K-Theory 4: 201–218, 1991.

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Equivariant Entire Cyclic Cohomology: I. Finite Groups*

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(Received: April 1990)

Abstract. We extend the framework of entire cyclic cohomology to the equivariant context.

Key words. Cyclic cohomology, group action.

1. Introduction

Entire cyclic cohomology was introduced by A. Connes [3] and studied further in [6, 5, 8, 4]. Its objective is to deal with aspects of noncommutative differential geometry [2] (see also [7]) of, somewhat loosely speaking, infinite-dimensional noncommutative spaces. Situations of this kind arise, for example, in studying global aspects of supersymmetric quantum field theory. Entire cyclic cohomology is believed to provide the right cohomological setup for studying topological invariants arising from models of supersymmetric quantum field theory, such as the index of the supercharge (which can be interpreted as a Dirac-like operator on loop space).

In this paper, we are concerned with the equivariant version of entire cyclic cohomology. We consider a triple $(\mathscr{A}, G, \rho)$, where $\mathscr{A}$ is a Banach algebra, where G is a group and where $\rho : G \to \mathrm{Aut}(\mathscr{A})$ is a G-action on $\mathscr{A}$. Infinite-dimensional examples of such a structure are provided, for example, by quantum field theory models with 'additional symmetries'. We believe that the natural cohomological setup for Witten's elliptic genera computations [14] is provided by equivariant entire cyclic cohomology of a suitable algebra $\mathscr{A}$ of functions on loop space with the symmetry group $G = \mathrm{U}(1)$. In this paper, we define the equivariant cyclic cohomology of $(\mathscr{A}, G, \rho)$ and discuss its functorial properties. We construct a natural pairing of the even cohomology group with the equivariant K-theory group $K_0^G(\mathscr{A})$. Finally, we discuss the connection between the equivariant entire cyclic cohomology and the ordinary cyclic cohomology of the Banach algebra $L^1_\rho(G, \mathscr{A})$.

* Supported in part by the Department of Energy under Grant DE-FG02-88ER25065.
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In the accompanying paper [9] we construct an equivariant entire cyclic cocycle, namely the equivariant Chern character. The framework of [9] is more differential geometric in spirit (rather than cohomological), and it assumes the existence of a G-invariant Dirac operator.

For conceptual clairty and technical simplicity, we deal with the case of G finite. We will discuss the case of a compact G in a future publication. The assumption that $\mathscr{A}$ is a unital Banach algebra is the only relevant technical assumption. Our constructions can be also carried through in a functorial way, if $\mathscr{A}$ is a unital $\mathbb{C}^*$-algebra. In this case, the algebra $L^1_\rho(G, \mathscr{A})$ is replaced by the crossed product $G \times_\rho \mathscr{A}$.

2. B-Dynamical Systems and the Algebra $L^1_\rho(G, \mathscr{A})$

In this section, we review the basic notions of the theory of Banach algebras with group actions (see [11] for details and references to the original literature).

2.1. B-DYNAMICAL SYSTEMS

Let $\mathscr{A}$ be a unital Banach algebra. By $\mathrm{Aut}(\mathscr{A})$ we denote the group of continuous automorphisms of $\mathscr{A}$. Let G be a finite group and let $\rho : G \to \mathrm{Aut}(\mathscr{A})$ be a homomorphism of groups. Clearly, for all $g \in G$ and $a \in \mathscr{A}$,

$$\|\rho_g(a)\| \leqslant C\|a\|, \tag{2.1}$$

where C is a constant. The triple $(\mathscr{A}, G, \rho)$ is called a B-dynamical system.

It is straightforward to define functorial operations in the category of B-dynamical systems and we will not elaborate on this point. In the following, we will need one particular functor in this category which we now describe. Let $\mathscr{V}$ be a finite-dimensional vector space and let $U : G \to \mathrm{End}(\mathscr{V})$ be a representation of G on $\mathscr{V}$. The algebra $\mathrm{End}(\mathscr{V})$ of endomorphisms of $\mathscr{V}$ has a natural G-action given by

$$U_g(m) := U_g m U_{g^{-1}}, \quad m \in \mathrm{End}(\mathscr{V}). \tag{2.2}$$

Then $(\mathscr{A} \otimes \mathrm{End}(\mathscr{V}), G, \rho \otimes U)$ is a B-dynamical system.

2.2. THE ALGEBRA $L^1_\rho(G, \mathscr{A})$

We consider the space of all functions $x : G \to \mathscr{A}$. For two such functions x and y we define

$$(x *_\rho y)(g) := \int_G x(h)\rho_h(y(h^{-1}g))\,\mathrm{d}h, \tag{2.3}$$

where for $x : G \to \mathscr{A}$

$$\int_G x(h)\,\mathrm{d}h := \frac{1}{|G|}\sum_{h \in G} x(h) \tag{2.4}$$

is the $\mathscr{A}$-valued Haar integral. Here $|G|$ denotes the order of G. The product (2.3) is associative. Let $L^1_\rho(G, \mathscr{A})$ denote the space of functions $x: G \to \mathscr{A}$ equipped with this product. This algebra is

(α) unital, with the identity element

$$\delta(g) := \begin{cases} 1, & \text{if } g = e, \\ 0, & \text{otherwise}; \end{cases} \tag{2.5}$$

(β) Banach, with the norm

$$\|x\|_1 := \int_G \|x(g)\| \, dg. \tag{2.6}$$

3. Equivariant Entire Cyclic Cohomology of a Banach Algebra

In this section, we introduce a cohomology framework generalizing the entire cyclic cohomology of [3] (see also [6, 4, 5, 8]) to the equivariant context. We consider a dynamical system $(\mathscr{A}, G, \rho)$.

3.1. THE SPACE $C^n_G(\mathscr{A})$

Let $\mathscr{F}(G)$ be the space of functions $f: G \to \mathbb{C}$ and let $\mathscr{L}^n(\mathscr{A}, \mathscr{F}(G))$ denote the linear space of n-linear mappings $f: \mathscr{A} \times \cdots \times \mathscr{A} \to \mathscr{F}(G)$ such that

$$\|f\| := \max_{g \in G} \sup_{\|a_j\| \leqslant 1} |f(a_1, \ldots, a_n)(g)| < \infty. \tag{3.1}$$

We define a G-action on $\mathscr{L}^n(\mathscr{A}, \mathscr{F}(G))$ by

$$(\rho^*_h f)(a_1, \ldots, a_n)(g) := f(\rho_h(a_1), \ldots, \rho_h(a_n))(g). \tag{3.2}$$

Note that $\rho^*_h : \mathscr{L}^n(\mathscr{A}, \mathscr{F}(G)) \to \mathscr{L}^n(\mathscr{A}, \mathscr{F}(G))$, as

$$\|\rho^*_h f\| \leqslant C^n \|f\|. \tag{3.3}$$

A mapping $f \in \mathscr{L}^n(\mathscr{A}, \mathscr{F}(G))$ is called G-equivariant if for all $h \in G$

$$(\rho^*_h f)(a_1, \ldots, a_n)(g) = f(a_1, \ldots, a_n)(h^{-1}gh). \tag{3.4}$$

We define $\mathscr{C}^n_G(\mathscr{A})$ to be the linear space of all G-equivariant $f \in \mathscr{L}^{n+1}(\mathscr{A}, \mathscr{F}(G))$.

We now define three basic complexes associated with $\mathscr{C}^n_G(\mathscr{A})$.

3.2. THE STANDARD $\mathbb{Z}_n$-COMPLEX

Let $\mathbb{Z}_n$ denote the cyclic group of n elements. We define for $f \in \mathscr{C}^n_G(\mathscr{A})$

$$(T_n f)(a_0, a_1, \ldots, a_n)(g) := (-1)^n f(\rho_{g^{-1}}(a_n), a_0, \ldots, a_{n-1})(g). \tag{3.5}$$

Note that

$$\|T_n f\| \leqslant C\|f\| \quad \text{and} \quad (\rho^*_h T_n f)(a_0, \ldots, a_n)(g) = (T_n f)(a_0, \ldots, a_n)(h^{-1}gh).$$

i.e. $T_n f \in C^n_G(\mathscr{A})$. The operator T_n defines a representation of $\mathbb{Z}_{n+1}$ on the space $\mathscr{C}^n_G(\mathscr{A})$. In fact, all we need to check is

$$T_n^{n+1} = I. \tag{3.6}$$

But, as a consequence of (3.4),

$$\begin{aligned}(T_n^{n+1} f)(a_0, a_1, \dots, a_n)(g) \\ &= (-1)^{n(n+1)} f(\rho_{g^{-1}}(a_0), \dots, \rho_{g^{-1}}(a_n))(g) \\ &= f(a_0, a_1, \dots, a_n)(g),\end{aligned}$$

which proves (3.6)

We define the norm operator

$$N_n := \sum_{j=0}^{n} T_n^j, \tag{3.7}$$

and observe that, as a consequence of (3.6),

$$N_n^2 = (n+1) N_n, \tag{3.8}$$

and

$$(I - T_n) N_n = N_n (I - T_n) = 0. \tag{3.9}$$

Identities (3.9) lead naturally to the following standard $\mathbb{Z}_{n+1}$-complex

$$\begin{tikzcd} \mathscr{C}^n_G(\mathscr{A}) \arrow[r, "N_n"] & \mathscr{C}^N_G(\mathscr{A}) \arrow[d, "I - T_n"] \\ \mathscr{C}^n_G(\mathscr{A}) \arrow[u, "I - T_n"] \arrow[r, "N_n"'] & \mathscr{C}^n_G(\mathscr{A}) \end{tikzcd} \tag{3.10}$$

This complex is acyclic, as we have the following homotopy equation:

$$M_n(1 - T_n) + \frac{1}{n+1} N_n = I, \tag{3.11}$$

where

$$M_n := -\frac{1}{n+1} \sum_{j=0}^{n} j T_n^j. \tag{3.12}$$

3.3. THE ACYCLIC HOCHSCHILD COMPLEX

We define for $f \in \mathscr{C}^n_G(\mathscr{A})$

$$(b'_n f)(a_0, \dots, a_n)(g) := \sum_{j=0}^{n} (-1)^j f(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1})(g). \tag{3.13}$$

Clearly, $b'_n f \in \mathscr{C}^{n+1}_G(\mathscr{A})$. It is well-known that $b'_{n+1} b'_n = 0$. This leads to the complex

$$\cdots \longrightarrow \mathscr{C}^{n-1}_G(\mathscr{A}) \xrightarrow{b'_{n-1}} \mathscr{C}^n_G(\mathscr{A}) \xrightarrow{b'_n} \mathscr{C}^{n+1}_G(\mathscr{A}) \longrightarrow \cdots. \tag{3.14}$$

The complex is acyclic with the homotopy operator $U_n : \mathscr{C}_G^n(\mathscr{A}) \to \mathscr{C}_G^{n-1}(\mathscr{A})$,

$$(U_n f)(a_0, \ldots, a_{n-1})(g) := (-1)^{n-1} f(a_0, \ldots, a_{n-1}, \mathbf{1})(g), \tag{3.15}$$

satisfying

$$b'_{n-1} U_n + U_{n+1} b'_n = I. \tag{3.16}$$

3.4. THE HOCHSCHILD COMPLEX

We set for $f \in \mathscr{C}_G^n(\mathscr{A})$

$$(V_n f)(a_0, \ldots, a_{n+1})(g) := (-1)^{n+1} f(\rho_{g^{-1}}(a_{n+1}) a_0, a_1, \ldots, a_n)(g), \tag{3.17}$$

and note that

$$\|V_n f\| \leqslant C \|f\|$$

and

$$(\rho_h^* V_n f)(a_0, \ldots, a_{n+1})(g) = (V_n f)(a_0, \ldots, a_{n+1})(h^{-1} g h),$$

i.e. $V_n f \in \mathscr{C}_G^{n+1}(\mathscr{A})$. Consider the operator $b_n : \mathscr{C}_G^n(\mathscr{A}) \to \mathscr{C}_G^{n+1}(\mathscr{A})$,

$$b_n := b'_n + V_n. \tag{3.18}$$

The operator is the equivariant version of the Hochschild coboundary operator. In fact, we claim that

$$b_{n+1} b_n = 0. \tag{3.19}$$

To prove this we compute

$$\begin{aligned}
&(b_{n+1} b_n f)(a_0, \ldots, a_{n+2})(g) \\
&\quad = (b'_{n+1} V_n + V_{n+1} b'_n + V_{n+1} V_n) f(a_0, \ldots, a_{n+2})(g) \\
&\quad = \sum_{j=0}^{n+1} (-1)^j (V_n f)(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+2})(g) + \\
&\qquad + (-1)^{n+2} (b' f)(\rho_{g^{-1}}(a_{n+2}) a_0, a_1, \ldots, a_{n+1})(g) + \\
&\qquad + (-1)^{n+2} (V_n f)(\rho_{g^{-1}}(a_{n+2}) a_0, a_1, \ldots, a_{n+1})(g) \\
&\quad = (-1)^{n+1} f(\rho_{g^{-1}}(a_{n+2}) a_0, a_1, a_2, \ldots, a_{n+1})(g) + \\
&\qquad + (-1)^{n+1} \sum_{j=1}^{n} (-1)^j f(\rho_{g^{-1}}(a_{n+2}) a_0, a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1})(g) + \\
&\qquad + f(\rho_{g^{-1}}(a_{n+1} a_{n+2}) a_0, a_1, \ldots, a_n)(g) + \\
&\qquad + (-1)^{n+2} f(\rho_{g^{-1}}(a_{n+2})(a_0, a_1, a_2, \ldots, a_{n+1})(g) + \\
&\qquad + (-1)^{n+2} \sum_{j=1}^{n} (-1)^j f(\rho_{g^{-1}}(a_{n+2}) a_0, a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1})(g) - \\
&\qquad - f(\rho_{g^{-1}}(a_{n+1} a_{n+2}) a_0, a_1, \ldots, a_n)(g) = 0.
\end{aligned}$$

This yields the complex

$$\cdots \longrightarrow \mathscr{C}_G^{n-1}(\mathscr{A}) \xrightarrow{b_{n-1}} \mathscr{C}_G^n(\mathscr{A}) \xrightarrow{b_n} \mathscr{C}_G^{n+1}(\mathscr{A}) \longrightarrow \cdots. \tag{3.20}$$

With the help of the above basic complexes we define the following cyclic complexes [10, 2].

3.5. THE EQUIVARIANT CYCLIC COMPLEX

We first observe that the following relations hold

$$b_n'(1 - T_n) = (1 - T_{n+1})b_n \tag{3.21}$$

and

$$N_{n+1}b_n' = b_n N_n. \tag{3.22}$$

The proof of (3.21) and (3.22) is a standard consequence of the following identities [13]:

$$b_n' = \sum_{j=0}^{n} T_{n+1}^{-(j+1)} V_n T_n^j \tag{3.23}$$

and

$$b_n = \sum_{j=0}^{n+1} T_{n+1}^{-(j+1)} V_n T_n^j. \tag{3.24}$$

Also, note that

$$U_{n+1} V_n = -I \tag{3.25}$$

and

$$b_{n-1}' U_n + U_{n+1} b_n = 0. \tag{3.26}$$

To prove (3.25) we compute

$$\begin{aligned}
&(U_{n+1} V_n f)(a_0, \ldots, a_n)(g)\\
&\quad = (-1)^n (V_n f)(a_0, \ldots, a_n, \mathbf{1})(g)\\
&\quad = (-1)^{2n+1} f(\rho_{g^{-1}}(\mathbf{1})a_0, a_1, \ldots, a_n)(g)\\
&\quad = -f(a_0, \ldots, a_n)(g).
\end{aligned}$$

Equation (3.26) is a consequence of (3.16) and (3.25)

$$b_{n-1}' U_n + U_{n+1} b_n = b_{n-1}' U_n + U_{n+1} b_n' + U_{n+1} U_n = 1 - 1 = 0.$$

We now consider the operator $B_n : \mathscr{C}_G^n(\mathscr{A}) \to \mathscr{C}_G^{n-1}(\mathscr{A})$ given by

$$B_n := N_{n-1} U_n (1 - T_n). \tag{3.27}$$

We claim that

$$B_{n-1}B_n = 0 \tag{3.28}$$

and

$$b_{n-1}B_n + B_{n+1}b_n = 0. \tag{3.29}$$

In fact, from (3.9)

$$B_{n-1}B_n = N_{n-2}U_{n-1}(1-T_{n-1})N_{n-1}U_n(1-T_n) = 0,$$

which is (3.28). To prove (3.29) we use (3.16)

$$\begin{aligned} &b_{n-1}B_n + B_{n+1}b_n \\ &\quad = b_{n-1}N_{n-1}U_n(1-T_n) + N_nU_{n+1}(1-T_{n+1})b_n \\ &\quad = N_nb'_{n-1}U_n(1-T_n) + N_nU_{n+1}b'_n(1-T_n) \\ &\quad = N_n(b'_{n-1}U_n + U_{n+1}b'_n)(1-T_n) = 0. \end{aligned}$$

This leads to the following complex: Set

$$\mathscr{C}_G^{\leqslant n}(\mathscr{A}) := \begin{cases} \bigoplus_{j=0}^{n/2} \mathscr{C}_G^{2j}(\mathscr{A}), & \text{if } n \text{ is even,} \\ \bigoplus_{j=0}^{(n-1)/2} \mathscr{C}_G^{2j+1}(\mathscr{A}); & \text{if } n \text{ is odd,} \end{cases} \tag{3.30}$$

and define an operator $\partial_n : \mathscr{C}_G^{\leqslant n}(\mathscr{A}) \to \mathscr{C}_G^{\leqslant n+1}(\mathscr{A})$ by

$$\begin{aligned} \partial_{2k}(f_0, f_2, \ldots, f_{2k}) &:= (b_0f_0 + B_2f_2, b_2f_2 + B_4f_4, \ldots, b_{2k}f_{2k}), \\ \partial_{2k+1}(f_1, f_3, \ldots, f_{2k+1}) &:= (B_1f_1, b_1f_1 + B_3f_3, \ldots, b_{2k+1}f_{2k+1}). \end{aligned} \tag{3.31}$$

As a consequence of (3.28), (3.29)

$$\partial_{n+1}\partial_n = 0 \tag{3.32}$$

and, thus,

$$\cdots \longrightarrow \mathscr{C}_G^{\leqslant n}(\mathscr{A}) \xrightarrow{\partial_n} \mathscr{C}_G^{\leqslant (n+1)}(\mathscr{A}) \longrightarrow \cdots \tag{3.33}$$

is a complex. It is the equivariant cyclic complex.

3.6. EQUIVALENT ENTIRE CYCLIC COMPLEX

Let $\mathscr{C}_G(\mathscr{A})$ be the space of sequences

$$f = \{f_n\}_{n=0}^{\infty}, \quad f_n \in \mathscr{C}_G^n(\mathscr{A})$$

such that

$$\lim_{n\to\infty} n^{1/2}\|f_n\|^{1/n} = 0. \tag{3.34}$$

We write

$$\mathscr{C}_G(\mathscr{A}) \cong \mathscr{C}_G^o(\mathscr{A}) \oplus \mathscr{C}_G^e(\mathscr{A}), \tag{3.35}$$

where $\mathscr{C}_G^o(\mathscr{A})$ consists of the sequences $\{f_{2k+1}\}_{k=0}^{\infty}$ obeying (3.34), whereas $\mathscr{C}_G^e(\mathscr{A})$ is the space of all $\{f_{2k}\}_{k=0}^{\infty}$ satisfying (3.34).

The spaces $\mathscr{C}_G^o(\mathscr{A})$, $\mathscr{C}_G^e(\mathscr{A})$ and $\mathscr{C}_G(\mathscr{A})$ are Fréchet space with the following set of norms. For $r = 1, 2, 3, \ldots$ we set

$$\|f\|_r := \sum_{n \geqslant 0} (n!)^{1/2} \|f_n\| r^n. \tag{3.36}$$

For $f \in \mathscr{C}_G^o(\mathscr{A})$ we set

$$\partial_o f := (B_1 f_1, b_1 f_1 + B_3 f_3, \ldots, b_{2k-1} f_{2k-1} + B_{2k+1} f_{2k+1}, \ldots). \tag{3.37}$$

It is easy to verify that

$$\|(\partial_o f)_{2k}\| \leqslant (2k+1)\|f_{2k-1}\| + 4k\|f_{2k+1}\|,$$

which implies that $\partial_o : \mathscr{C}_G^o(\mathscr{A}) \to \mathscr{C}_G^e(\mathscr{A})$ is a continuous homomorphism. Likewise, for $f \in \mathscr{C}_G^e(\mathscr{A})$, we set

$$\partial_e f := (b_0 f_0 + B_2 f_2, b_2 f_2 + B_4 f_4, \ldots, b_{2k} f_{2k} + B_{2k+2} f_{2k+2}, \ldots). \tag{3.38}$$

Then

$$\|(\partial_e f)_{2k+1}\| \leqslant 2(k+1)\|f\|_{2k} + 2(2k+1)\|f_{2k+2}\|,$$

i.e., $\partial_e : \mathscr{C}_G^e(\mathscr{A}) \to \mathscr{C}_G^o(\mathscr{A})$ is a continuous homomorphism. We also note that

$$\partial_e \partial_o = \partial_o \partial_e = 0. \tag{3.39}$$

The cohomology of the complex

$$\begin{array}{ccc} \mathscr{C}_G^o(\mathscr{A}) & \xrightarrow{\partial_o} & \mathscr{C}_G^e(\mathscr{A}) \\ {\scriptstyle \partial_e}\uparrow & & \downarrow{\scriptstyle \partial_e} \\ \mathscr{C}_G^e(\mathscr{A}) & \xleftarrow{\partial_o} & \mathscr{C}_G^o(\mathscr{A}) \end{array} \tag{3.40}$$

is called the equivariant entire cyclic cohomology of $(\mathscr{A}, G, \rho)$. The corresponding cohomology groups are denoted by $\mathscr{H}_G^o(\mathscr{A})$ and $\mathscr{H}_G^e(\mathscr{A})$.

The operations T, N, M, U and V defined above can be naturally defined on $\mathscr{C}_G(\mathscr{A})$. We set

$$Tf := (T_0 f_0, T_1 f_1, \ldots, T_n f_n, \ldots), \tag{3.41}$$

etc., and verify easily that the corresponding operations are continuous homomorphisms of the entire complex. Also, to simplify the notation, we will suppress the subscripts in ∂_o and ∂_e and write simply ∂.

3.7. NORMALIZED COCYCLES

An equivariant entire cocycle f is normalized (in the sense of Connes [3]), if

$$B_0 f = A B_0 f, \tag{3.42}$$

where

$$B_0 := U(1 - T), \tag{3.43}$$

and where

$$(Af)_n = \frac{1}{n+1} N_n f_n. \tag{3.44}$$

We prove below that every equivariant entire cohomology class has a normalized representative. Our proof is somewhat different from Connes' original proof. The referee of this paper called our attention to the fact that a very similar construction (in the context of $\mathbb{Z}_2$ graded entire cyclic cohomology) appeared in [8].

LEMMA. *A cocycle $f \in \mathscr{C}_G(\mathscr{A})$ is normalized if and only if*

$$(1 - T) B_0 f = 0. \tag{3.45}$$

Proof. Applying $1 - T$ to (3.42) and using (3.9) we obtain (3.45). Suppose now that f satisfies (3.45). Then using (3.11),

$$(I - A) B_0 f = M(1 - T) B_0 f = 0. \qquad \square$$

THEOREM. *Every equivariant entire cohomology class has a normalized representative.*

Proof. Let f be a cocycle. Set

$$g := MU(1 - T) f = M B_0 f \tag{3.46}$$

and

$$f' := f - \partial g. \tag{3.47}$$

Clearly, $f' \in \mathscr{C}_G(\mathscr{A})$. We claim that f' is normalized. By the lemma, we have to show that $(1 - T) B_0 f' = 0$. But since $(1 - T) B = (1 - T) N B_0 = 0$, this is equivalent to showing that

$$(1 - T) B_0 f - (1 - T) B_0 b M B_0 f = 0 \tag{3.48}$$

(Note that (3.48) corresponds to the property $v^{\perp} \mathbb{B}_o (1 - \lambda) = 0$ in Remark 1.8 of [8].)

The second term on the left-hand side of (3.48) can be written as

$$\begin{aligned}&-(1-T)U(1-T)bMB_0f\\&\quad=-(1-T)Ub'(1-T)MB_0f\\&\quad=-(1-T)Ub(1-T)MB_0f+(1-T)UV(1-T)MB_0f\\&\quad=-(1-T)Ub(1-A)B_0f-(1-T)(1-A)B_0f\\&\quad=-(1-T)Ub(1-A)B_0f-(1-T)B_0f.\end{aligned}$$

The last term in this identity cancels the first term in (3.48) and thus (3.48) is reduced to showing that

$$(1-T)Ub(1-A)B_0f=0. \tag{3.49}$$

But f is a cocycle, i.e. $bf+Bf=0$ and, thus,

$$AB_0f'=0. \tag{3.50}$$

Moreover,

$$\begin{aligned}&(1-T)UbB_0f\\&\quad=(1-T)UbU(1-T)f\\&\quad=(1-T)Ub'U(1-T)f-(1-T)U(1-T)f\\&\quad=(1-T)U^2b'(1-T)f=(1-T)U^2(1-T)bf\\&\quad=-(1-T)U^2(1-T)Bf=0,\end{aligned}$$

as $(1-T)Bf=(1-T)NB_0f=0$. This and (3.50) imply (3.49). □

4. Examples and Computations

This section contains a number of examples of equivariant entire cyclic cohomologies. Of particular importance for the constructions of Section 5 is example 4.5 of this section. We show in this example that there is a cochain homomorphism between the equivariant entire cyclic complex of $\mathscr{A}$ and the ordinary entire cyclic complex of $L^1(\mathscr{A}, G, \rho)$. In this section, $R(G)$ denotes the space of central functions on G.

4.1. THE EQUIVARIANT ENTIRE CYCLIC COHOMOLOGY OF $\mathbb{C}$

Let $\mathscr{A}=\mathbb{C}$ and let G be any finite group. Since $\mathrm{Aut}(\mathbb{C})=\{1\}$, any group action on $\mathbb{C}$ is trivial. For $f_n\in\mathscr{C}^n_G(\mathscr{A})$ we set

$$\lambda_n(g):=f_n(1,1,\ldots,1)(g). \tag{4.1}$$

By linearity, f_n is completely specified by λ_n. Equivariance of f_n means that λ_n is a central function on G. Simple computations show that:

(α) Any sequence $\{\lambda_{2k}\}_{k=0}^{\infty}$ of central functions on G satisfying

$$\lim_{n\to\infty} n^{1/2}\{\max_G |\lambda_n(g)|\}^{1/n} = 0, \tag{4.2}$$

determines a cocycle. A sequence $\{\lambda_{2k}\}_{k=0}^{\infty}$ determines a coboundary only if

$$\lambda_{2k}(g) = \mu_{2k-1}(g) + 2(2k+1)\mu_{2k+1}(g), \tag{4.3}$$

for some sequence $\{\mu_{2k+1}\}_{k=0}^{\infty}$ of central functions obeying (4.2).

(β) A sequence $\{\lambda_{2k+1}\}_{k=0}^{\infty}$ of central functions on G satisfying (4.2) determines a cocycle if

$$\lambda_{2k-1}(g) + 2(2k+1)\lambda_{2k+1}(g) = 0. \tag{4.4}$$

Such a sequence determines a coboundary only if $\lambda_{2k+1} = 0$, $k = 0, 1, \ldots$.

Solving (4.4) we find that

$$\lambda_{2k+1}(g) = (-1)^k \frac{k!}{(2k+1)!} \lambda_1(g), \tag{4.5}$$

i.e.

$$\lim_{k\to\infty} (2k+1)^{1/2} \{\max_G |\lambda_{2k+1}(g)|\}^{1/2k+1} = (e/2)^{1/2}. \tag{4.6}$$

Therefore, we are forced to take $\lambda_1 = 0$ and thus $\mathscr{H}_G^o(\mathbb{C}) = 0$.

Let us now determine $\mathscr{H}_G^e(\mathbb{C})$. As a consequence of (4.2) the functions

$$\lambda_g(z) := \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} \lambda_{2k}(g) z^k \tag{4.7}$$

and

$$\mu_g(z) := \sum_{k=0}^{\infty} (-1)^k \frac{(2k+2)!}{(k+1)!} \mu_{2k+1}(g) z^k \tag{4.8}$$

are entire. Equation (4.3) is equivalent to

$$\lambda_g(z) = (z-1)\mu_g(z), \tag{4.9}$$

which has a solution if and only if $\lambda_g(1) = 0$. Consequently, $\mathscr{H}_G^e(\mathbb{G}) \cong R(G)$, with the isomorphism

$$\mathscr{H}_G^e(\mathbb{C}) \ni \{f_{2n}\} \to \lambda_g(1) \in R(G). \tag{4.10}$$

We have thus proved the following theorem.

THEOREM. *Let G be an arbitrary finite group. Then $\mathscr{H}_G^o(\mathbb{C}) = 0$ and $\mathscr{H}_G^e(\mathbb{C}) = R(G)$.*

4.2. $G = \{0\}$. If G is the trivial group, then $\mathscr{H}_G^*(\mathscr{A})$ reduces to the entire cyclic cohomology of a trivially graded Banach algebra [3] which we denote here by $\mathscr{H}^*(\mathscr{A})$.

4.3. $G = \mathbb{Z}_2$. For $G = \mathbb{Z}_2$, the equivariant entire cyclic cohomology defined above is related to the cohomology group defined in [6] as follows. If $\mathbb{Z}_2 = \{0, 1\}$, then

$$\rho_0(a) = a, \qquad \rho_1(a) = a^{\Gamma}, \tag{4.11}$$

and

$$\mathscr{H}^*_{\mathbb{Z}_2}(\mathscr{A}) \cong \mathscr{H}^*(\mathscr{A}) \oplus H^*_+(\mathscr{A}), \tag{4.12}$$

where $\mathscr{H}^*(\mathscr{A})$ is the ordinary entire cyclic cohomology of $\mathscr{A}$ [3] (see also Appendix K of [7]), and where $H^*_+(\mathscr{A})$ is the cohomology group defined in [6].

4.4. TRIVIAL GROUP ACTION

If G acts trivially on $\mathscr{A}$, i.e., $\rho_g(a) = a$, for all $g \in G$ and $a \in \mathscr{A}$, then $\mathscr{H}^*_G(\mathscr{A})$ can be expressed in terms of $\mathscr{H}^*(\mathscr{A})$.

THEOREM. *Let G act trivially on $\mathscr{A}$. Then*

$$\mathscr{H}^*_G(\mathscr{A}) \cong \mathscr{H}^*(G) \otimes R(G). \tag{4.13}$$

Proof. Since $\rho_h = 1$, also $\rho_h^* = 1$, and as a consequence of (4.4),

$$G \ni g \to f_n(a_0, a_1, \dots, a_n)(g) \tag{4.14}$$

is a central function on G. Let $\hat{G}$ denote the set of equivalence classes of irreducible representations of G. for $\sigma \in \hat{G}$, let χ_σ denote the character of σ. Let $\sigma \in \hat{G}$ and let $f = \{f_n\} \in \mathscr{C}^*(\mathscr{A})$ be an entire cochain on (the trivially graded Banach algebra) $\mathscr{A}$. Then

$$\mathscr{C}^*(\mathscr{A}) \otimes R(G) \ni f \otimes \chi_\sigma \to \{f_n \otimes \chi_\sigma\} \in \mathscr{C}^*_G(\mathscr{A}) \tag{4.15}$$

extends to an isomorphism of $\mathscr{C}^*(\mathscr{A}) \otimes R(G)$ and $\mathscr{C}^*_G(\mathscr{A})$. This isomorphism is easily seen to be a cochain homomorphism which implies (4.13). □

4.5. CONNECTION WITH $L^1_\rho(G, \mathscr{A})$

Let now $(\mathscr{A}, G, \rho)$ be a B-dynamical system. Since $L^1_\rho(G, \mathscr{A})$ is a trivially graded unital Banach algebra, we can consider its entire cyclic cohomology $\mathscr{H}^*(L^1_\rho(G, \mathscr{A}))$. Let $\varphi \in R(G)$. We define a mapping $\Phi_\varphi : \mathscr{C}^*_G(\mathscr{A}) \to \mathscr{C}^*(L^1_\rho(G, \mathscr{A}))$ by

$$\begin{aligned}(\Phi_{\varphi n} f_n)&(x_0, x_1, \dots, x_n) \\ &:= \int f_n(x_0(h_0), \rho_{h_0}(x_1(h_1)), \dots, \rho_{h_0 h_1 \cdots h_{n-1}}(x_n(h_n)))(h_0 h_1 \cdots h_n) \times \\ &\quad \times \varphi(h_0 h_1 \cdots h_n)\, \mathrm{d}^{n+1} h.\end{aligned} \tag{4.16}$$

LEMMA. *The above homomorphism has the following properties*:

$$(\alpha) \quad \Phi T = T\Phi, \tag{4.17}$$

$$(\beta) \quad \Phi U = U\Phi, \tag{4.18}$$

$$(\gamma) \quad \Phi V = V\Phi. \tag{4.19}$$

Proof. (α) We have

$$(\Phi_{\varphi n} T_n f_n)(x_0, x_1, \ldots, x_n)$$

$$= \int (T_n f_n)(x_0(h_0), \rho_{h_0}(x_1(h_1)), \ldots, \rho_{h_0 h_1 \cdots h_{n-1}}(x_n(h_n))(h_0 h_1 \cdots h_n) \times$$

$$\times \varphi(h_0 h_1 \cdots h_n)\, \mathrm{d}^{n+1} h$$

$$= (-1)^n \int f_n(\rho_{h_n^{-1}}(x_n(h_n)), x_0(h_0, \ldots, \rho_{h_0 h_1 \cdots h_{n-2}}(x_{n-1}(h_{n-1}))) \times$$

$$\times (h_0 h_1 \cdots h_n) \varphi(h_0 h_1 \cdots h_n)\, \mathrm{d}^{n+1} h.$$

Using the fact that f_n is equivariant and that φ is central we can write this as

$$(-1)^n \int f_n(x_n(h_n), \rho_{h_n}(x_0(h_0)), \ldots \rho_{h_n h_0 \cdots h_{n-2}}(x_{n-1}(h_{n-1}))) \times$$

$$\times (h_n h_0 \cdots h_{n-1}) \varphi(h_n h_0 \cdots h_{n-1})\, \mathrm{d}^{n+1} h$$

$$= (T_n \Phi_{\varphi n} f_n)(x_0, x_1, \ldots, x_n).$$

which is our claim.

(β) We have

$$(\Phi_{\varphi n} U_n f_n)(x_0, x_1, \ldots, x_{n-1})$$

$$= \int (U_n f_n)(x_0(h_0), \rho_{h_0}(x_1(h_1)), \ldots, \rho_{h_0 h_1 \cdots h_{n-2}}(x_{n-1}(h_{n-1}))) \times$$

$$\times (h_0 h_1, \ldots h_{n-1}) \varphi(h_0 h_1 \cdots h_{n-1})\, \mathrm{d}^n h$$

$$= (-1)^{n-1} \int f_n(x_0(h_0), \ldots, \rho_{h_0 h_1 \cdots h_{n-2}}(x_{n-1}(h_{n-1})), \mathbf{1})(h_0 h_1 \cdots h_{n-1}) \times$$

$$\times \varphi(h_0 h_1 \cdots h_{n-1})\, \mathrm{d}^n h$$

$$= (-1)^{n-1} \int f_n(x_0(h_0), \ldots, \rho_{h_0 h_1 \cdots h_{n-2}}(x_{n-1}(h_{n-1})), \rho_{h_0 h_1 \ldots h_{n-1}}(\delta(h_n))) \times$$

$$\times (h_0 h_1 \cdots h_n) \varphi(h_0 h_1 \cdots h_n)\, \mathrm{d}^{n+1} h$$

$$= (U_n \Phi_{\varphi n} f_n)(x_0, x_1, \ldots, x_{n-1}),$$

as claimed.

(γ) Using equivariance of f_n and cyclicity of φ we have

$$
\begin{aligned}
&(\Phi_{\varphi n} V_n f_n)(x_0, x_1, \dots, x_{n+1})\\
&\quad = \int (V_n f_n)(x_0(h_0), \rho_{h_0}(x_1(h_1)), \dots, \rho_{h_0 h_1 \cdots h_n}(x_{n+1})))(h_0 h_1 \cdots h_{n+1}) \times\\
&\qquad \times \varphi(h_0 h_1 \cdots h_{n+1})\, \mathrm{d}^{n+2}h\\
&\quad = (-1)^{n+1} \int f_n(\rho_{h_{n+1}^{-1}}(x_{n+1}(h_{n+1}))x_0(h_0), \rho_{h_0}(x_1(h_1)), \dots, \rho_{h_0 \cdots h_{n-1}} \times\\
&\qquad \times (x_n(h_n)))(h_0 h_1 \cdots h_{n+1})\varphi(h_0 h_1 \cdots h_{n+1})\, \mathrm{d}^{n+2}h\\
&\quad = (-1)^{n+1} \int f_n(x_{n+1}(h_{n+1})\rho_{h_{n+1}}(x_0(h_0)), \rho_{h_{n+1}h_0}(x_1(h_1)), \dots,\\
&\qquad \dots, \rho_{h_{n+1}h_0 \cdots h_{n-1}}(x_n(h_n)))(h_{n+1}h_0 \cdots h_n)\varphi(h_{n+1}h_0 \cdots h_n)\, \mathrm{d}^{n+2}h.
\end{aligned}
$$

Substituting $h_0 \to h_{n+1}^{-1} h_0$ we find that this is equal to

$$
\begin{aligned}
&(-1)^{n+1} \int f_n(x_{n+1} *_\rho x_0(h_0), \rho_{h_0}(x_1(h_1)), \dots, \rho_{h_0 h_1 \cdots h_{n-1}}(x_n(h_n))) \times\\
&\quad \times (h_0 h_1 \cdots h_n)\varphi(h_0 h_1 \cdots h_n)\, \mathrm{d}^{n+1}h\\
&= (V_n \Phi_{\varphi n} f_n)(x_0, x_1, \dots, x_{n+1}),
\end{aligned}
$$

as claimed. □

THEOREM. *For each* $\varphi \in R(G)$, $\Phi_\varphi : \mathscr{C}_G^*(\mathscr{A}) \to \mathscr{C}^*(L_\rho^1(G, \mathscr{A}))$ *is a continuous chain homomorphism which maps normalized cocycles into normalized cocycles. Consequently,* Φ_φ *defines a homomorphism*

$$\Phi_\varphi : \mathscr{H}_G^*(\mathscr{A}) \to \mathscr{H}^*(L_\rho^1(G, \mathscr{A})). \tag{4.20}$$

Proof. From (4.16), (2.3) and (2.7),

$$|\Phi_{\varphi n} f_n)(x_0, x_1, \dots, x_n)| \leqslant \|f_n\| \max_{g \in G} |\varphi(g)| \prod_{j=0}^{n} \|x_j\|_1,$$

which implies that $\Phi_\varphi : \mathscr{C}_G^*(\mathscr{A}) \to \mathscr{C}^*(L_\rho^1(G, \mathscr{A}))$ is continuous. As a consequence of the lemma,

$$\partial \Phi_\varphi = \Phi_\varphi \partial, \tag{4.21}$$

i.e. Φ_φ is a chain homomorphism. This means that Φ_φ projects to a homomorphism of $\mathscr{H}_G^*(\mathscr{A})$ into $\mathscr{H}^*(L_\rho^1(G, \mathscr{A}))$. □

Remark. We conjecture that the induced homomorphism $\Phi : \mathscr{H}_G^*(\mathscr{A}) \otimes R(G) \to \mathscr{H}^*(L_\rho^1(G, \mathscr{A}))$ is in fact an isomorphism.

4.6. TENSORING $\mathscr{A}$ WITH $\mathrm{End}(\mathscr{V})$

Let $(\mathscr{V}, U)$ be a finite-dimensional representation of G. We will now construct a chain homomorphism

$$L : \mathscr{C}^*_G(\mathscr{A}) \to \mathscr{C}^*_G(\mathscr{A}) \otimes \mathrm{End}(\mathscr{V})). \tag{4.22}$$

For $f_n \in \mathscr{C}^n_G(\mathscr{A})$ we set

$$\begin{aligned}(L_n f_n)(a_0 \otimes m_0, \ldots, a_n \otimes m_n)(g) \\ := f_n(a_0, \ldots, a_n)(g) \operatorname{tr}(m_0 \cdots m_1 U(g)).\end{aligned} \tag{4.23}$$

Clearly, L_n maps $\mathscr{C}^n_G(\mathscr{A})$ into $\mathscr{C}^n_G(\mathscr{A} \otimes \mathrm{End}(\mathscr{V}))$, as

$$\begin{aligned}&(L_n f_n)(\rho_h(a_0) \otimes U_h(m_0), \ldots, \rho_h(a_n) \otimes U_h(m_n))(g) \\ &\quad = f_n(\rho_h(a_0), \ldots, \rho_h(a_n))(g) \operatorname{tr}(U_h m_0 m_1 \cdots m_n U_{h^{-1}} U_g) \\ &\quad = f_n(a_0, \ldots, a_n)(h^{-1}gh) \operatorname{tr}(m_0 \cdots m_n U_{h^{-1}gh}) \\ &\quad = (L_n f_n)(a_0 \otimes m_0, \ldots, a_n \otimes m_n)(h^{-1}gh).\end{aligned}$$

It is also clear that $L : \mathscr{C}_G(\mathscr{A}) \to \mathscr{C}_G(\mathscr{A} \otimes \mathrm{End}(\mathscr{V}))$ is continuous.

LEMMA. *The above homomorphism satisfies*

$$(\alpha) \quad LT = TL, \tag{4.24}$$

$$(\beta) \quad LU = UL, \tag{4.25}$$

$$(\gamma) \quad LV = VL. \tag{4.26}$$

Proof. We compute

$$\begin{aligned}(\alpha) \quad &(L_n T_n f_n)(a_0 \otimes m_0, \ldots, a_n \otimes m_n)(g) \\ &= (-1)^n f_n(\rho_{g^{-1}}(a_n), a_0, \ldots, a_{n-1})(g) \operatorname{tr}(m_0 m_1 \cdots m_n U_g) \\ &= (-1)^n f_n(\rho_{g^{-1}}(a_n), a_0, \ldots, a_{n-1})(g) \operatorname{tr}(U_{g^{-1}} m_n U_g m_0 m_1 \cdots m_{n-1} U_g) \\ &= (-1)^n (L_n f_n)(\rho_{g^{-1}}(a_n) \otimes U_{g^{-1}}(m_n), a_0 \otimes m_0, \ldots, a_{n-1} \otimes m_{n-1})(g) \\ &= (T_n L_n f_n)(a_0 \otimes m_0, \ldots, a_n \otimes m_n)(g);\end{aligned}$$

$$\begin{aligned}(\beta) \quad &(L_n U_n f_n)(a_0 \otimes m_0, \ldots, a_{n-1} \otimes m_{n-1})(g) \\ &= (-1)^{n-1} f_n(a_0, \ldots, a_{n-1}, \mathbf{1})(g) \operatorname{tr}(m_0 \cdots m_{n-1} U_g) \\ &= (-1)^{n-1} (L_n f_n)(a_0 \otimes m_0, \ldots, a_{n-1} \otimes m_{m-1}, \mathbf{1} \otimes \mathbf{1})(g) \\ &= (U_n L_n f_n)(a_0 \otimes m_0, \ldots, a_{n-1} \otimes m_{n-1})(g);\end{aligned}$$

$$\begin{aligned}(\gamma) \quad &(L_n V_n f_n)(a_0 \otimes m_0, \ldots, a_{n+1} \otimes m_{n+1})(g) \\ &= (-1)^{n+1} f_n(\rho_{g^{-1}}(a_{n+1}) a_0, a_1, \ldots, a_n)(g) \operatorname{tr}(m_0 \cdots m_{n+1} U_g) \\ &= (-1)^{n+1} f_n(\rho_{g^{-1}}(a_{n+1}) a_0, a_1, \ldots, a_n)(g) \operatorname{tr}(U_{g^{-1}} m_{n+1} U_g m_0 \cdots m_n U_g) \\ &= (-1)^{n+1} (L_n f_n)(\rho_{g^{-1}}(a_{n+1}) a_0 \otimes U_{g^{-1}}(m_{n+1}) m_0, \\ &\quad a_1 \otimes m_1, \ldots, a_n \otimes m_n)(g) \\ &= (V_n L_n f_n)(a_0 \otimes m_0, \ldots, a_{n+1} \otimes m_{n+1})(g).\end{aligned}$$

□

An immediate consequence of this lemma is the following theorem.

THEOREM. *The mapping* $L:\mathscr{C}_G^*(\mathscr{A})\to\mathscr{C}_G^*(\mathscr{A}\otimes\mathrm{End}(\mathscr{V}))$ *is a continuous chain homomorphism of equivariant complexes which maps normalized cocycles into normalized cocyles. Consequently, L defines a homomorphism*

$$L:\mathscr{H}_G^*(\mathscr{A})\to\mathscr{H}_G^*(\mathscr{A}\otimes\mathrm{End}(\mathscr{V})). \tag{4.27}$$

5. Pairing with $K_0^G(\mathscr{A})$

Let $K_0^G(\mathscr{A})$ be the equivariant Grothendieck group of the B-dynamical system $(\mathscr{A}, G, \rho)$. Recall that to define $K_0^G(\mathscr{A})$ one considers algebras of the form $\mathscr{A}\otimes\mathrm{End}(\mathscr{V})$, where $(\mathscr{V}, U)$ is a finite-dimensional representation of G. We define $K_0^G(\mathscr{A})$ to be the set of suitable equivalence classes of G-invariant idempotents $e\in\mathscr{A}\otimes\mathrm{End}(\mathscr{V})$ (i.e., $(\rho\otimes U)_g(e)=e$), see [12] for details. Our aim is to construct a pairing

$$K_0^G(\mathscr{A})\times\mathscr{H}_G^e(\mathscr{A})\to R(G), \tag{5.1}$$

generalizing the pairing constructed by Connes in [3].

5.1. DIRECT CONSTRUCTION

Let $L:\mathscr{C}_G^*(\mathscr{A})\to\mathscr{C}_G^*(\mathscr{A}\otimes\mathrm{End}(\mathscr{V}))$ be the chain homomorphism constructed in Section 4.6. For $e\in\mathscr{A}\otimes\mathrm{End}(\mathscr{V})$, a G-invariant idempotent, and a normalized cocycle $f\in\mathscr{C}_G^e(\mathscr{A})$ we set

$$\langle e,f\rangle(g):=\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{k!}(L_{2k}f_{2k})(e,e,\ldots,e)(g). \tag{5.2}$$

As a consequence of (3.34), this series converges absolutely. Furthermore, since e is G-invariant,

$$\langle e,f\rangle(h^{-1}gh)=\langle e,f\rangle(g), \tag{5.3}$$

i.e.

$$\langle e,f\rangle\in R(G). \tag{5.4}$$

THEOREM. *Formula* (5.2) *defines a map*

$$(\cdot,\cdot):K_0^G(\mathscr{A})\times\mathscr{H}_G^e(\mathscr{A})\to R(G) \tag{5.5}$$

such that $e\to\langle e,f\rangle$ *is additive and* $f\to\langle e,f\rangle$ *is linear.*

To prove this theorem one has to show that $\langle e,f\rangle(g)$ is independent of the choice of e representing a class in $K_0^G(\mathscr{A})$ and of the choice a normalized cocycle f representing a cohomology class in $\mathscr{H}_G^e(\mathscr{A})$. [By Section 3.7 each cohomology class has a normalized representative.] The proof is a repetition of the proofs of Lemma 7 and Theorem 8 in [3] and we do not reproduce it here.

Observe that if $G = \{0\}$, then (5.2) reduces to the pairing defined by Connes in [3].

5.2. CONSTRUCTION USING JULG'S THEOREM

The celebrated theorem of Julg's (see, e.g., [12, 1]) states that there is an isomorphism

$$j : K_0^G(\mathscr{A}) \cong K_0(L^1_\rho(G, \mathscr{A})). \tag{5.6}$$

The isomorphism is defined as follows. If e is a G-invariant idempotent in $\mathscr{A} \otimes \mathrm{End}(\mathscr{V})$, then $j(e) \in L^1_{\rho\otimes U}(G, \mathscr{A} \otimes \mathrm{End}(\mathscr{V})) \cong L^1_\rho(G, \mathscr{A}) \otimes \mathrm{End}(\mathscr{V})$ defined by

$$j(e)(g) := e, \quad g \in G, \tag{5.7}$$

is an idempotent. [In fact $j(e) *_{\rho\otimes U} j(e) = \int e(\rho \otimes u)_h(e)\, \mathrm{d}h = \int e^2\, \mathrm{dh} = e = j(e)$.] Let $\langle \cdot, \cdot \rangle^*$ be the pairing

$$\langle \cdot, \cdot \rangle : K_0(L^1_\rho(G, \mathscr{A})) \times \mathscr{H}^e(L^1_\rho(G, \mathscr{A})) \to \mathbb{C}. \tag{5.8}$$

THEOREM. *With the above definitions*

$$\langle j(e), \Phi_\varphi f \rangle^* = \int_G \langle e, f \rangle(g)\varphi(g)\, \mathrm{d}g, \tag{5.9}$$

for $f \in \mathscr{C}^e_G(\mathscr{A})$ *and* $\varphi \in R(G)$.

Proof. By functoriality (Section 4.6) we can replace $\mathscr{A} \otimes \mathrm{End}(\mathscr{V})$ by $\mathscr{A}$. Then

$$\begin{aligned}
\langle j(e), \Phi_\varphi f \rangle^* &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} (\Phi_\varphi f_{2k})(j(e), j(e), \ldots, j(e)) \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} \int f_{2k}(e, e, \ldots, e) \times \\
&\quad \times (h_0 h_1 \cdots h_n)\varphi(h_0 h_1 \cdots h_n)\, \mathrm{d}^{n+1}h \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} \int f_{2k}(e, e, \ldots, e)(g)\varphi(g)\, \mathrm{d}g \\
&= \int_G \langle e, f \rangle(g)\varphi(g)\, \mathrm{d}g. \qquad \square
\end{aligned}$$

Acknowledgement

W. K. would like to thank Arthur Jaffe for his hospitality at Harvard.

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