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Ground State Structure in Supersymmetric Quantum Mechanics*

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We present a rigorous analysis of the vacuum structure of two models of supersymmetric quantum mechanics. They are the quantum mechanics versions of the two-dimensional $N = 1$ and $N = 2$ Wess–Zumino quantum field models. We find that the $N = 2$ quantum mechanics has degenerate vacua. The space of vacuum states is bosonic, and its dimension is determined by the topological properties of the superpotential.

1. SUPERSYMMETRY

In this paper we address the ground state structure of supersymmetric quantum theory models. We study the simplest class of such models, anharmonic oscillators, in order to illustrate the phenomena which are qualitatively different from the usual ones.

Supersymmetric quantum mechanics is defined here by a quadruple $(\mathscr{H}, H, \gamma, Q)$. The pair $(\mathscr{H}, H)$ defines a quantum mechanics with a self-adjoint Hamiltonian H acting on a Hilbert space $\mathscr{H}$. We require also the existence of a unitary, self-adjoint operator γ (the grading operator) and a self-adjoint operator called the supercharge Q such that

$$H = Q^2 \qquad \text{and} \qquad Q\gamma + \gamma Q = 0. \tag{1.1}$$

Let

$$E = \inf \text{spectrum } H. \tag{1.2}$$

In our examples E is an eigenvalue of H. Let $P = P_E$ denote the projection onto the eigenspace of E.

DEFINITION 1. *Unbroken supersymmetry means* $E = 0$; *broken supersymmetry means* $E > 0$.

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Let λ be an eigenvalue of H and let P_λ denote the orthogonal projection of $\mathscr{H}$ onto the corresponding eigenspace. We identify the projection P_λ and the subspace $P_\lambda \mathscr{H}$, when no ambiguity can result.

PROPOSITION 2. *Let $\lambda > 0$ be an eigenvalue of H. Then*

$$P_\lambda = P_+ \oplus P_- = \Gamma_+ \oplus \Gamma_- . \tag{1.3}$$

Here $P_\pm$ are isomorphic eigenspaces of Q with eigenvalues $\pm\sqrt{\lambda}$, and $\Gamma_\pm$ are isomorphic eigenspaces of γ with eigenvalues ± 1.

Proof. Note that Q and γ commute with H and hence with P_λ. Let $P_\pm = (2\sqrt{\lambda})^{-1}(\lambda^{1/2} \pm Q)\, P_\lambda$. Clearly $P_\pm$ are projections onto orthogonal subspaces of P_λ. They are eigenspaces of Q with eigenvalues $\pm\lambda^{1/2}$. Also

$$P_\lambda = P_+ + P_-$$
$$\gamma P_\pm \gamma = P_\mp .$$

Thus P_+ and P_- are isomorphic and span P_λ as claimed. Also $\Gamma_\pm = \frac{1}{2}(1 \pm \gamma)\, P_\lambda$ are the projections onto the eigenspaces of γ in P_λ. Thus $P_\lambda = \Gamma_+ + \Gamma_-$. Also

$$\frac{Q}{\lambda^{1/2}}\, \Gamma_\pm \frac{Q}{\lambda^{1/2}} = \Gamma_\mp .$$

But $Q\lambda^{-1/2} \mid P_\lambda$ is unitary and self-adjoint. Hence Γ_+ and Γ_- are isomorphic.

Remark. If supersymmetry is broken, the ground state is degenerate, as follows by Proposition 2. The converse is not in general true; the ground state may be degenerate even though supersymmetry is unbroken, as the example in Section 4 illustrates.

The operator γ plays a fundamental role. It is natural to decompose $\mathscr{H}$ into the eigenspaces of γ,

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_- , \tag{1.4}$$

where $\mathscr{H}_\pm$ are the eigenspaces of γ corresponding to eigenvalues ± 1. Thus $\Gamma_\pm \subset \mathscr{H}_\pm$, and each linear operator on $\mathscr{H}$ has a 2×2 matrix representation with respect to the decomposition (1.4). We write this representation for γ, Q, H as

$$\gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} Q_- Q_+ & 0 \\ 0 & Q_+ Q_- \end{pmatrix}. \tag{1.5}$$

In the literature, $\mathscr{H}_+$ is sometimes referred to as the bosonic subspace and $\mathscr{H}_-$ as the fermionic subspace. An arbitrary linear operator B on $\mathscr{H}$ has the representation

$$B = B_e + B_0, \tag{1.6}$$

where B_e is diagonal (even), and B_0 is off-diagonal (odd). The even operator is called bosonic, and it leaves $\mathscr{H}_+$ and $\mathscr{H}_-$ invariant. The odd operator is called fermionic, and it interchanges $\mathscr{H}_\pm$. Furthermore,

$$\gamma B_e \gamma = B_e, \qquad \gamma B_0 \gamma = -B_0, \tag{1.7}$$

so the even elements form an algebra.

Witten [1] proposed an interpretation of unbroken supersymmetry in terms of the index of Q, defined as follows: Assume that Q_+ is Fredholm, namely Q_+ is closed, has closed range, and its kernel and cokernel are finite dimensional. Let $n_\pm = \dim \text{kernel } Q_\pm$ and define

$$i(Q_+) = n_+ - n_-. \tag{1.8}$$

A zero mode of Q is defined as an element of the kernel of Q, namely an eigenvector of H with energy zero. The index (1.8) is the number of linearly independent bosonic zero modes n_+, minus the number of linearly independent fermionic zero modes n_-. Clearly $i(Q_+) \neq 0$ is a sufficient condition for unbroken supersymmetry. If $n_+ = 0$ or $n_- = 0$, then $i(Q_+) \neq 0$ is a necessary and sufficient condition for unbroken supersymmetry.

A standard formula for the index, when $\exp(-tH)$ is trace class, is

$$i(Q_+) = \operatorname{Tr}(\gamma e^{-tH}); \tag{1.9}$$

see, for example, [2]. This follows from Proposition 2, which ensures that each nonzero eigenspace of H gives a vanishing contribution to (1.9). On the other hand, zero modes of H are zero modes of Q, and (1.9) follows.

We consider here two examples with qualitatively different vacuum structures. The first model is a quantum mechanics version of the $N=1$ Wess–Zumino field theory [1]. This model as well as its generalizations was studied by many authors; see, for example, [3–9]. In this example supersymmetry is broken or unbroken depending on the asymptotics of the superpotential at infinity, and is characterized by its degree: $i(Q_+) = \pm \deg V (\text{mod } 2)$. In the unbroken case, there is a unique ground state; it belongs to $\mathscr{H}_+$ ($n_+ = 1$, $n_- = 0$) or to $\mathscr{H}_-$ ($n_+ = 0$, $n_- = 1$), according to the additional $\mathbb{Z}_2$ symmetry of the superpotential. In the case of broken supersymmetry, there are exactly two ground states and $n_+ = n_- = 1$. We have proved that similar results are true in the corresponding $d=2$ quantum field models in a finite volume [10], for which this paper serves as a toy model.

The second model is a one-dimensional version of the $N=2$ Wess–Zumino model. This model leads to the introduction of a holomorphic quantum mechanics. For this model, supersymmetry is unbroken for any polynomial superpotential. Nevertheless, the ground state is degenerate—except when the superpotential is quadratic. For a superpotential of degree n, it follows that $n_+ = n - 1 = i(Q_+)$, $n_- = 0$. We establish analytic properties of the index using functional analysis methods, which display continuity in certain parameters. The equality $i(Q_+) =$

$n-1$ has been established in the corresponding two-dimensional, finite volume Wess–Zumino quantum field model [10].

We end this paper with a brief description of a phenomenological pion system, described by the holomorphic quantum theory.

2. Perturbation Theory

Let us assume that $Q = Q_0$ is a self-adjoint Fredholm operator, and assume that Ω_0 is a normalized, zero mode for Q_0. We saw in Section 1 that it is no loss of generality to assume that Ω_0 is an eigenstate of γ, namely $\gamma\Omega_0 = \alpha\Omega_0$, where $\alpha = \pm 1$. Here we assume that the space of all zero modes of Q_0 is either bosonic ($n_- = 0$) or fermionic ($n_+ = 0$).

Let us perturb Q_0 by εQ_1, where ε is a small, real parameter. Thus

$$Q = Q(\varepsilon) = Q_0 + \varepsilon Q_1, \qquad H = H(\varepsilon) = Q(\varepsilon)^2. \tag{2.1}$$

We seek a family of ground states $\Omega(\varepsilon)$ of $H(\varepsilon)$ which are continuous perturbations of Ω_0 and which satisfy

$$Q(\varepsilon)\,\Omega(\varepsilon) = \lambda(\varepsilon)\,\Omega(\varepsilon). \tag{2.2}$$

Here $\lambda(\varepsilon)$ is an eigenvalue of $Q(\varepsilon)$, $E(\varepsilon) = \lambda(\varepsilon)^2$, $\lambda(0) = 0$, and $\Omega(0) = \Omega_0$. In this section we consider $\Omega(\varepsilon)$, $\lambda(\varepsilon)$ as formal power series in ε, namely

$$\lambda(\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k \lambda_k, \qquad \Omega(\varepsilon) = \Omega_0 + \sum_{k=1}^{\infty} \varepsilon^k \Omega_k. \tag{2.3}$$

Proposition 3. *Let Ω_0 be a zero model of Q_0 as above, and suppose that for $\varepsilon = 0$, either $n_+ = 0$ or $n_- = 0$. Then as formal power series in ε,*

$$\lambda(\varepsilon) = 0, \qquad \textit{and} \qquad \gamma\Omega(\varepsilon) = \alpha\Omega(\varepsilon). \tag{2.4a}$$

In other words,

$$E(\varepsilon) = 0, \qquad \text{and either} \qquad n_+(\varepsilon) = 0 \qquad \text{or} \qquad n_-(\varepsilon) = 0. \tag{2.4b}$$

Remark. The proposition states that if the unperturbed ground states are all bosonic or all fermionic, then supersymmetry is never broken in perturbation theory. In fact, $n_+ = 0$ or $n_- = 0$ in all the examples we study, so this proposition applies. Thus if supersymmetry actually is broken for such an example, we can conclude that $E(\varepsilon)$ is not an analytic function of ε at $\varepsilon = 0$. Thus we expect that broken supersymmetry is accompanied by instanton-like effects.

Proof. We proceed to establish (2.4) by induction on the order of the power series (2.3). By assumption (2.4) holds in zero order. Assume that (2.4) holds for power series of order $\leqslant j-1$. Then the coefficient of (2.2) in order j is

$$Q_0\Omega_j + Q_1\Omega_{j-1} = \lambda_j\Omega_0. \tag{2.5}$$

Taking an inner product with Ω_0 yields $\lambda_j = \langle \Omega_0, Q_1 \Omega_{j-1} \rangle$. However, since $\gamma^2 = I$, and $\gamma = \gamma^*$, we can use our induction hypothesis $\gamma\Omega_{j-1} = \alpha\Omega_{j-1}$ with $\alpha^2 = 1$ to ensure

$$\begin{aligned}\lambda_j &= \langle \Omega_0, Q_1 \Omega_{j-1} \rangle = \langle \gamma^2 \Omega_0, Q_1 \Omega_{j-1} \rangle = \langle \gamma\Omega_0, \gamma Q_1 \Omega_{j-1} \rangle \\ &= -\langle \gamma\Omega_0, Q_1 \gamma\Omega_{j-1} \rangle = -\alpha^2 \langle \Omega_0, Q_1 \Omega_{j-1} \rangle = -\lambda_j.\end{aligned}$$

Hence $\lambda_j = 0$. Furthermore, it follows from (2.5) that

$$Q_0 \gamma \Omega_j = -\gamma Q_0 \Omega_j = \gamma Q_1 \Omega_{j-1} = -Q_1 \gamma \Omega_{j-1} = -\alpha Q_1 \Omega_{j-1} = \alpha Q_0 \Omega_j,$$

or

$$Q_0 (\gamma - \alpha)\, \Omega_j = 0.$$

Thus, $(\gamma - \alpha)\, \Omega_j$ is a zero mode of Q_0. By assumption, all zero modes of Q_0 belong to $\mathscr{H}_\alpha$. However, $\frac{1}{2}(\gamma - \alpha)$ is the orthogonal projection onto $\mathscr{H}_{-\alpha}$. Thus $(\gamma - \alpha)\, \Omega_j \in \mathscr{H}_\alpha \cap \mathscr{H}_{-\alpha} = 0$. Hence $\Omega_j \in \mathscr{H}_\alpha$, as desired, and the proof of (2.4) is complete.

3. $N = 1$ Supersymmetric Quantum Mechanics

3.1. *The Model*

We study here a single bosonic degree of freedom x with a supersymmetric interaction determined by a superpotential $V(x)$. We assume that V is a polynomial of degree $n \geqslant 2$; the choice $V = \frac{1}{2}\mu x^2 + \frac{1}{3}\lambda x^3$ corresponds to the $N = 1$ Wess–Zumino model.

The fermionic degree of freedom is described by the "field operator" $\psi(t)$ whose conjugate we denote by $\bar{\psi}(t)$. We assume the equal-time anticommutation relations

$$\{\psi, \bar{\psi}\} = 1, \qquad \{\psi, \psi\} = 0, \qquad \{\bar{\psi}, \bar{\psi}\} = 0. \tag{3.1}$$

A simple 2×2 matrix representation of this algebra can be given in terms of the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$. We have $\sigma_j^2 = 1$, $\sigma_1 \sigma_2 = i\sigma_3$. Then defining

$$\psi = \tfrac{1}{2}(\sigma_1 + i\sigma_2), \qquad \bar{\psi} = \tfrac{1}{2}(\sigma_1 - i\sigma_2) \tag{3.2}$$

yields (3.1), and $[\psi, \bar{\psi}] = \sigma_3$.

The Lagrangian that we study is

$$\mathscr{L} = \tfrac{1}{2}\dot{x}^2 + i\bar{\psi}\dot{\psi} + \tfrac{1}{2}(\bar{\psi}\psi - \psi\bar{\psi})\, V'' - \tfrac{1}{2}(V')^2. \tag{3.3}$$

The corresponding action $\int \mathscr{L}\, dt$ is invariant with respect to supersymmetric transformation. The infinitesimal supersymmetry transformation is defined by the linear part of a formal power series in a Grassmann parameter ε:

$$\delta x = \bar{\varepsilon}\psi + \bar{\psi}\varepsilon, \qquad \delta\psi = -i\varepsilon\dot{x} + \varepsilon V', \qquad \delta\bar{\psi} = i\bar{\varepsilon}\dot{x} + \bar{\varepsilon} V', \tag{3.4}$$

Corresponding to this variation, there are two conserved charges,

$$Q_+ = \frac{1}{\sqrt{2}}(\dot{x} + iV'(x))\bar{\psi}, \qquad Q_- = \frac{1}{\sqrt{2}}(\dot{x} - iV'(x))\psi. \tag{3.5}$$

The Hamiltonian corresponding to (3.1) is

$$H = \tfrac{1}{2}(p^2 + (\psi\bar{\psi} - \bar{\psi}\psi)V'' + (V')^2). \tag{3.6}$$

We can rewrite the above expressions in terms of $p = \dot{x}$ and the matrices σ_j. The self-adjoint charges

$$Q_r = Q_+ + Q_-, \qquad Q_i = \frac{1}{i}(Q_+ - Q_-) \tag{3.7}$$

have the representation

$$Q_r = \frac{1}{\sqrt{2}}(p\sigma_1 + V'\sigma_2), \qquad Q_i = \frac{1}{\sqrt{2}}(-p\sigma_2 + V'\sigma_1) = -i\sigma_3 Q_r. \tag{3.8}$$

Also,

$$H = Q_r^2 = Q_i^2 = \tfrac{1}{2}(p^2 + \sigma_3 V'' + (V')^2). \tag{3.9}$$

Thus choosing $\gamma = \sigma_3$ and $Q = Q_r$, or $Q = Q_i$, we have the desired algebra (1.1).

The Hilbert space of our model is a direct sum (1.4) with $\mathscr{H}_\pm = L_2(\mathbf{R})$. Define the Sobolev space

$$L_2^*(\mathbf{R}) = \left\{ f : f \in L_2,\ \|f\|_*^2 = \left\| \frac{d^2}{dx^2} f \right\|^2 + \|(1+|x|)^{2(n-1)} f\|^2 < \infty \right\}. \tag{3.10}$$

Here n is the degree of $V(x)$. Then

$$\mathscr{H}^* = L_2^* \oplus L_2^* \tag{3.11}$$

is a domain for Q and for H.

PROPOSITION 4. *The operator H is self-adjoint on $\mathscr{H}^*$ and Q is essentially self-adjoint. Both H and Q have compact resolvents.*

Proof. The operator $\tilde{H} = H - \frac{1}{2}\sigma_3 V''$ is clearly self-adjoint on $\mathscr{H}^*$. Furthermore since degree $V \geqslant 2$, $\deg(V')^2 \geqslant 2$ and there exist positive constants a, b such that

$$\tilde{H}_0 \leqslant a\tilde{H} + b, \tag{3.12}$$

where $\tilde{H}_0$ is $\tilde{H}$ for $V = x^2$. Since $\tilde{H}_0$ is a direct sum of oscillator Hamiltonians, it has a compact resolvent. But inequality (3.12) shows that $\tilde{H}$ is relatively compact with respect to $\tilde{H}_0$, and hence it too has a compact resolvent.

The Hamiltonian H is a perturbation of $\tilde{H}$ by $\frac{1}{2}\sigma_3 V''$. But deg $V'' < \deg(V')^2$, and

$$-|V''| \leqslant \sigma_3 V'' \leqslant |V''|.$$

So $\frac{1}{2}\sigma_3 V''$ satisfies the following estimate: Given $\tilde{\alpha} > 0$, there exists $\tilde{b} < \infty$ such that

$$\pm \tfrac{1}{2}\sigma_3 V'' \leqslant \tilde{a}\tilde{H}_0 + \tilde{b} \leqslant \tilde{a}(a\tilde{H} + b) + \tilde{b}. \tag{3.13}$$

Choose $\tilde{a}$ so $\tilde{a}a = \frac{1}{2}$. Thus with $c = \tilde{a}b + \tilde{b}$,

$$\tfrac{1}{2}\tilde{H} \leqslant H + c, \tag{3.14}$$

which shows that H is relatively compact with respect to $\tilde{H}$. Furthermore, H has a compact resolvent. Inequality (3.13) ensures $D(H) = D(\tilde{H}) = \mathscr{H}^*$. Finally, note that H and Q have the same null spaces, so using Proposition 2, we conclude that the resolvent of Q is compact. Since the eigenstates of H belong to $\mathscr{H}^*$, Q is essentially self-adjoint on this domain.

3.2. *The Supersymmetric Oscillator*

Let us consider the superpotential

$$V = V_0(x) = \tfrac{1}{2}x^2 \tag{3.15}$$

yielding the supersymmetric harmonic oscillator. Using the representation (1.5), we write

$$Q_- = Q_{0^-} = \frac{1}{\sqrt{2}}(p - ix), \tag{3.16}$$

which is the standard annihilation operator. Thus the eigenfunctions of Q_0 can be written in closed form, in terms of the normalized, zero mode of Q_-, namely

$$\Omega(x) = \pi^{-1/4} \exp(-x^2/2).$$

There is exactly one zero mode of Q_0, and this mode is fermionic,

$$\varphi_0 = (0, 1)\,\Omega. \tag{3.17}$$

Furthermore, for each positive integer $j = 1, 2, \ldots$, there are two eigenfunctions $\varphi_j^\pm$ of Q_0 with eigenvalues $\pm\sqrt{j/2}$. Explicitly

$$\varphi_j^\pm = \frac{1}{\sqrt{2}}(\mp ih_{j-1}, h_j)\,\Omega, \tag{3.18}$$

where h_j is the jth Hermite polynomial, normalized with respect to Ω^2. Note that

$$\sigma_3 \varphi_j^\pm = -\varphi_j^\mp.$$

Here supersymmetry is unbroken, $E=0$, $n_+=0$, $n_-=1$. Note that if V were chosen equal to $-\frac{1}{2}x^2$, then we obtain a similar model, but in that case $n_+=1$, $n_-=0$.

3.3. *Perturbation Theory is Misleading*

Consider a perturbation of the form (2.1), given by a perturbation of the superpotential $V=V_0=\frac{1}{2}x^2$ of the form

$$V=V(\varepsilon)=V_0+\varepsilon W, \tag{3.19}$$

where W is a polynomial. Then the perturbation of (3.16) is

$$Q_-=Q_{0^-}-\frac{i}{\sqrt{2}}\varepsilon W'. \tag{3.20}$$

Consider $H=Q^2$, as in (1.5).

PROPOSITIONS 5. *Let Ω be a ground state of H with eigenvalue E. Then the following statements are equivalent*:

(i) *Ω is a zero mode, i.e., $E=0$.*

(ii) *E is a simple eigenvalue.*

(iii) $i(Q_+)=\pm 1$.

(iv) $\deg V=0 \pmod 2$.

Remarks. The proposition states that for this example, supersymmetry is unbroken iff $i(Q_+)=\pm 1$. Since the index vanishes when supersymmetry is broken, the only possible values of $i(Q_+)$ for this model are 0, ± 1. Furthermore, the index vanishes iff V has odd degree. Thus supersymmetry is broken for any anharmonic oscillator, such as the quartic oscillator, when the superpotential has odd degree $2m+1$. (The ordinary potential has degree $4m$, $m=1, 2, ...$) Since $n_+=0$ for $W=0$, Proposition 3 applies and perturbation theory is misleading; it suggests unbroken supersymmetry. Note further that the index $i(Q_+)$ is not necessarily continuous in ε, even when supersymmetry is unbroken for $\varepsilon\neq 0$. If W is a polynomial of even degree $\geqslant 4$, then $i(Q_+)$ changes sign across $\varepsilon=0$; at $\varepsilon=0$, $i(Q_+)=-1$.

Proof. (i)$\Rightarrow$(ii), and (i)$\Rightarrow$(iv). If Ω is a zero mode, then $Q\Omega=0$. Using the representation (3.8) we can integrate $(p\sigma_1+V'\sigma_2)\,\Omega=0$ to obtain

$$\Omega(x)=\exp(V\sigma_3)\,\Omega(0). \tag{3.21}$$

Here $\Omega(0)=\alpha_+\Omega_++\alpha_-\Omega_-$, where $\alpha_\pm$ are constants and $\Omega_\pm$ are unit vectors in $\mathscr{H}_\pm$, respectively. Then

$$\Omega=\alpha_+\exp(V)\,\Omega_++\alpha_-\exp(-V)\,\Omega_-. \tag{3.22}$$

Since $|V| \to \infty$ as $|x| \to \infty$, exactly one of the two terms in (3.28) remains bounded if deg V is even; neither term remains bounded if deg V is odd. Thus $E=0$ iff deg V is even; in that case E is simple.

(ii) $\Rightarrow$ (i), (iii). If E is a simple eigenvalue of H, then by Proposition 2, it follows that $E=0$. The definition of $i(Q_+)$ ensures that $i(Q_+) = \pm 1$.

(iii) $\Rightarrow$ (i). If $i(Q_+) \neq 0$, then $H\Omega = 0$ has a nonzero solution in H. Since $H \geqslant 0$, Ω is a zero mode.

4. Holomorphic Quantum Mechanics

4.1. $N=2$, *Wess–Zumino Quantum Mechanics*

This model describes the interaction between a complex bosonic degree of freedom $z(t)$ and two fermionic degrees of freedom ψ_1, ψ_2. The fermions are assumed to satisfy the following anticommutation relations at equal time:

$$\begin{aligned} \{\bar{\psi}_1, \psi_2\} = \{\bar{\psi}_2, \psi_1\} &= 1, \\ \{\psi_i, \psi_j\} = \{\bar{\psi}_i, \bar{\psi}_j\} &= 0. \end{aligned} \tag{4.1}$$

We realize this algebra in terms of the following Euclidean γ matrices,

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma_j = \begin{pmatrix} 0 & i\sigma_j \\ -i\sigma_j & 0 \end{pmatrix}, \qquad j = 1, 2, 3. \tag{4.2}$$

The grading operator γ is defined as

$$\gamma = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{4.3}$$

namely the "γ_5" matrix for the generators (4.2) of the Clifford algebra. In terms of these matrices, we define the time-zero fermions as

$$\begin{aligned} \psi_1 &= \tfrac{1}{2}(\gamma_0 - i\gamma_3), \qquad & \bar{\psi}_1 &= \tfrac{1}{2}(\gamma_1 + i\gamma_2), \\ \psi_2 &= \tfrac{1}{2}(\gamma_1 - i\gamma_2), \qquad & \bar{\psi}_2 &= \tfrac{1}{2}(\gamma_0 + i\gamma_3). \end{aligned} \tag{4.4}$$

Note that $\psi_2^* = \bar{\psi}_1$ and $\psi_1^* = \bar{\psi}_2$, so $\bar{\psi}_1\psi_2$ and $\bar{\psi}_2\psi_1$ are positive. The Lagrangian of our model is

$$\mathscr{L} = |\dot{z}|^2 + i(\bar{\psi}_1\dot{\psi}_2 + \bar{\psi}_2\dot{\psi}_1) + \bar{\psi}_1\psi_1\,\partial^2 V + \bar{\psi}_2\psi_2(\partial^2 V)^* - |\partial V|^2. \tag{4.5}$$

Here we assume that $V = V(z)$ is a polynomial of degree n. Note that $\bar{\psi}_1\psi_1$ and $\bar{\psi}_2\psi_2$ are adjoints, so $\mathscr{L}$ is real. Critical points of the action $\int \mathscr{L}\,dt$ are invariant under the following infinitesimal supersymmetry transformation

$$\begin{aligned} \delta z &= \bar{\psi}_1 \varepsilon, \qquad & \delta\bar{z} &= \bar{\varepsilon}\psi_2, \qquad & \delta\psi_1 &= -(\partial V)^*\,\varepsilon, \\ \delta\bar{\psi}_1 &= i\dot{\bar{z}}\bar{\varepsilon}, \qquad & \delta\psi_2 &= i\dot{z}\varepsilon, \qquad & \delta\bar{\psi}_2 &= (\partial V)\,\varepsilon. \end{aligned} \tag{4.6}$$

The corresponding conserved charges are

$$Q_1 = i\bar{\psi}_1 \, \partial - i\bar{\psi}_2(\partial V)^*, \qquad Q_2 = i\psi_2 \, \bar{\partial} + i\psi_1 \, \partial V, \tag{4.7}$$

and it is useful to study the self-adjoint charge

$$Q = Q_1 + Q_2. \tag{4.8}$$

With these conventions, the Hamiltonian can be written

$$H = Q^2 = -\partial\bar{\partial} - \bar{\psi}_1\psi_1 \, \partial^2 V - \bar{\psi}_2\psi_2(\partial^2 V)^* + |\partial V|^2, \tag{4.9}$$

which is hermitian, as $\bar{\psi}_1\psi_1 = (\bar{\psi}_2\psi_2)^*$. As in Section 3, we show that H and Q are essentially self-adjoint operators with compact resolvents.

Following the convention (1.5) for Q, we find that

$$Q_- = i(\sigma_+ \, \partial + \sigma_- \, \bar{\partial}) + \tfrac{1}{2}(1+\sigma_3)(\partial V) - \tfrac{1}{2}(1-\sigma_3)(\partial V)^*, \tag{4.10}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 + i\sigma_2)$. This can also be written

$$Q_- = \begin{pmatrix} \partial V & i\partial \\ i\bar{\partial} & -(\partial V)^* \end{pmatrix}, \qquad Q_+ = \begin{pmatrix} (\partial V)^* & i\partial \\ i\bar{\partial} & -\partial V \end{pmatrix}. \tag{4.11}$$

From this we can compute

$$h_1 = Q_- Q_+ = h_2 + \begin{pmatrix} 0 & -i(\partial^2 V) \\ i(\partial^2 V)^* & 0 \end{pmatrix} \tag{4.12}$$

and

$$h_2 = Q_+ Q_- = (-\partial\bar{\partial} + |\partial V|^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.13}$$

4.2. *There Are No Fermionic Zero Modes, $n_- = 0$*

By (1.5), the equation satisfied by a fermionic zero mode $\Omega_- \in \mathscr{H}_-$ is $Q_-\Omega_- = 0$, or equivalently $h_2\Omega_- = 0$, with h_2 given by (4.13).

PROPOSITION 6. *The operator h_2 has no zero modes.*

Proof. Clearly h_2 is the sum of two positive operators, and $h_2\Omega_- = 0$ ensures

$$\partial V \Omega_- = \bar{\partial}\Omega_- = 0.$$

This is possible only if $\Omega_- = 0$.

4.3. *For $V = \lambda z^n$, There Are $n-1$ Zero Modes*

Here we compute $i(Q_+)$ for monomial, holomorphic potentials.

PROPOSITION 7. *Let $n \in \mathbb{Z}_+$, $\lambda \neq 0$, and $V(z) = \lambda z^n$. Then $n_+ = n - 1 = i(Q_+)$.*

Proof. We show that the null space of Q_+ is $(n-1)$-dimensional. Hence $n_+ = n-1$ and by Proposition 6, $n_+ = i(Q_+)$. It is equivalent to study the null space of h_1, defined in (4.12). Write $\Omega_+ = (f, -ig) \in \mathscr{H}_+$, where $f, g \in L_2(C)$. Then the equation $Q_+ \Omega_+ = 0$ takes the form

$$\partial g + (\partial V)^* f = 0, \qquad \bar{\partial} f + (\partial V)\, g = 0. \tag{4.14}$$

It is no loss of generality to assume $n \geqslant 2$, since for $n = 0, 1$ we have $h_1 = h_2$ and h_2 has no zero modes by Proposition 6.

LEMMA 8. *Every zero mode Ω_+ of h_1 arises from a pair of functions $f, g \in L_2(C)$, which satisfy the equations*

$$(-\partial\bar{\partial} + |\partial V|^2)\, f + (\partial^2 V)(\partial V)^{-1}\, \bar{\partial} f = 0 \tag{4.15}$$

and

$$(-\partial\bar{\partial} + |\partial V|^2)\, g + ((\partial^2 V)/\partial V)^*\, \partial g = 0. \tag{4.16}$$

Remark. The equations (4.15) and (4.16) are complex conjugate equations, so the set of solutions f of (4.15) is just the complex conjugates of the solutions g to (4.16).

Proof. We first derive (4.15)–(4.16) from (4.14). In fact, differentiating the second equation (4.14) yields

$$-\partial\bar{\partial} f - (\partial^2 V)\, g - (\partial V)\, \partial g = 0.$$

Using (4.14) to eliminate g yields (4.15). Similarly, we derive (4.16). Conversely, suppose that g satisfies (4.16), and define

$$f \equiv -(1/\partial V)^*\, \partial g.$$

Differentiating, we find

$$\begin{aligned} \bar{\partial} f &= [\partial^2 V/(\partial V)^2]^*\, \partial g - (1/\partial V)^*\, \partial\bar{\partial} g \\ &= -(1/\partial V)^*\, (-|\partial V|^2\, g) = -(\partial V)\, g, \end{aligned}$$

so (4.14) holds for the pair (f, g). Since (4.15) follows from (4.14), f is also a solution to (4.15). If f, g are $L_2(\mathbf{C})$, then $\Omega_+ = (f, -ig) \in \mathscr{H}_+$ and Ω_+ is a zero mode as claimed.

We now remark that the equations (4.15)–(4.16) can be solved in closed form for monomial, holomorphic V. The solutions are hypergeometric functions. For $V = \lambda z^n$, Eq. (4.16) can be written

$$-\partial\bar{\partial} g + (n-1)\, \bar{z}^{-1}\, \partial g + |n\lambda z^{n-1}|^2\, g = 0. \tag{4.17}$$

It is convenient to express g in polar coordinates (r, θ), where

$$\frac{1}{\bar{z}}\frac{\partial}{\partial z}=(2r)^{-1}\left(\frac{\partial}{\partial r}-i\frac{1}{r}\frac{\partial}{\partial \theta}\right),$$

so (4.17) becomes

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right)-\frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}+\frac{2(n-1)}{r}\left(\frac{\partial g}{\partial r}-\frac{i}{r}\frac{\partial g}{\partial \theta}\right)+4n^2|\lambda|^2\, r^{2(n-1)}g=0. \tag{4.18}$$

Expressing g as a Fourier series in the angular variable

$$g(r, \theta)=\sum_m u_m(r)\, e^{-im\theta}, \qquad m\in \mathbb{Z}, \tag{4.19}$$

we can separate variables and obtain an ordinary differential equation for u_m,

$$-\frac{1}{r}(ru_m')'+\frac{2(n-1)}{r}u_m'+\frac{1}{r^2}\left[4n^2\lambda^2 r^{2n}+m(m-2n+2)\right]u_m=0. \tag{4.20}$$

In general there are two linearly independent solutions for each value of m, namely $u_m^{(1)}$ and $u_m^{(2)}$, and the desired solution (4.19) will be given by a linear combination u_m of $u_m^{(1)}$ and $u_m^{(2)}$ for each m. The requirement that $g\in L_2(\mathbf{R}^2)$ is related to u_m by

$$\|g\|_{L_2}^2=2\pi\sum_m\int_0^\infty |u_m|^2\, r\, dr, \tag{4.21}$$

so we desire

$$r^{1/2}u_m\in L_2([0, \infty], dr), \qquad \|r^{1/2}u_m\|_{L_2}\in l_2(m). \tag{4.22}$$

We now study the regularity of $u_m(r)$ at $r=0$ and at $r=\infty$ which is compatible with (4.22). We will find that for given n these conditions limit us to $n-1$ linearly independent solutions u_m and hence to $n-1$ linearly independent g's.

First, we remark that the equation (4.20) is invariant under the substitution

$$m=(n-1)-\delta\rightarrow m=(n-1)+\delta, \tag{4.23}$$

so it is no loss of generality to study only solutions for which

$$m\leqslant (n-1). \tag{4.24}$$

Second, we remark that for each m, only one solution u_m to (4.20) decays as $r\rightarrow\infty$. In fact, $r=\infty$ is an "irregular singular point" of the equation. The equation has an exponentially growing solution and one which decays as $\exp(-2\lambda r^n)$ [11]. We choose the decaying solution.

Next consider the asymptotics at $r=0$. This is a regular singular point with behavior $u_m^{(j)}\sim r^{\alpha_j}$ as $r\rightarrow 0$, for $j=1, 2$, with α_j the roots of the indicial polynomial

$\alpha^2-2(n-1)\alpha-m(m-2n+2)$. These roots are $\alpha_1=m$, $\alpha_2=2n-2-m$. (Note that for $m=n-1$ the roots are degenerate and the second solution has an $r^m \ln r$ part. This logarithm does not affect our analysis of local square integrability of $r^{1/2}u_m$, so we ignore it.) Thus up to the logarithms described above, as $r \to 0$,

$$u_m^{(1)} \sim r^m, \qquad u_m^{(2)} \sim r^{2n-2-m}. \tag{4.25}$$

We claim that local square integrability, combined with (4.20), ensures $0 \leqslant m \leqslant (n-2)$; so there are exactly $n-1$ allowed solutions u_m for given n. This additional restriction on m follows by taking the L_2 inner product of (4.20) with u_m, with respect to the measure $r\,dr$. Then

$$\int_0^\infty [|u_m'|^2+4n^2\lambda^2 r^{2n-2}|u_m|^2+m(m-2n+2)|u_m|^2]\,r\,dr=(n-1)|u_m(0)|^2. \tag{4.26}$$

Here we use the fact that

$$2\operatorname{Re}\int_0^\infty \bar{u}_m u_m'\,dr=|u_m(0)|^2. \tag{4.27}$$

Every term in (4.20) is hermitian in the inner product $L_2([0,\infty], r\,dr)$, except the term containing $\int_0^\infty \bar{u}_m u_m'\,dr$. Thus

$$\operatorname{Im}\int_0^\infty \bar{u}_m u_m'\,dr=0,$$

and (4.26) holds.

Case 1. $u_m(0)=0$. In this case, it follows from (4.26) that $m(m-2n+2)<0$, or else $u_m \equiv 0$. Using (4.24) we conclude that $m>0$, so in this case

$$1 \leqslant m \leqslant (n-1). \tag{4.28}$$

This inequality assures that both solutions (4.25) are locally $L_2(r\,dr)$, so u_m can be chosen to decay at $r=\infty$.

Case 2. $u_m(0) \neq 0$. The only way that $u_m(0)$ is nonzero, yet $r^{1/2}u_m(r)$ is locally L_2, is the case $m=0$. In this case (4.26) provides no further constraint, and again both solutions are regular at $r=0$.

We have now determined all the possible solutions to the radial equation (4.20), and this gives us solutions to the second order equations for g and f, namely

$$g_m^{(1)}(r,\theta)=u_m(r)\,e^{-im\theta}, \qquad g_m^{(2)}(r,\theta)=u_m(r)\,e^{-i(2n-2-m)\theta},$$

and

$$f_m^{(1)}(r,\theta)=\overline{u_m(r)}\,e^{im\theta},\qquad f_m^{(2)}(r,\theta)=\overline{u_m(r)}\,e^{i(2n-2-m)\theta},\tag{4.29}$$

where $0\leqslant m\leqslant(n-1)$.

As a consequence of Lemma 8, we now require that solutions (4.29) to (4.15)–(4.16) yield L_2 solutions to (4.14). In fact, the $r=0$ asymptotics (4.30) yield at $r=0$ the asymptotics

$$\begin{aligned}&\partial g_m^{(1)}\sim r^{2(n-2-m)}\bar{z}^{m+1}, &&\partial g_m^{(2)}\sim r^{2(m-n-2)}\bar{z}^{(2n-1-m)},\\ &\partial f_m^{(1)}\sim r^{2m}\bar{z}^{(n-m-1)}, &&\partial f_m^{(2)}\sim r^{4(n-1)}\bar{z}^{-(n-1-m)}.\end{aligned}\tag{4.30}$$

Comparing with (4.18), we see that of the possible pairings, only

$$(f_{n-2-m}^{(1)},\,g_m^{(1)})\tag{4.31}$$

yield solutions to (4.14). Furthermore, local regularity of f ensures $n-2-m\geqslant 0$, which rules out $m=n-1$ and restricts us to

$$0\leqslant m\leqslant(n-2).\tag{4.32}$$

Hence there are exactly $(n-1)$ linearly independent solutions, as claimed. This completes the proof of Proposition 7.

Let us define normalized zero modes

$$\Omega_m=(f_m^{(1)},\,g_{n-2-m}^{(1)})=(\bar{u}_m e^{im\theta},\,u_{n-2-m}e^{-(n-2-m)\theta}),\tag{4.33}$$

where $j=0,1,\dots,n-2$. These vacuum states clearly satisfy

$$(\bar{z}\Omega_{m+1},\Omega_m)\neq 0,\qquad (z\Omega_m,\Omega_{m+1})\neq 0,\qquad m=0,\dots,n-3.$$

4.4. *Regular Deformations Preserve the Index*

PROPOSITION 9. *Let $V(z)$ be a polynomial of degree n. Then $i(Q_+)=n_+=n-1$.*

Proof. Write $V(z)=\lambda z^n+q(z)$, where $\lambda\neq 0$, $\deg q<n$. We show that $q(z)$ is a regular deformation of V in the following sense: Let $Q_+^{(0)}$ denote the transformation Q_+ defined by (4.11) with $V=V_0=\lambda z^n$. Then

$$Q_+=Q_+^{(0)}+R,\qquad R=\begin{pmatrix}\partial q & 0\\ 0 & -(\partial q)^*\end{pmatrix}.\tag{4.34}$$

We claim that R is a relatively compact perturbation of $Q_+^{(0)}$. In other words, whenever $\{\Omega_n\} \subset \text{Domain}(A_0)$ satisfies

$$\|\Omega_n\|^2 + \|Q_+^{(0)}\Omega_n\|^2 = \|\Omega_n\|^2 + \|\bar{\partial}\Omega_n\|^2 + \|\partial V_0 \Omega_n\|^2 \leqslant 1, \tag{4.35}$$

it follows that $\{R\Omega_n\}$ has a strongly convergent subsequence. Now, $Q_+^{(0)}$ is a Fredholm operator. By a standard result, Theorem 5.26 of [12], Q_+ is Fredholm and $i(Q_+) = i(Q_+^{(0)})$. But $i(Q_+^{(0)}) = n_+ = n-1$, so Proposition 9 follows.

In order to establish relative compactness, note that $h_2^{(0)} = -\partial\bar{\partial} + |n\lambda z^{n-1}|^2$ satisfies

$$\|\Omega_n\|^2 + \|Q_+^{(0)}\Omega_n\|^2 = \|(h_2^{(0)}+1)^{1/2}\,\Omega_n\|^2 \leqslant 1. \tag{4.36}$$

In fact, Ω_n satisfying (4.36) are uniformly continuous on compact sets and decay uniformly with a polynomial rate at infinity. So the sequence $\{\Omega_n\}$ has a convergent subsequence and $h_2^{(0)}+1$ has a compact inverse. We claim that, if $\{\Omega_m\}$ is this subsequence, then $\{R\Omega_m\}$ is convergent. We estimate $\|R(\Omega_m - \Omega_{m'})\|$as follows: Write

$$R\Omega_m = \chi R\Omega_m + (1-\chi)\,R\Omega_m,$$

where χ is the characteristic function of $|z| \leqslant a$. Thus

$$\begin{aligned}\|(1-\chi)\,R\Omega_m\|_{L_2} &= \|(1-\chi)\,R(\partial V_0)^{-1}\,\partial V_0 \Omega_m\|_{L_2}\\ &\leqslant \|(1-\chi)\,R(\partial V_0)^{-1}\|_{L_\infty}\,\|\partial V_0\Omega_m\|_{L_2}.\end{aligned}$$

Using the fact that V_0 is of degree n and q is of degree $<n$, it follows that

$$M \equiv \|(1-\chi)\,R(\partial V_0)^{-1}\|_{L_\infty} = O(a^{-1}).$$

Given ε, choose the constant a sufficiently large that $M < \varepsilon$. Thus

$$\|(1-\chi)\,R(\Omega_m - \Omega_{m'})\| \leqslant 2\varepsilon.$$

Since $\{\Omega_m\}$ converges in L_2, and R is bounded on compact sets, $\{R\Omega_m\}$ also converges on the compact set $|z| \leqslant a$. Thus for m, m' sufficiently large, $\|\chi R(\Omega_m - \Omega_{m'})\| \leqslant \varepsilon$. Altogether this yields

$$\|R(\Omega_m - \Omega_{m'})\| \leqslant 3\varepsilon,$$

and $\{R\Omega_m\}$ converges as claimed.

5. Physical Interpretation of Holomorphic Quantum Mechanics

In the remaining section we briefly discuss the possibilities of a physical interpretation of holomorphic quantum mechanics in terms of the nonrelativistic motion of a spin $\frac{1}{2}$ particle in an external $SU(2)$ gauge field. Models of this type find application in atomic, molecular, and nuclear physics.

5.1. *Spin $\frac{1}{2}$ Particle in an External $SU(2)$ Gauge Field*

The nonrelativistic motion of a particle in an external non-abelian gauge field has appealing formal aspects. External gauge potentials may arise in the problems of atomic and molecular physics as a result of an adiabatic elimination of certain slowly varying, internal degrees of freedom. Examples of such potentials realized in the framework of the Born–Oppenheimer method applied to diamagnetic molecules are discussed, e.g., in [13].

Consider an $SU(2)$ gauge field $A_\mu(x) = A_\mu^a \tau_a$, where $\mu = 1, 2, 3$ is a spatial index while τ_a, $a = 1, 2, 3$, are the generators of the $SU(2)$ group. Let us choose A_μ to be real and such that (i) $A_1(x) = A_2(x) = 0$, (ii) $A_3^3(x) = 0$, (iii) $A_3^1(x)$ and $A_3^2(x)$ depend only on x_1 and x_2, and (iv) $A_3^+(x) = (1/\sqrt{2})(A_3^1(x_1, x_2) - A_3^2(x_1, x_2))$ is holomorphic in $z = x_1 + ix_2$. The nonrelativistic minimal coupling Hamiltonian takes the form

$$H = \tfrac{1}{2}(p - gA)^2 + \tfrac{1}{2}\sigma \cdot \operatorname{curl} A, \tag{5.1}$$

where g is a coupling constant and $\frac{1}{2}\sigma$ is the spin operator of the spin $\frac{1}{2}$ particle. Note that the second term on the right-hand side of (5.1) does not involve a commutator, since $A_1 = A_2 = 0$. Using explicit formulas for τ_a and σ_μ we obtain that

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}p_3^2 - gp_3 A_3 + i\sqrt{2}\, g \begin{pmatrix} 0 & 0 & 0 & \partial A_3^+ \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\bar{\partial} A_3^- & 0 & 0 & 0 \end{pmatrix}$$
$$+ g^2 A_3^- A_3^+, \tag{5.2}$$

where $A_3^- = (A_3^+)^*$, and $\partial = \frac{1}{2}(\partial/\partial x_1 - i\partial/\partial x_2)$, $\bar{\partial} = \frac{1}{2}(\partial/\partial x_1 + i\partial/\partial x_2)$. p_3 is a constant of motion. Changing to a reference frame in which $p_3 = 0$ and rearranging the components, we can rewrite (5.2) in the form (4.9) with $A_3^+ = -\sqrt{2}\,\partial V$.

5.2. *Interpretation of Holomorphic Quantum Mechanics in Pion Physics*

The motion of nucleons in an external pion field arises in the study of nuclear matter interacting with a π condensate (see discussion and references in [8]). We show that holomorphic quantum mechanics describes such motion.

The pseudoscalar pion–nuclear coupling is described phenomenologically by the chiral Yukawa–Lagrangian $\mathscr{L} = g\bar{\psi}(x)\gamma_5 \tau \cdot \pi(x)\psi(x)$, where $(\bar{\psi}, \psi)$ is the isodoublet of the nucleon field and $\pi = (\pi_1, \pi_2, \pi_3)$ is the real pion field. The field $\pi^{(0)} = \pi_3$ corresponds to neutral pions, while $\pi^{(\pm)} = (1/\sqrt{2})(\pi_1 \pm i\pi_2)$ describe the charged pions. Also $\tau = (\tau_1, \tau_2, \tau_3)$ are the isotopic Pauli matrices. In the nonrelativistic approximation the Hamiltonian takes the form

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}g\sigma \cdot \nabla(\tau \cdot \pi(x)) + \tfrac{1}{2}g^2 \pi(x)^2. \tag{5.3}$$

Consider a pion field such that (i) $\pi_3 = 0$ (no neutral pions), (ii) π_1 and π_2 depend only on x_1 and x_2, (iii) $\pi^{(+)}$ is holomorphic in $z = x_1 + ix_2$. Then H takes the form

$$H = -2\partial\bar{\partial} + \sqrt{2}\, g \begin{pmatrix} 0 & 0 & 0 & \partial\pi^{(+)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{\partial}\pi^{(-)} & 0 & 0 & 0 \end{pmatrix} + g^2 \pi^{(-)}\pi^{(+)} \tag{5.4}$$

Identifying $\pi^{(+)}$ with $-i\sqrt{2}\,\partial U$ and rearranging the components we see that (5.4) is equivalent to (4.9).

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