ldots,k_{l}}\prod_{j=1}^{l}\widetilde{f}_{k_{j}}(m+n)U^{k_{j}}$$
$$-\exp\left(-\frac{\lambda}{4}[(m+n)^{2}+2i(m^{2}+mn)]\right)$$
$$\times\sum_{l=0}^{\infty}\frac{(2\pi i\epsilon)^{l}(m+n)^{l}}{l!}\sum_{k_{1},\ldots,k_{l}}\prod_{j=1}^{l}f_{k_{j}}U^{k_{j}}$$
$$\times\exp-\frac{\lambda}{4}[(m+k_{1}+\cdots+k_{l})^{2}+2i(m+n)(k_{1}+\cdots+k_{l})]\Bigg\|,$$

where we have used the fact that $\|U\|=\|V\|=1$. We next use a simple argument to eliminate the exponentials preceding the sums in both terms on the right-hand side of the above expression. We see that

$$
\begin{aligned}
\|K^{\dagger}T_{\hbar}(\sigma_{mn})K-T_{\hbar}(\sigma_{mn}\circ\kappa)\| \leqslant & \left\| \left(1-\exp-\frac{\lambda}{4}\,(n^2+m^2+2inm-2im^2)\right) \right. \\
& \left. \times \sum_{l=0}^{\infty} \frac{(2\pi i\epsilon)^l(m+n)^l}{l!} \sum_{k_1,\ldots,k_l} \prod_{j=1}^{l} \tilde{f}_{k_j}(m+n)U^{k_j} \right\| \\
& + \left\| \sum_{l=0}^{\infty} \frac{(2\pi i\epsilon)^l(m+n)^l}{l!} \sum_{k_1,\ldots,k_l} \prod_{j=1}^{l} U^{k_j} \left(\sum_{j=1}^{l} \tilde{f}_{k_j}(m+n) - \prod_{j=1}^{l} f_{k_j} \right. \right. \\
& \left. \left. \times \exp-\frac{\lambda}{4}\,[(m+k_1+\cdots+k_l)^2+2i(m+n)(k_1+\cdots+k_l)] \right) \right\| \\
& + \left\| \left(1-\exp-\frac{\lambda}{4}\,((m+n)^2+2i(m^2+mn))\right) \right. \\
& \times \sum_{l=0}^{\infty} \frac{(2\pi i\epsilon)^l(m+n)^l}{l!} \sum_{k_1,\ldots,k_l} \prod_{j=1}^{l} f_{k_j}U^{k_j} \\
& \left. \times \exp-\frac{\lambda}{4}\,((m+k_1+\cdots+k_l)^2+2i(m+n)(k_1+\cdots+k_l)) \right\| .
\end{aligned}
$$

The first and third terms on the right-hand side are both of the form $\|(1-\exp-a\lambda)F\|$, where F is a bounded operator as $\lambda\to 0$ and $\mathrm{Re}(a)>0$. From the mean value theorem, it follows that

$$
|1-\exp(-a\lambda)| \leqslant |a|\lambda, \tag{19}
$$

so that these two terms are both $O(\lambda)$. Thus

$$
\begin{aligned}
& \|K^{\dagger}T_{\hbar}(\sigma_{mn})K-T_{\hbar}(\sigma_{mn}\circ\kappa)\| \\
& \quad \leqslant O(\lambda) + \left\| \sum_{l=0}^{\infty} \frac{(2\pi i\epsilon)^l(m+n)^l}{l!} \sum_{k_1,\ldots,k_l} U^{k_j} \left(\prod_{j=1}^{l} \tilde{f}_{k_j}(m+n) - \prod_{j=1}^{l} f_{k_j} \right. \right. \\
& \quad \left. \left. \times \exp-\frac{\lambda}{4}\,((m+k_1+\cdots+k_l)^2+2i(m+n)(k_1+\cdots+k_l)) \right) \right\| .
\end{aligned}
$$

An only slightly more complicated telescoping argument eliminates the exponentials inside the remaining sums. We see

$$
\begin{aligned}
& \|K^{\dagger}T_{\hbar}(\sigma_{mn})K-T_{\hbar}(\sigma_{mn}\circ\kappa)\| \\
& \quad \leqslant O(\lambda) + \left\| \sum_{l=0}^{\infty} \frac{(2\pi i\epsilon)^l(m+n)^l}{l!} \sum_{k_1,\ldots,k_l} U^{k_j} \left(\prod_{j=1}^{l} \tilde{f}_{k_j}(m+n) - \prod_{j=1}^{l} f_{k_j} \right) \right\| \\
& \quad + \left\| \sum_{l=0}^{\infty} \frac{(2\pi i\epsilon)^l(m+n)^l}{l!} \sum_{k_1,\ldots,k_l} \prod_{j=1}^{l} U^{k_j} f_{k_j} \right. \\
& \quad \left. \times \left[1-\exp-\frac{\lambda}{4}\,((m+k_1+\cdots+k_l)^2+2i(m+n)(k_1+\cdots+k_l)) \right] \right\| .
\end{aligned}
$$

Consider the last set of terms only in the above expression. We have, using (19) once again

$$\Bigg\| \sum_{l=0}^{\infty} \frac{(2\pi i\epsilon)^l (m+n)^l}{l!}$$
$$\times \sum_{k_1,\ldots,k_l} \prod_{j=1}^{l} U^{k_j} f_{k_j} \left[1 - \exp - \frac{\lambda}{4} \left((m+k_1+\cdots+k_l)^2 + 2i|m+n|(k_1+\cdots+k_l) \right) \right] \Bigg\|$$
$$\leqslant \frac{\lambda}{4} \sum_{l=0}^{\infty} \frac{(2\pi\epsilon)^l (|n|+|m|)^l}{l!} \sum_{k_1,\ldots,k_l} \prod_{j=1}^{l} |f_{k_j}|$$
$$\times |(m+k_1+\cdots+k_l)^2 + 2(|m|+|n|)(k_1+\cdots+k_l)|$$
$$\leqslant \frac{\lambda}{4} \sum_{l=0}^{\infty} \frac{(2\pi\epsilon)^l (|n|+|m|)^l}{l!} \sum_{k_1,\ldots,k_l} \prod_{j=1}^{l} |f_{k_j}|$$
$$\times (m^2 + l(k_1^2+\cdots+k_l^2) + (4|m|+2|n|)(k_1+\cdots+k_l))$$
$$\leqslant \frac{\lambda}{4} \sum_{l=0}^{\infty} \frac{(2\pi\epsilon)^l (|n|+|m|)^l}{l!} (m^2+4|m|+2|n|+l)\|f\|_2^l$$
$$\leqslant \frac{\lambda}{4} (m^2+4|m|+2|n|+2\pi\epsilon(|n|+|m|)\|f\|_2) \exp 2\pi\epsilon(|n|+|m|)\|f\|_2 = O(\lambda).$$

Combining all of this, we have reduced our estimate to

$$\|K^\dagger T_\hbar(\sigma_{mn})K - T_\hbar(\sigma_{mn}\circ\kappa)\|$$
$$\leqslant O(\lambda) + \sum_{l=0}^{\infty} \frac{(2\pi\epsilon)^l (|n|+|m|)^l}{l!}$$
$$\times \sum_{k_1,\ldots,k_l} \left| 1 - \left(\prod_{j=1}^{l} \frac{1-\exp(-i\lambda k_j(m+n))}{i\lambda k_j(m+n)} \right) \right| |f_{k_1}|\cdots|f_{k_l}|. \tag{20}$$

By Taylor's theorem with remainder, we know that for $\alpha \in \mathbb{R}$,

$$|1 - i\alpha - e^{-i\alpha}| = \left| \frac{\alpha^2}{2} (\cos\beta_1 - i\cos\beta_2) \right| \leqslant |\alpha^2|, \tag{21}$$

where β_1 and β_2 are between 0 and α. Similarly, $|1-e^{-i\alpha}| \leqslant |\alpha|$. Using these two bounds, we see that

$$\left| 1 - \prod_{j=1}^{l} \frac{1-\exp(-i\lambda k_j(m+n))}{i\lambda k_j(m+n)} \right| \leqslant \left| 1 - \frac{1-\exp(-i\lambda k_1(m+n))}{i\lambda k_1(m+n)} \right| + \left| \frac{1-\exp(-i\lambda k_1(m+n))}{i\lambda k_1(m+n)} \right|$$
$$\times \left| 1 - \frac{1-\exp(-i\lambda k_2(m+n))}{i\lambda k_2(m+n)} \right| + \cdots$$
$$+ \left| \frac{1-\exp(-i\lambda k_1(m+n))}{i\lambda k_1(m+n)} \right| \cdots \left| 1 - \frac{1-\exp(-i\lambda k_l(m+n))}{i\lambda kl(m+n)} \right|$$
$$\leqslant \lambda(m+n)(|k_1|+\cdots+|k_l|).$$

Putting this back in (20) yields

$$\|K^\dagger T_\hbar(\sigma_{mn})K - T_\hbar(\sigma_{mn}\circ\kappa)\| \leqslant O(\lambda) + \lambda(2\pi\epsilon)(|n|+|m|)^2 \|f\|_2 e^{2\pi\epsilon(|n|+|m|)\|f\|_2} = O(\lambda).$$

The right-hand side vanishes as $\lambda \to 0$. This completes the proof for a pure harmonic σ_{mn}.

Step 3. We can now complete the proof for any classical symbol $\sigma\in C(\mathbb{T}^2)$ with a simple application of the Stone–Weierstrass theorem. For such σ there exists a trigonometric polynomial P such that given $\delta>0$,

$$\|\sigma-P\|_\infty\leqslant\delta/3,$$

where $\|\sigma\|_\infty=\sup_z|\sigma(z)|$ is the sup-norm. Now, since the operator norm of a Toeplitz operator does not exceed the sup-norm of the symbol, i.e., $\|T_\hbar(\sigma)\|\leqslant\|\sigma\|_\infty$ (see Ref. 10), we find

$$\|T_\hbar(\sigma)-T_\hbar(P)\|\leqslant\delta/3.$$

Since P is a linear combination of finitely many simple harmonics, by part Step 2, we can find γ (depending on P) such that for $\hbar<\gamma$

$$\|K^\dagger T_\hbar(P)K-T_\hbar(P\circ\kappa)\|\leqslant\delta/3.$$

Thus

$$\begin{aligned}\|K^\dagger T_\hbar(\sigma)K-T_\hbar(\sigma\circ\kappa)\|\leqslant&\|K^\dagger T_\hbar(\sigma)K-K^\dagger T_\hbar(P)K\|+\|K^\dagger T_\hbar(P)K-T_\hbar(P\circ\kappa)\|+\|T_\hbar(P\circ\kappa)\\&-T_\hbar(\sigma\circ\kappa)\|\leqslant\|T_\hbar(\sigma)-T_\hbar(P)\|+\delta/3+\|(P-\sigma)\circ\kappa\|_\infty\leqslant\delta.\end{aligned}$$

This completes the proof of the theorem. ■

IV. GENERALIZED HARPER MAPS

We complete the class of quantizations of toral automorphisms with the generalized Harper maps. Exponentiating (6), we find

$$e^{2\pi ix'}=e^{2\pi i(x+\epsilon_2 g(p'))},\quad e^{2\pi ip'}=e^{2\pi i(p+\epsilon_1 f(x))}. \tag{22}$$

As before, we let (22) give the quantum evolution. That is, we are led to the following definition of the quantum Harper dynamics:

$$U'=\exp\left(2\pi i\epsilon_2\sum_{k\in\mathbb{Z}}(V')^k\widetilde{g}_k\right)U,\quad V'=V\exp\left(2\pi i\epsilon_1\sum_{k\in\mathbb{Z}}\widetilde{f}_kU^k\right), \tag{23}$$

where

$$\widetilde{g}_k=\left(\frac{1-\exp(-i\lambda k)}{i\lambda k}\right)g_k,\quad\widetilde{f}_k=\left(\frac{1-\exp(-i\lambda k)}{i\lambda k}\right)f_k,$$

and

$$g(\hat{p})=\frac{dv}{dp}(\hat{p})=\sum_{k\in\mathbb{Z}}g_kV^k,\quad f(\hat{x})=\frac{du}{dx}(\hat{x})=\sum_{k\in\mathbb{Z}}f_kU^k.$$

Calculations similar to the kick map case show that

$$U'=e^{2\pi i(\hat{x}+\epsilon_2 g(\hat{p}+\epsilon_1 f(\hat{x})))},\quad V'=e^{2\pi i(\hat{p}+\epsilon_1 f(\hat{x}))}.$$

We have the following theorem, which shows that the map above indeed yields a well-defined quantum dynamics.

Theorem 5: *(1) There exists a unitary quantum propagator which implements (23), namely,*

$$H=\exp(i\epsilon_2 v(\hat{p})/\hbar)\exp(-i\epsilon_1 u(\hat{x})/\hbar).$$

(2) The quantum Harper map defined by (23) extends to an automorphism of the von Neumann algebra $\mathfrak{A}_\hbar$ of observables on the quantized torus.

(3) For maps with f and g satisfying

$$\sum_j j^2|f_j|<\infty, \quad \sum_k k^2|g_k|\exp\left(2\pi\epsilon_1 k\sum_j |f_j|\right)<\infty,$$

the quantum propagator has a well-defined classical limit:

$$\|H^\dagger T_\hbar(\sigma)H-T_\hbar(\sigma\circ\eta)\|\to 0 \quad \text{as} \quad \hbar\to 0,$$

where $\eta(p,x)=(p',x')$ is the classical evolution defined in (6).

Proof: (1) Observe

$$\begin{aligned}H^\dagger UH&=\exp(i\epsilon_l u(\hat{x})/\hbar)\exp(-i\epsilon_2 v(\hat{p})/\hbar)U\ \exp(i\epsilon_2 v(\hat{p})/\hbar)\exp(-i\epsilon_1 u(\hat{x})/\hbar)\\&=\exp(i\epsilon_1 u(\hat{x})/\hbar)\exp(2\pi i(\hat{x}+\epsilon_2 g(\hat{p})))\exp(-i\epsilon_1 u(\hat{x})/\hbar)=\exp(2\pi i(\hat{x}+\epsilon_2 g(\hat{p}'))).\end{aligned}$$

Similarly we see $H^\dagger VH=V'$.

(2) The proof that the mapping is an automorphism of the algebra $\mathfrak{A}_\hbar$ follows that of the kick map. We omit the details here.

(3) We break the proof of (3) into the same steps as in the proof for the kick map.

Step 1. First we calculate $H^\dagger T_\hbar(\sigma_{mn})H=(V')^n(U')^m\ \exp-\pi^2\hbar(m^2+n^2+2imn)$. Then using Trotter's formula, we get

$$(V')^n=V^n\ \exp 2\pi i\epsilon_1 n\sum_{k\in\mathbb{Z}}\tilde{f}_k(n)U^k,$$

and

$$(U')^m=\exp\left(2\pi i\epsilon_2 m\sum_{k\in\mathbb{Z}}\tilde{g}_k(m)(V')^k\right)U^m,$$

where $\tilde{f}_k(n)$ and $\tilde{g}_k(n)$ are defined as in (18). Thus

$$\begin{aligned}H^\dagger T_\hbar(\sigma_{mn})H=&\sum_{l=0}^\infty\sum_{k_1,\ldots,k_l}\frac{(2\pi i\epsilon_2 m)^l}{l!}\tilde{g}_{k_1}(m)\cdots\tilde{g}_{k_l}(m)V')^{n+k_1+\cdots+k_l}U^m\\&\times\exp(-\pi^2\hbar(m^2+n^2+2imn))\\=&\sum_{l=0}^\infty\sum_{k_1,\ldots,k_l}\frac{(2\pi i\epsilon_2 m)^l}{l!}\tilde{g}_{k_1}(m)\cdots\tilde{g}_{k_l}(m)V^{n+k_1+\cdots+k_l}\\&\times\sum_{q=0}^\infty\sum_{j_1,\ldots,j_q}\frac{(2\pi i\epsilon_1(n+k_1+\cdots+k_l))^q}{q!}\left(\prod_{i=1}^q\tilde{f}_{j_i}(n+k_1+\cdots+k_l)\right)\\&\times U^{m+j_1+\cdots+j_q}\ \exp(-\pi^2\hbar(m^2+n^2+2imn)).\end{aligned}$$

Next, we calculate

$$T_\hbar(\sigma_{mn}\circ\eta)=T_\hbar(\exp[2\pi i(mx'+np')])$$

$$=\sum_{l=0}^{\infty}\sum_{k_1,\dots,k_l}\sum_{q=0}^{\infty}\sum_{j_1,\dots,j_q} T_\hbar\Bigg(\frac{(2\pi i\epsilon_2 m)^l}{l!}\, g_{k_1}\cdots g_{k_l}$$

$$\times\frac{(2\pi i\epsilon_1(n+k_1+\cdots+k_l))^q}{q!}\, f_{j_1}\cdots f_{j_q}$$

$$\times\exp\{2\pi i((m+j_1+\cdots+j_q)x+(n+k_1+\cdots+k_l)p)\}\Bigg)$$

$$=\sum_{l=0}^{\infty}\sum_{k_1,\dots,k_l}\sum_{q=0}^{\infty}\sum_{j_1,\dots,j_q}\frac{(2\pi i\epsilon_2 m)^l}{l!}\, g_{k_1}\cdots g_{k_l}$$

$$\times\frac{(2\pi i\epsilon_1(n+k_1+\cdots+k_l))^q}{q!}\, f_{j_1}\cdots f_{j_q}V^{n+k_1+\cdots+k_l}U^{m+j_1+\cdots-j_q}$$

$$\times\exp[(n+k_1+\cdots+k_l)^2+(m+j_1+\cdots+j_q)^2$$

$$+2i(n+k_1+\cdots+k_l)(m+j_1+\cdots+j_q)].$$

It is obvious that term by term (in powers of U^pV^q) that the previous two expressions approach each other as $\hbar\to 0$. The proof that the sums converge is very similar to the analogous proof for the kick map in the previous section. An outline of the results is as follows. We find that

$$\|H^\dagger T_\hbar(\sigma_{mn})H-T_\hbar(\sigma_{mn}\circ\kappa)\|\leqslant D_1+D_2+D_3,$$

where (writing $\overline{n}=n+k_1+\cdots+k_l$, $\overline{m}=m+j_1+\cdots+j_q$)

$$D_1=\Bigg\|\sum_{l=0}^{\infty}\sum_{k_1,\dots,k_l}\sum_{q=0}^{\infty}\sum_{j_1,\dots,j_q}\frac{(2\pi i\epsilon_2 m)^l(2\pi i\epsilon_1\overline{n})^q}{l!\,q!}\, V^{\overline{n}}U^{\overline{m}}$$

$$\times g_{k_1}\cdots g_{k_l}f_{j_1}\cdots f_{j_q}(e^{-\pi^2\hbar(\overline{n}^2+\overline{m}^2+2i\overline{n}\overline{m})}-1)\Bigg\|,$$

$$D_2=\Bigg\|\sum_{l=0}^{\infty}\sum_{k_1,\dots,k_l}\sum_{q=0}^{\infty}\sum_{j_1,\dots,j_q}\frac{(2\pi i\epsilon_2 m)^l(2\pi i\epsilon_1\overline{n})^q}{l!\,q!}\, V^{\overline{n}}U^{\overline{m}}g_{k_1}\cdots g_{k_l}f_{j_1}\cdots f_{j_q}$$

$$-\tilde{g}_{k_1}(m)\cdots\tilde{g}_{k_l}(m)\tilde{f}_{j_1}(\overline{n})\cdots\tilde{f}_{j_q}(\overline{n})\Bigg\|,$$

$$D_3=\Bigg\|\sum_{l=0}^{\infty}\sum_{k_1,\dots,k_l}\sum_{q=0}^{\infty}\sum_{j_1,\dots,j_q}\frac{(2\pi i\epsilon_2 m)^l(2\pi i\epsilon_1\overline{n})^q}{l!\,q!}\, V^{\overline{n}}U^{\overline{m}}\tilde{g}_{k_1}(m)\cdots\tilde{g}_{k_l}\tilde{f}_{j_1}(\overline{n})\cdots\tilde{f}_{j_q}(\overline{n})$$

$$\times(e^{-\pi^2\hbar(m^2+n^2+2inm)}-1)\Bigg\|.$$

Using (19) once again, we see that D_3 is $O(\lambda)$. To show that D_1 and D_2 go to zero we set

$$\|g\|_{2,f}:=\sum_{k\in\mathbb{Z}}(1+|k|)^2|g_k|\exp(2\pi\epsilon_1 k\|f\|_2)<\infty,$$

and with these bounds,

$$
\begin{aligned}
D_1 \leqslant & \frac{\lambda}{4} \sum_{l=0}^{\infty} \sum_{k_1,\ldots,k_l} \sum_{q=0}^{\infty} \sum_{j_1,\ldots,j_q} \frac{(2\pi\epsilon_2|m|)^l (2\pi\epsilon_1|\overline{n}|)^q}{l!q!} |g_{k_1}|\cdots|g_{k_l}||f_{j_1}|\cdots|f_{j_q}| \\
& \times(\overline{n}^2+m^2+2m(j_1+\cdots+j_q)+q(j_1^2+\cdots+j_q^2)+2\overline{n}(m+j_1+\cdots+j_q)) \\
\leqslant & \frac{\lambda}{4} \sum_{l=0}^{\infty} \sum_{k_1,\ldots,k_l} \frac{(2\pi\epsilon_2|m|)^l}{l!} |g_{k_1}|\cdots|g_{k_6}| \exp(2\pi\epsilon_1|\overline{n}|\|f\|_2) \\
& \times(\overline{n}^2+m^2+2|m|+2\pi\epsilon_1\overline{n}\|f\|_2+2|\overline{n}|(|m|+1)) \\
\leqslant & \frac{\lambda}{4} \exp(2\pi\epsilon_1|n|\|f\|_2)\exp(2\pi\epsilon_2 m\|g\|_{2,f})((|m|+|n|)^2+4(|n|+|m|)+2\pi\epsilon_2|m|\|g\|_{2,f} \\
& +2\pi\epsilon_1\|f\|_2(|n|+1)+2)=O(\lambda),
\end{aligned}
$$

while using (21),

$$
\begin{aligned}
D_2 \leqslant & \sum_{l=0}^{\infty} \sum_{k_1,\ldots,k_l} \sum_{q=0}^{\infty} \sum_{j_1,\ldots,j_q} \frac{(2\pi\epsilon_2|m|)^l (2\pi\epsilon_1|\overline{n}|)^q}{l!q!} |g_{k_1}|\cdots|g_{k_l}||f_{j_1}|\cdots|f_{j_q}| \\
& \times\left(\left|1-\prod_{r=1}^{l} \left|\frac{\widetilde{g}_{k_r}(m)}{g_{k_r}}\right|\right| + \left|1-\prod_{r=1}^{l} \left|\frac{\widetilde{g}_{k_r}(m)}{g_{k_r}}\right|\right| \left|1-\prod_{r=1}^{l} \left|\frac{\widetilde{f}_{j_s}(\overline{n})}{f_{j_s}}\right|\right| + \left|1-\prod_{r=1}^{l} \left|\frac{\widetilde{f}_{j_s}(\overline{n})}{f_{j_s}}\right|\right| \right) \\
\leqslant & \sum_{l=0}^{\infty} \sum_{k_1,\ldots,k_l} \sum_{q=0}^{\infty} \sum_{j_1,\ldots,j_q} \frac{(2\pi\epsilon_2|m|)^l (2\pi\epsilon_1|\overline{n}|)^q}{l!q!} |g_{k_1}|\cdots|g_{k_l}||f_{j_1}|\cdots|f_{j_q}|(\lambda|m|(|k_1|+\cdots \\
& +|k_l|)+\lambda|\overline{n}|(|j_1|+\cdots+|j_q|)\lambda m(|k_1|+\cdots+|k_l|)+\lambda|\overline{n}|(|j_1|+\cdots+|j_q|)) \\
= & \lambda \sum_{l=0}^{\infty} \sum_{k_1,\ldots,k_l} \frac{(2\pi\epsilon_2|m|)^l}{l!} |g_{k_1}|\cdots|g_{k_l}| \exp(2\pi\epsilon_1\overline{n}\|f\|_2)(|m|(|k_1|+\cdots+|k_l|) \\
& +\lambda|\overline{n}|m(|k_1|+\cdots+|k_l|))+|\overline{n}| \\
= & \lambda\ \exp(2\pi\epsilon_1|n|\|f\|_2)\exp(2\pi\epsilon_2|m|\|g\|_{2,f})(|m|+\lambda|m|(|n|+2\pi\epsilon_2 m\|g\|_{2,f}) \\
& +(|n|+1))=O(\lambda).
\end{aligned}
$$

We have thus shown that $\|H^{\dagger}T_{\hbar}(\sigma_{mn})H-T_{\hbar}(\sigma_{mn}\circ\eta)\|=O(\lambda)$. The proof is completed for a general symbol $\sigma\in C(\mathbb{T}^2)$ by referring to Step 3 in the proof of the analogous theorem for the kick map. ■

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