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A Golden–Thompson Inequality in Supersymmetric Quantum Mechanics*

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Abstract. We establish a supersymmetric version of the Golden–Thompson inequality

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1. Introduction

The celebrated Golden–Thompson inequality [8, 3, 9] states that the quantum partition function of a system does not exceed its classical partition function. A generalization of this inequality to systems in external magnetic fields was established in [1]. Apart from its usefulness in studying the classical limit of a quantum system, the Golden–Thompson bound provides a useful criterion for the discreteness of energy levels of the system. In particular, it allows us to prove, in certain cases, that the Hamiltonian $H = -\frac{1}{2}\Delta + V(x)$ has a compact resolvent, even though the potential $V(x)$ does not increase as $|x| \to \infty$ along certain directions.

In this Letter, we study a supersymmetric version of this inequality. The Hamiltonian in supersymmetric quantum mechanics has the form

$$H = -\tfrac{1}{2}\Delta + \tfrac{1}{2}\sum \nabla^2_{jk} V(x)[b_j^*, b_k] + \tfrac{1}{2}|\nabla V(x)|^2,$$

where V is the superpotential. The main result of this Letter is the following inequality

$$\operatorname{tr}(e^{-tH}) \leqslant (2\pi t)^{-n/2} \int \det(I + e^{-t\nabla^2 V(x)})\, e^{-(1/2)t[|\nabla V(x)|^2 - \Delta V(x)]}\, dx$$

(see Section III for precise formulations). This inequality shows that H may have a compact resolvent even when V has flat directions. This improves on the result of Section IV of [4]. Our proof of the inequality follows essentially [8] and [7], with some technical twists due to the presence of fermions.

The Letter is organized as follows. Section 2 contains certain introductory results of technical character. In Section 3, we state and prove the inequality.

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2. Technical Preliminaries

In this section, we prove two results (Proposition 2.2 and Proposition 2.3) of a technical character which will be used in Section 3 to prove our main result. The first of the two propositions establishes the convexity of a certain matrix function. The second proposition provides a criterion of trace classness of the heat kernel of a positive operator on a Hilbert space.

In the lemma below, $\mathscr{L}_h(V)$ denotes the space of Hermitian operators on a finite-dimensional complex vector space V.

LEMMA 2.1. *The function* $f\colon \mathscr{L}_h(\mathbb{C}^M) \to \mathbb{R}$ *defined by*

$$f(A) := \log \det(I + \mathrm{e}^A), \tag{2.1}$$

is convex.

Proof. We have the well known expansion

$$\begin{aligned}\det(I + \mathrm{e}^A) &= I + \sum_{m \geqslant 1} \frac{1}{m!} \operatorname{tr}_{\bigwedge^m(\mathbb{C}^M)}\left(\bigwedge^m \mathrm{e}^A\right)\\ &= I + \sum_{m \geqslant 1} \frac{1}{m!} \operatorname{tr}_{\bigwedge^m(\mathbb{C}^M)}(\mathrm{e}^{A^{(m)}}),\end{aligned} \tag{2.2}$$

where $\bigwedge^m(V)$ is the mth exterior power of V, and where

$$A^{(m)} := A \wedge I \wedge \cdots \wedge I + I \wedge A \wedge I \wedge \cdots \wedge I + \cdots + I \wedge \cdots \wedge I \wedge A. \tag{2.3}$$

Using the inequality (see [5], p. 27)

$$\operatorname{tr}(\mathrm{e}^{\alpha S + (1-\alpha)T}) \leqslant \{\operatorname{tr}(\mathrm{e}^S)\}^\alpha \{\operatorname{tr}(\mathrm{e}^T)\}^{1-\alpha}, \tag{2.4}$$

valid for $S, T \in \mathscr{L}_h(V)$, and $0 \leqslant \alpha \leqslant 1$, we obtain from (2.2)

$$\begin{aligned}&\det(I + \mathrm{e}^{\alpha A + (1-\alpha)B})\\ &\quad\leqslant 1 + \sum_{m \geqslant 1} \left\{\frac{1}{m!} \operatorname{tr}_{\bigwedge^m(\mathbb{C}^M)}(\mathrm{e}^{A^{(m)}})\right\}^\alpha \left\{\frac{1}{m!} \operatorname{tr}_{\bigwedge^m(\mathbb{C}^M)}(\mathrm{e}^{B^{(m)}})\right\}^{1-\alpha}.\end{aligned} \tag{2.5}$$

Applying Hölder's inequality with $p = 1/\alpha$, $q = 1/(1-\alpha)$ $(0 < \alpha < 1)$ to the right-hand side of (2.5) yields

$$\begin{aligned}&\det(I + \mathrm{e}^{\alpha A + (1-\alpha)B})\\ &\quad\leqslant \left\{1 + \sum_{m \geqslant 1} \frac{1}{m!} \operatorname{tr}_{\bigwedge^m(\mathbb{C}^M)}(\mathrm{e}^{A^{(m)}})\right\}^\alpha \left\{1 + \sum_{m \geqslant 1} \frac{1}{m!} \operatorname{tr}_{\bigwedge^m(\mathbb{C}^M)}(\mathrm{e}^{B^{(m)}})\right\}^{1-\alpha}\\ &\quad= \{\det(I + \mathrm{e}^A)\}^\alpha \{\det(I + \mathrm{e}^B)\}^{1-\alpha},\end{aligned}$$

i.e., $f(\alpha A + (1 - \alpha B) \leqslant \alpha f(A) + (1-\alpha) f(B)$, as claimed. □

PROPOSITION 2.2. *The function* $g: \mathscr{L}_h(\mathbb{C}^n) \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$g(A, x) := \det(I + e^A)\, e^x, \tag{2.6}$$

is convex.

Proof. Since $\mathbb{R} \ni y \to e^y$ is a monotonically increasing convex function, and the composition of a convex function with a monotonically increasing convex function is convex, it suffices to show that

$$\log g(A, x) = \log \det(I + e^A) + x, \tag{2.7}$$

is convex. But this is a consequence of Lemma 2.1. □

In the proposition below, $\mathscr{H}$ denotes a complex Hilbert space. If H is a symmetric operator, then $D(H)$ denotes its domain and $Q(H)$ denotes the domain of the corresponding bilinear form. If H is positive and self-adjoint, then $Q(H)$ is closed in the topology defined by the norm $\|x\|_* := \|x\| + (x, Hx)^{1/2}$.

PROPOSITION 2.3. *Let H and U be two positive self-adjoint operators on $\mathscr{H}$, and let H_ε, $\varepsilon > 0$, be the form sum*

$$H_\varepsilon := H + \varepsilon U. \tag{2.8}$$

Assume that

(i) *$Q(H) \cap Q(U)$ is dense in $Q(H)$ in the topology of $Q(H)$,*
(ii) *for some $t > 0$,*

$$\operatorname{tr}(e^{-tH_\varepsilon}) \leqslant C, \tag{2.9}$$

where C is a constant independent of ε.

Then the heat kernel of H is trace class and

$$\lim_{\varepsilon \searrow 0} \operatorname{tr}(e^{-tH_\varepsilon}) = \operatorname{tr}(e^{-tH}). \tag{2.10}$$

Proof. Let $\{\mu_n(H_\varepsilon)\}_{n=1}^{\infty}$ be the sequence of numbers given by the minimax principle ([6], Chapter XIII.1). Since by (ii) H_ε has a compact resolvent, $\mu_n(H_\varepsilon)$ are the eigenvalues of H_ε. Furthermore, as a consequence of (i) and Theorem 10.2 in [2], we have

$$\mu_n(H_\varepsilon) \searrow \mu_n(H), \tag{2.11}$$

as $\varepsilon \searrow 0$. Therefore, by (2.9),

$$C \geqslant \lim_{\varepsilon \searrow 0} \operatorname{tr}(e^{-tH_\varepsilon}) = \lim_{\varepsilon \searrow 0} \sum_{n \geqslant 1} e^{-t\mu_n(H_\varepsilon)}$$

$$= \sum_{n \geqslant 1} e^{-t\mu_n(H)}.$$

Consequently, $\mu_n(H) \to \infty$, as $n \to \infty$, and so by Theorem XIII.64 of [6], H has a compact resolvent, and its spectrum consists of the numbers $\mu_n(H)$, $n \geqslant 1$. But then (2.10) follows from (2.12). □

3. The Inequality

In this section we formulate and prove our supersymmetric Golden–Thompson inequality. We begin with a brief review of supersymmetric quantum mechanics.

The Hilbert space of supersymmetric quantum mechanics is $\mathscr{H} := L^2(\mathbb{R}^n) \otimes \bigwedge(\mathbb{C}^n)$, where $\bigwedge(\mathbb{C}^n)$ denotes the Grassmann algebra over $\mathbb{C}^n$. We choose an orthonormal basis for $\mathbb{C}^n$, and introduce the fermionic creation and annihilation operators b_j^*, b_j, $1 \leqslant j \leqslant n$, in the usual fashion. Let $V: \mathbb{R}^n \to \mathbb{R}$ be a polynomial of algebraic degree $N \geqslant 2$; V is called a superpotential (we will be concerned with polynomial superpotentials only). The supercharge Q is a symmetric operator defined on $C_0^\infty(\mathbb{R}^n) \otimes \bigwedge(\mathbb{C}^n)$ by

$$Q := \frac{i}{\sqrt{2}} \sum_j (b_j^* \nabla_j + b_j \nabla_j) + \frac{i}{\sqrt{2}} \sum_j (b_j^* \nabla_j V(x) - b_j \nabla_j V(x)). \tag{3.1}$$

Its square is the Hamiltonian,

$$H := Q^2 = -\tfrac{1}{2}\Delta + \tfrac{1}{2} \sum_{j,k} \nabla_{jk}^2 V(x)[b_j^*, b_k] + \tfrac{1}{2} \sum_j |\nabla_j V(x)|^2. \tag{3.2}$$

Clearly, $H \geqslant 0$ on $C_0^\infty(\mathbb{R}^n) \otimes \bigwedge(\mathbb{C}^n)$. The Friedrichs extension of H defines a positive self-adjoint operator which we will also denote by H.

The main result of this paper is the following theorem.

THEOREM 3.1. *Assume that*

$$\int \det(I + \exp\{-t\nabla^2 V(x)\}) \exp\{-\tfrac{1}{2}t(|\nabla V(x)|^2 - \Delta V)\} \, \mathrm{d}^n x < \infty,$$

for some $t > 0$. *Then*, $\exp\{-tH\}$ *is trace class, and*

$$\operatorname{tr}(\mathrm{e}^{-tH}) \leqslant (2\pi t)^{-n/2} \int \det(I + \mathrm{e}^{-t\nabla^2 V(x)}) \, \mathrm{e}^{-1/2t(|\nabla V(x)|^2 - \Delta V)} \, \mathrm{d}^n x. \tag{3.3}$$

Proof. For $\varepsilon > 0$, we set $H_\varepsilon = H + \varepsilon U$, where $U := \tfrac{1}{2}|x|^{2N}$. This defines an essentially self-adjoint operator on $D(H) \cap D(U)$. Indeed, $C_0^\infty(\mathbb{R}^n) \otimes \bigwedge(\mathbb{C}^n) \subset D(H) \cap D(U)$ and the operator

$$T_\varepsilon := -\tfrac{1}{2}\Delta + \tfrac{1}{2}(|\nabla V(x)|^2 - \Delta V) + \tfrac{1}{2}\varepsilon|x|^{2N}, \tag{3.4}$$

is essentially self-adjoint of $C_0^\infty(\mathbb{R}^n) \otimes \bigwedge(\mathbb{C}^n)$. Since

$$S := \sum_{j,k} \nabla_{jk}^2 V(x) b_j^* b_k \tag{3.5}$$

is an infinitesimal perturbation (in the sense of Kato) of T_ε, it follows that H_ε is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n) \otimes \bigwedge(\mathbb{C}^n)$. We let H_ε denote its closure. A simple perturbation theory argument shows also that $\exp\{-tH_\varepsilon\}$ is trace class for $t > 0$.

We now claim that the following Duhamel expansion converges in the first Schatten norm $\|\cdot\|_1$:

$$\begin{aligned} e^{-tH_\varepsilon} &= e^{-t(T_\varepsilon + S)} \\ &= \sum_{m \geq 0} (-t)^m \int_{\sigma_m^1} e^{-s_1 T_\varepsilon} S\, e^{-(s_2 - s_1)T_\varepsilon} \cdots e^{-(s_m - s_{m-1})T_\varepsilon} S\, e^{-(1 - s_m)T_\varepsilon}\, d^m s, \end{aligned} \tag{3.6}$$

where

$$\sigma_m^t := \{s \in \mathbb{R}^n : 0 \leq s_1 \cdots \leq s_m \leq t\}.$$

Indeed, using Hölder's inequality on first Schatten class,

$$\begin{aligned} &\left\| \int_{\sigma_m^1} e^{-s_1 T_\varepsilon} S \cdots e^{-(s_m - s_{m-1})T_\varepsilon} S\, e^{-(1 - s_m)T_\varepsilon}\, d^m s \right\|_1 \\ &\quad \leq \int_{\sigma_m^1} \prod_{1 \leq j \leq m} \| e^{-1/2(s_j - s_{j-1})T_\varepsilon} S\, e^{-1/2(s_{j+1} - s_j)T_\varepsilon} \|_{1/(s_{j+1} - s_j)}\, d^m s, \end{aligned}$$

where $s_0 = 0$, $s_{m+1} = 1$. But since

$$\pm S \leq C(T_\varepsilon + I)^{1/2}, \tag{3.7}$$

it follows that for $0 < \sigma, \tau < 1$,

$$\| e^{-\sigma T_\varepsilon} S\, e^{-\tau T_\varepsilon} \| \leq C(\sigma\tau)^{-1/4}, \tag{3.8}$$

and, consequently,

$$\begin{aligned} &\prod_{1 \leq j \leq m} \| e^{-1/2(s_j - s_{j-1})T_\varepsilon} S\, e^{-1/2(s_{j+1} - s_j)T_\varepsilon} \|_{1/(s_{j+1} - s_j)} \\ &\quad \leq C^m \operatorname{tr}(e^{-1/4 T_\varepsilon}) \prod_{1 \leq j \leq m} (s_{j+1} - s_j)^{-1/2}. \end{aligned}$$

This implies that

$$\begin{aligned} &\sum_{m \geq 0} t^m \left\| \int_{\sigma_m^1} e^{-s_1 T_\varepsilon} S \cdots e^{-(s_m - s_{m-1})T_\varepsilon} S\, e^{-(1 - s_m)T_\varepsilon}\, d^m s \right\|_1 \\ &\quad \leq \operatorname{tr}(e^{-T_\varepsilon/4}) \sum_{m \geq 0} (Ct)^m \int_{\sigma_m^1} \prod_{1 \leq j \leq m} (s_{j+1} - s_j)^{-1/2}\, d^m s < \infty, \end{aligned}$$

i.e. the series on the right-hand side of (3.6) converges in $\|\cdot\|_1$. The identity (3.6) is a standard consequence of this convergence.

We can now write

$$\operatorname{tr}(e^{-tH_\varepsilon}) = \sum_{m\geqslant 0}(-1)^m \sum_{\{j\}\{k\}} \operatorname{tr}(b_{j_1}^* b_{k_1}\cdots b_{j_m}^* b_{k_m}) \times \\ \times \int_{\sigma_m^t} \operatorname{tr}(e^{-s_1 T_\varepsilon}\nabla^2_{j_1k_1}V(x)\cdots e^{-(s_m-s_{m-1})T_\varepsilon}\nabla^2_{j_mk_m}V(x)\times \\ \times e^{-(t-s_m)T_\varepsilon})\,d^m s. \tag{3.9}$$

By standard arguments (see e.g., [7], Chapter III),

$$\operatorname{tr}(e^{-s_1 T_\varepsilon}\nabla^2_{j_1k_1}V(x)\cdots e^{-(s_m-s_{m-1})T_\varepsilon}\nabla^2_{j_mk_m}V(x)\,e^{-(t-s_m)T_\varepsilon}) \\ = \int d^n x \int \prod_{1\leqslant i\leqslant m}\nabla^2_{j_ik_i}V(\omega(s_i))\exp\left\{-\frac{1}{2}\int_0^t [|\nabla V(\omega(s))|^2 - \Delta V(\omega(s)) + \\ + \varepsilon|\omega(s)|^{2N}]\,ds\right\} d\mu^t_{x,x}(\omega),$$

where $d\mu^t_{x,y}(\omega)$ denotes the conditional Wiener measure. This and the elementary identity

$$\operatorname{tr}_{\wedge(\mathbb{C}^n)}\left(\exp\left\{\sum_{j,k} A_{jk} b_j^* b_k\right\}\right) = \det(I + e^A), \tag{3.10}$$

valid for $A \in \mathscr{L}_h(\mathbb{C}^n)$, yield

$$\operatorname{tr}(e^{-tH_\varepsilon}) = \int d^n x \int \det\left(I + \exp\left\{-\int_0^t \nabla^2 V(\omega(s))\,ds\right\}\right)\times \\ \times \exp\left\{-\frac{1}{2}\int_0^t [|\nabla V(\omega(s))|^2 - \Delta V(\omega(s)) + \varepsilon|\omega(s)|^{2N}]\,ds\right\} d\mu^t_{x,x}(\omega). \tag{3.11}$$

Interchanging the sum and the integral is legitimate, since for arbitrary $C > 0$,

$$\int d^n x \int \exp\left\{-\frac{1}{2}\int_0^t [|\nabla V(\omega(s))|^2 + \varepsilon|\omega(s)|^{2N}]\,ds + \\ + C\sum_{j,k}\int_0^t |\nabla^2_{jk}V(\omega(s))|\,ds\right\} d\mu^t_{x,x}(\omega) < \infty,$$

and Lebesque's dominated convergence theorem can be applied.

Observe now that

$$\exp\left\{-\frac{1}{2}\int_0^t [|\nabla V(\omega(s))|^2 - \Delta V(\omega(s)) + \varepsilon|\omega(s)|^{2N}]\,ds\right\} \\ \nearrow \exp\left\{-\frac{1}{2}\int_0^t [|\nabla V(\omega(s))|^2 - \Delta V(\omega(s))]\,ds\right\},$$

as $\varepsilon \searrow 0$. Therefore, using Lebesque's monotone convergence theorem, we obtain the following identity:

$$\lim_{\varepsilon \searrow 0} \operatorname{tr}(e^{-tH_\varepsilon}) = \int d^n x \int \det\left(I + \exp\left\{-\int_0^t \nabla^2 V(\omega(s))\, ds\right\}\right) \times$$
$$\times \exp\left\{-\frac{1}{2}\int_0^t [|\nabla V(\omega(s))|^2 - \Delta V(\omega(s))]\, ds\right\} d\mu^t_{x,x}(\omega). \quad (3.12)$$

We now use Proposition 2.2 and Jensen's inequality to bound the right-hand side of (3.12) by

$$\frac{1}{t}\int_0^t ds \int d^n x \int \det(I + \exp\{-t\nabla^2 V(\omega(s))\}) \times$$
$$\times \exp\{-\tfrac{1}{2}t[|\nabla V(\omega(s))|^2 - \Delta V(\omega(s))]\}\, d\mu^t_{x,x}(\omega).$$

Writing $\omega(s) = x + \tilde{\omega}(s)$, where $\tilde{\omega}(s)$ is the Wiener process with $\tilde{\omega}(0) = \tilde{\omega}(t) = 0$, and then substituting $x + \tilde{\omega}(s) \to x$, we rewrite the above integral as

$$\int \det(I + \exp\{-t\nabla^2 V(x)\}) \exp\{-\tfrac{1}{2}t[|\nabla V(x)|^2 - \Delta V(x)]\}\, d^n x \int d\mu^t_{0,0}(\tilde{\omega})$$
$$= (2\pi t)^{-n/2} \int \det(I + \exp\{-t\nabla^2 V(x)\}) \exp\{-\tfrac{1}{2}t[|\nabla V(x)|^2 - \Delta V(x)]\}\, d^n x.$$

As a consequence,

$$\lim_{\varepsilon \searrow 0} \operatorname{tr}(e^{-tH_\varepsilon}) \leqslant (2\pi t)^{-n/2} \int \det(I + e^{-t\nabla^2 V(x)})\, e^{-(1/2)t[|\nabla V(x)|^2 - \Delta V(x)]}\, d^n x.$$

Since $C_0^\infty(\mathbb{R}^n) \otimes \bigwedge(\mathbb{C}^n) \subset Q(H) \cap Q(U)$ and is dense in $Q(H)$, we can use Proposition 2.3 to conclude the proof. □

COROLLARY 3.2. *Assume that*

$$\int \exp\left\{-\frac{t}{2}[|\nabla V(x)|^2 - \operatorname{tr}(|\nabla^2 V(x)|)]\right\} d^m x < \infty, \quad (3.13)$$

where $|A| := (A^*A)^{1/2}$ *denotes the absolute value of the matrix* A. *Then,* $\exp\{-tH\}$ *is trace class, and*

$$\operatorname{tr}(e^{-tH}) \leqslant (2/\pi t)^{n/2} \int \exp\left\{-\frac{t}{2}[|\nabla V(x)|^2 - \operatorname{tr}(|\nabla^2 V(x)|)]\right\} d^n x. \quad (3.14)$$

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A \in \mathscr{L}_h(V)$. Then

$$\det(I + e^{-A})\, e^{1/2\operatorname{tr}(A)} = \prod_{1 \leqslant j \leqslant n} (e^{(1/2)\lambda_j} + e^{-(1/2)\lambda_j}) \leqslant 2^n \exp\{\tfrac{1}{2}\operatorname{tr}(|A|)\}. \quad (3.15)$$

The claim is a consequence of this inequality and Theorem 3.1. □

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