Hermitian symmetric superspaces of type IV

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A construction of the Hermitian symmetric superspaces which are the supermanifold analogs of the Cartan domains of type IV is presented herein. Natural generalizations of the Jordan triple product and Bergman operator for the superdomains are defined, and their properties are studied.

I. INTRODUCTION

Cartan superdomains are natural $\mathbb{Z}_2$-graded (super)generalizations of Cartan domains. The latter arise in the theory of Hermitian symmetric spaces:[1] each irreducible Hermitian symmetric space of noncompact type is equivalent to a Cartan domain. Cartan domains form four infinite series of "classical domains" and two "exceptional domains." Each Cartan domain is a symplectic manifold and thus the phase space of a mechanical system. A general framework for quantization of all Cartan domains was presented in Ref. 2.

Explicit constructions of the Cartan superdomains of types I–III ("the matrix superdomains") were presented in Ref. 3. This work also contained a general framework for the quantization of these domains. It was based on a class of deformed measures on the superdomain which satisfied a positivity property similar to the reflection positivity of Euclidean field theory and statistical mechanics.

We consider in this work a family of homogeneous supermanifolds based on the type IV Cartan domains, which we call the Cartan superdomains of type IV. Our goal is to present an explicit construction of these superdomains and to exhibit their properties as analogs of the ordinary domains. We base our discussion on the supergeneralizations of the Jordan triple product, the Bergman operator, and the Jordan triple determinant. These objects are determined by applying the constructions based on ordinary Lie algebras to the appropriate Lie superalgebras. We believe that there is a rich theory behind the concept of a super-Jordan triple product which should lead to a general theory of Cartan superdomains, just the way the theory of ordinary Jordan triple products provides a natural framework for studying the Cartan domains.[4] We should remark here that the type IV superdomains do not satisfy the positivity property of Ref. 3, and so their quantization is an open problem.

The definition of a supermanifold which we adopt in this work is that of Kostant–Berezin–Leites,[5–7] enhanced by the use of the projective tensor products as in Ref. 8. Recall that a smooth supermanifold $\mathscr{M}$ is a ringed space $(M,\mathscr{O}_M)$, where M is an ordinary smooth manifold (called the base of $\mathscr{M}$), and where $\mathscr{O}_M$ is a sheaf of supercommutative algebras (over $\mathbb{R}$) satisfying the following conditions:

(i) the quotient sheaf $\mathscr{O}_M/[\mathscr{O}_{M,1}+(\mathscr{O}_{M,1})^2]$, where $\mathscr{O}_{M,1}$ is the odd part of $\mathscr{O}_M$, is isomorphic to the sheaf of smooth functions on the base M;

(ii) every point of M has a neighborhood U such that

$$\mathscr{O}_M|U \cong C^\infty(U) \otimes \wedge(E), \tag{1.1}$$

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where $\wedge(E)$ is the Grassmann algebra over a finite-dimensional real vector space E. We let $C^\infty(\mathscr{M})$ denote the superalgebra of global sections of $\mathscr{O}_M$ and refer to its elements as smooth functions on $\mathscr{M}$. A set of generators of $C^\infty(\mathscr{M})$ will often be referred as the coordinates of a "point on $\mathscr{M}$." The definition of a complex supermanifold is analogous to the real case. The pair $(n_0|n_1)$, where $n_0=\dim_{\mathbb{C}} M$, $n_1=\dim_{\mathbb{C}} E$, is called the (complex) dimension of $\mathscr{M}$. We equip each $\mathscr{O}_M(U)$ with the usual topology of a Frechet space. Then $\mathscr{O}_M$ becomes a sheaf of nuclear Frechet algebras. A morphism in the category of supermanifolds is a pair $(\varphi,\varphi^{\#})$ where $\varphi:M\rightarrow N$ is a smooth map of the base manifolds and where $\varphi^{\#}:\mathscr{O}_N\rightarrow\varphi_*\mathscr{O}_M$ is a continuous map of sheaves of algebras over N ($\varphi_*\mathscr{O}_M$ denotes the direct image of $\mathscr{O}_M$ under φ). A direct product $\mathscr{M}\times\mathscr{N}$ of two supermanifolds is a product object in the category of supermanifolds. Clearly, $\mathscr{M}\times\mathscr{N}=(M\times N,\mathscr{O}_M\hat{\otimes}_\pi\mathscr{O}_N)$, where $\hat{\otimes}_\pi$ is the completed projective tensor product.

This article is organized as follows. In Sec. II we give a brief review of our conventions for superlinear algebra. Section III contains the basic construction of the noncompact type IV superdomains, including the super-Harish–Chandra map, the super-Jordan triple, and the super-Bergman operators. In Sec. IV we compute the triple determinant and the basic properties of these objects. In Sec. V we discuss the alternatives available in defining type IV superdomains, involving two possible kinds of involution for the complex conjugation.

II. SOME SUPERLINEAR ALGEBRA CONVENTIONS

This article involves a good deal of explicit computations with supermatrices, and so it is useful to review here our conventions. For the most part we follow the conventions of Ref. 6. For ordinary matrices, which will typically be denoted by lower case Roman letters, we use the standard notations of $\bar{a}$ and a^t to denote the complex conjugate and transpose. Matrices with purely odd entries will be denoted by lower case Greek letters, and conjugation and transposition will be defined just as for ordinary matrices. Note, however, that

$$\overline{\alpha\beta}=-\bar{\alpha}\bar{\beta},\quad (\alpha\beta)^t=-\beta^t\alpha^t. \tag{2.1}$$

Capital Roman letters will denote supermatrices. We use $*$ to denote the Hermitian adjoint for these cases.

An $m|n\times k|l$ supermatrix has the form

$$A=\begin{matrix} & k & l \\ m & a & \alpha \\ n & \beta & b\end{matrix}\;\text{ i.e. }\; A=\begin{pmatrix} a & \alpha \\ \beta & b\end{pmatrix}, \tag{2.2}$$

where a and b are ordinary matrices and α and β have purely odd entries. If $l=0$ we will write $m|n\times k$ for the dimension, and if $n=0$ the dimension will be $m\times k|l$, i.e., single dimensions always refer to an even component. The superanalogs of conjugation and transposition are defined as follows:

$$A^c:=\begin{pmatrix} \bar{a} & -\bar{\alpha} \\ \bar{\beta} & \bar{b}\end{pmatrix}, \tag{2.3}$$

$$A^T:=\begin{pmatrix} a^t & \beta^t \\ -\alpha^t & b^t\end{pmatrix}. \tag{2.4}$$

Note that $T^2 \neq 1$. The Hermitian adjoint of a supermatrix is given by $A^* := (A^c)^T$. We use the same symbol as for ordinary matrices because the same transformation is performed

$$A^* = \begin{pmatrix} a^* & \beta^* \\ \alpha^* & b^* \end{pmatrix} \tag{2.5}$$

The Berezinian[6] of a square supermatrix is defined by the formula

$$\mathrm{Ber}\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} := \frac{\det(a - \alpha b^{-1}\beta)}{\det b} \tag{2.6}$$

Sometimes we find it convenient to write supermatrices in a nonstandard form

$$\gamma = \begin{matrix} & n|q & m \\ n|q & A & B \\ m & C & D \end{matrix}, \tag{2.7}$$

where A, B, C, and D are subsupermatrices. In this case the Berezinian is

$$\mathrm{Ber}\,\gamma = \det D\,\mathrm{Ber}(A - BD^{-1}C). \tag{2.8}$$

For convenience we state here a formula for the inverse of a (super)matrix which we will use frequently. For any ordinary matrix or supermatrix in block form, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}. \tag{2.9}$$

III. EVEN SUPERDOMAINS OF TYPE IV

In this section we describe the main object of our study, namely, the type IV Cartan superdomains. The ordinary n-dimensional Cartan domain of type IV is defined to be the following space:

$$D_n := \{z \in \mathrm{Mat}_{n,1}(\mathbb{C}) : 1 - z^*z > 0, 1 - 2z^*z + |z^tz|^2 > 0\} \cong \mathrm{SO}_o(n,2)/\mathrm{SO}(n) \times \mathrm{SO}(2), \tag{3.1}$$

where the subscript o on SO denotes the identity component. The covering group of $\mathrm{Aut}(D_n)$, the group of holomorphic automorphisms of D_n, is $\mathrm{SO}(n,2)$. We will write $\gamma \in \mathrm{SO}(n,2)$ in block form

$$\gamma = \begin{matrix} & n & 2 \\ n & a & b \\ 2 & c & d \end{matrix}, \tag{3.2}$$

with a, b, c, and d real matrices of dimensions as indicated, such that

$$a^ta - c^tc = I_n, \quad a^tb = c^td, \quad d^td - b^tb = I_2. \tag{3.3}$$

For $z \in D_n$ we define an n-vector $\xi_1(z)$ and a two-vector $\xi_2(z)$ by

$$\xi_1(z) := z,$$

$$\xi_2(z) := \frac{1}{2}\begin{pmatrix} z^t z + 1 \\ i(z^t z - 1) \end{pmatrix}. \tag{3.4}$$

The action of γ on D_n is

$$\gamma:\ z \mapsto \frac{a\xi_1(z) + b\xi_2(z)}{(1,i)[c\xi_1(z) + d\xi_2(z)]}. \tag{3.5}$$

This action can be understood best in terms of the Harish–Chandra embedding $\psi: D_n \to \mathbb{C}P^{n+1}$, where $\mathbb{C}P^{n+1}$ is the complex projective space, defined as the composition of maps

$$D_n \to \mathbb{C}^{n+2} \to \mathbb{C}P^{n+1},$$

$$z \to \xi(z) \to [\xi(z)],$$

where

$$\xi(z) := \begin{pmatrix} \xi_1(z) \\ \xi_2(z) \end{pmatrix}$$

and where $[\xi(z)]$ denotes the equivalence class of $\xi(z)$. The first of these maps takes D_n into the set $\{\xi \in \mathbb{C}^{n+2}: \xi_1^T \xi_1 - \xi_2^T \xi_2 = 0\}$, which is invariant under the natural action of $SO(n,2)$.

A type IV Cartan superdomain is a supermanifold $\mathscr{D}_{n|q} := (D_n, \mathscr{O})$, where $\mathscr{O}$ is the sheaf of superalgebras on D_n whose space of global sections is

$$C^\infty(\mathscr{D}_{n|q}) := C^\infty(D_n) \otimes \wedge(\mathbb{C}^q), \tag{3.6}$$

where the integer q is even. We organize the standard generators of $\wedge(\mathbb{C}^q)$ into $q \times 1$ matrices $\theta = \{\theta_j\}$ and $\bar\theta = \{\bar\theta_j\}$. The "points" of $\mathscr{D}_{n|q}$ are then represented by the supermatrices

$$Z = \begin{pmatrix} z \\ \theta \end{pmatrix}. \tag{3.7}$$

We require that $\theta_j \to \bar\theta_j$ defines an involution of the first kind (i.e., its square is the identity map). The adjective "even" in the section title refers to this property. Observe that $\mathscr{D}_{n|q}$ has a natural structure of a complex supermanifold. Let $\mathrm{Aut}(\mathscr{D}_{n|q})$ denote the supergroup of superholomorphic automorphisms of $\mathscr{D}_{n|q}$.

The type IV superdomain admits an action of the Lie supergroup $SO(n|q,2)$, which is defined as follows. The base manifold is $SO(n,2) \times Sp(q,\mathbb{R})$ and the structure sheaf is generated by γ_{jk}, $1 \leqslant j, k \leqslant n+2+q$, with the following parity assignments:

$$p(\gamma_{jk}) = \begin{cases} 0, & \text{if } 1 \leqslant j,k \leqslant n+2 \quad \text{or} \quad n+2 < j,k \leqslant n+2+q, \\ 1, & \text{otherwise} \end{cases} \tag{3.8}$$

and with the following relations. We write γ as a block supermatrix

$$\gamma=\begin{matrix} & n & 2 & q \\ n & a & b & \rho \\ q & c & d & \delta \\ q & \alpha & \beta & e \end{matrix}, \tag{3.9}$$

where a, b, c, d, and e are even matrices and α, β, ρ, and δ are odd matrices of the dimensions indicated. We require that γ be real in the supermatrix sense

$$\gamma^c=\gamma. \tag{3.10}$$

In addition, we have the requirements that

$$\text{Ber }\gamma=1 \tag{3.11}$$

and that

$$\gamma^T M\gamma=M, \tag{3.12}$$

where

$$M=\begin{pmatrix} I_n & 0 & 0 \\ 0 & -I_2 & 0 \\ 0 & 0 & \tau_q \end{pmatrix} \tag{3.13}$$

and τ_q is given by

$$\tau_q:=\begin{pmatrix} 0 & iI_{q/2} \\ -iI_{q/2} & 0 \end{pmatrix}. \tag{3.14}$$

Note that $\tau=\tau^*=\tau^{-1}$. Equation (3.12) is equivalent to the following set of relations:

$$\begin{gathered} a^ta-c^tc+\alpha^t\tau\alpha=I_n, \quad a^tb-c^td+\alpha^t\tau\beta=0, \\ a^t\rho-c^t\delta+\alpha^t\tau e=0, \quad b^tb-d^td+\beta^t\tau\beta=-I_2, \\ b^t\rho-d^t\delta+\beta^t\tau e=0, \quad -\rho^t\rho+\delta^t\delta+e^t\tau e=\tau \end{gathered} \tag{3.15}$$

and Eq. (3.10) is equivalent to the relations

$$\begin{gathered} a=\bar{a}, \quad b=\bar{b}, \quad c=\bar{c}, \quad d=\bar{d}, \quad \bar{e}=e, \\ \bar{\alpha}=\alpha, \quad \bar{\beta}=\beta, \quad \bar{\rho}=-\rho, \quad \bar{\delta}=-\delta. \end{gathered} \tag{3.16}$$

The Hopf algebra structure is defined in the obvious way.

In order to connect with the framework of ordinary type IV domains, we will find it convenient to write γ in the nonstandard form

$$\gamma=\begin{matrix} & n|q & 2 \\ n|q & A & B \\ 2 & C & D \end{matrix}, \tag{3.17}$$

where

$$A=\begin{pmatrix} a & \rho \\ \alpha & e \end{pmatrix}, \quad B=\begin{pmatrix} b \\ \beta \end{pmatrix}, \quad C=(c,\delta), \quad D=d. \tag{3.18}$$

In terms of these submatrices, the conditions (3.11), (3.12), and (3.10) are expressed as

$$\begin{gathered} \mathrm{Ber}(A-BD^{-1}C)\det D=1, \\ A^TSA-C^TC=S, \quad A^TSB=C^TD, \quad D^TD-B^TSB=I_2, \\ A^c=A, \quad B^c=B, \quad C^c=C, \quad D^c=D, \end{gathered} \tag{3.19}$$

where

$$S:=\begin{pmatrix} I_n & 0 \\ 0 & \tau_q \end{pmatrix}. \tag{3.20}$$

Note that $S=S^*=S^{-1}$.

To define the action of this supergroup on $\mathscr{D}_{n|q}$, we construct the super-Harish–Chandra embedding. This is a morphism

$$\psi:\ \mathscr{D}_{n|q}\rightarrow \mathbb{C}P^{n+1|q}, \tag{3.21}$$

where $\mathbb{C}P^{n+1|q}$ denotes the projective superspace.[7] Since $\mathrm{SO}(n|q,2)$ acts naturally on $\mathbb{C}P^{n+1|q}$, we will then define the action of $\gamma\in \mathrm{SO}(n|q,2)$ on $\mathscr{D}_{n|q}$ by the composition of morphisms of supermanifolds $\psi^{-1}\circ\gamma\circ\psi$.

Let $\mathfrak{g}$ denote the Lie superalgebra of $\mathrm{SO}(n|q,2)$. We decompose $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, where $\mathfrak{k}$ is the Lie superalgebra of the isotropy subsupergroup of the origin, $\mathrm{SO}(n|q)\times \mathrm{SO}(2)$. An arbitrary element $x\in\mathfrak{g}$ is represented by a real matrix

$$x=\begin{pmatrix} A & B \\ B^TS & D \end{pmatrix}, \tag{3.22}$$

where $A^T=-SAS$ and $D^T=-D$. For $y\in\mathfrak{p}$ we have

$$y=\begin{pmatrix} 0 & B \\ B^TS & 0 \end{pmatrix}. \tag{3.23}$$

We identify the supervector space $\mathbb{C}^{n|q}$ in which $\mathscr{D}_{n|q}$ is embedded with $\mathfrak{p}$ by the map $p{:}Z\mapsto y$, where y is of the form (3.23) with $B=(Z+\bar{Z},\ i(Z-\bar{Z}))$.

The superalgebra $\mathfrak{p}$ is naturally identified with $T_0\mathscr{D}_{n|q}$, the tangent space at 0. Thus the almost complex structure of $\mathscr{D}_{n|q}$ acts as a transformation $J{:}\mathfrak{p}\rightarrow\mathfrak{p}$. For y as in Eq. (3.23) this J takes $B=(b_1,b_2)$ to $(b_2,-b_1)$, corresponding to $Z\mapsto iZ$ in the supermanifold. The complexification $\mathfrak{p}^{\mathbb{C}}$ decomposes into $\pm i$ eigenspaces of J, $\mathfrak{p}^{\mathbb{C}}=\mathfrak{p}_-\oplus\mathfrak{p}_+$. There is a canonical isomorphism $\mathfrak{p}\cong\mathfrak{p}_+$, which takes $p(Z)\in\mathfrak{p}$ to $p_+(Z)$ given by

$$p_+(Z)=\begin{pmatrix} 0 & B \\ B^TS & 0 \end{pmatrix}, \quad B=(Z,\ iZ). \tag{3.24}$$

The Lie subsuperalgebra $\mathfrak{p}_+$ acts on $\mathbb{C}^{n+2|q}$ and $\mathbb{C}P^{n+1|q}$ through the exponential map. We choose a base point $x_0 \in \mathbb{C}^{n+2|q}$ such that $[x_0]$, the class of x_0, is preserved by the action of $\mathfrak{p}_-$. Following the usual convention for ordinary domains,[9] we set

$$x_0 = \begin{matrix} n|q \\ 2 \end{matrix}\begin{pmatrix} 0 \\ \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} \end{pmatrix}. \tag{3.25}$$

We now define the Harish–Chandra map as

$$\psi(Z) = [\exp p_+(Z) x_0] \tag{3.26}$$

and we can easily compute its explicit form. The exponential of $p_+(Z)$ has the form

$$\exp p_+(Z) = \begin{pmatrix} 1 & B \\ B^T S & 1 + \frac{1}{2} B^T S B \end{pmatrix}, \tag{3.27}$$

where $B = (Z,\ iZ)$ (note that $BB^T = 0$). Thus

$$\psi(Z) = [\xi(Z)], \tag{3.28}$$

where

$$\xi_1(Z) = Z, \quad \xi_2(Z) = \frac{1}{2}\begin{pmatrix} Z^T S Z + 1 \\ i(Z^T S Z - 1) \end{pmatrix}, \quad \xi(Z) = \begin{pmatrix} \xi_1(Z) \\ \xi_2(Z) \end{pmatrix}. \tag{3.29}$$

The inverse of ψ is given by

$$\psi^{-1}([\xi]) = \frac{\xi_1}{(1,i)\xi_2}. \tag{3.30}$$

The action of $\mathrm{SO}(n|q,2)$ on $\mathscr{D}_{n|q}$ by superholomorphic automorphisms is thus given by

$$Z \mapsto Z' := \frac{A\xi_1(Z) + B\xi_2(Z)}{(1,i)[C\xi_1(Z) + D\xi_2(Z)]}. \tag{3.31}$$

For future reference, we note that

$$\begin{aligned} \xi_1(Z') &= \frac{A\xi_1(Z) + B\xi_2(Z)}{(1,i)[C\xi_1(Z) + D\xi_2(Z)]}, \\ \xi_2(Z') &= \frac{C\xi_1(Z) + D\xi_2(Z)}{(1,i)[C\xi_1(Z) + D\xi_2(Z)]}. \end{aligned} \tag{3.32}$$

Proposition III.1. The above morphism (3.31) defines a transitive action of $\mathrm{SO}(n|q,2)$ *on* $\mathscr{D}_{n|q}$. *Furthermore*

$$\mathscr{D}_{n|q} \cong \mathrm{SO}_0(n|q,2)/\mathrm{SO}(n|q) \times \mathrm{SO}(2). \tag{3.33}$$

Proof: Clearly $\mathrm{SO}(n|q,2)$ acts transitively on the base of $\mathscr{D}_{n|q}$, since it contains the subgroup $\mathrm{SO}(n,2)$ which acts as the group of holomorphic automorphisms on the base. The covering space in $\mathbb{C}^{n+2|q}$ of the image of ψ in $\mathbb{C}P^{n+1|q}$ is defined by the algebraic condition

$$\xi^T M \xi = 0. \tag{3.34}$$

This is the only condition satisfied by the fermionic generators, and it is clearly preserved under the action of $\mathrm{SO}(n|q,2)$ because of Eq. (3.12). The relation (3.33) holds because the isotropy group of 0 consists of matrices

$$\gamma = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \tag{3.35}$$

where $A^c = A$, $A^T S A = S$, $A^c = A$, $D^c = D$, and $D^T D = I_2$. □

Since the superdomain is a homogeneous superspace, it is naturally parametrized by elements of the group. That is, to each $\gamma \in \mathrm{SO}(n|q,2)$, we associate the point $\gamma(0)$. If

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.36}$$

then the relation between the group coordinates and our parametrization in terms of Z takes the form

$$Z = \frac{1}{2h} B \begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{3.37}$$

where

$$h := \frac{1}{2} (1,i) D \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{3.38}$$

Note that the parametrization is not bijective; group elements which are related by conjugation with the isotropy group of zero will correspond to the same point Z.

Remark: The usual duality between Hermitian symmetric spaces of compact and noncompact-type carries over to the case of Hermitian symmetric superspaces. Corresponding to the type IV superdomain we have defined is the homogeneous supermanifold

$$\mathrm{Osp}(n+2|q)/\mathrm{SO}(n|q) \times \mathrm{SO}(2), \tag{3.39}$$

which has compact base. Defining $\mathfrak{p}_+$ as above, the orbit of x_0 under the action of $\exp(i\mathfrak{p}_+)$ generates a supermanifold isomorphic to Eq. (3.39).

We now discuss the super-Jordan triple product for the type IV Cartan superdomains. We view the Lie superalgebra $\mathfrak{p}$ as the space of holomorphic vector fields on $\mathscr{D}_{n|q}$. Under this identification, we associate to the element of $\mathfrak{p}$

$$p(Z) = \begin{pmatrix} 0 & B \\ B^T S & 0 \end{pmatrix},$$

where $B = (Z + \bar{Z},\ i(Z - \bar{Z}))$, the supervector field $X_Z \in \mathfrak{p}$ which is the infinitesimal transformation determined by the action of $\exp p(Z)$. From formula (3.31) we see that, to the leading order,

$$\exp p(Z) Y = \frac{Y + B\xi_2(Y)}{1 + (1,i) B^T S Y} = Y + B\xi_2(Y) - Y(1,i) B^T S Y \tag{3.40}$$

and thus

$$X_Z = B\xi_2(Y) - Y(1,i)B^TSY = Z + (Y^TSY)\bar{Z} - 2(Z^*SY)Y. \tag{3.41}$$

We now follow the procedure familiar from the theory of Cartan domains.[4] From the supervector field X_Z (which is defined by the property that it is equal to Z at the origin), we define $Q_Z(Y) := Z - X_Z$, which is quadratic in Y and antilinear in Z. The super-Jordan triple product is defined as the polarization of $Q_Z(Y)$

$$\{WZ^*Y\} := \tfrac{1}{2}[Q_Z(W+Y) - Q_Z(W) - Q_Z(Y)] \tag{3.42}$$

[note that $Q_Z(Y) = \{YZ^*Y\}$]. Explicitly, from Eq. (3.41) we read off that

$$\{YZ^*Y\} = 2(Z^*SY)Y - (Y^TSY)\bar{Z} \tag{3.43}$$

and thus

$$\{ZW^*Y\} = (W^*SY)Z + (W^*SZ)Y - (Z^TSY)\bar{W}. \tag{3.44}$$

The super-Jordan triple product has the familiar properties of a Jordan triple product:

Theorem III.2: $\{WZ^*Y\}$ *has the following properties:*

$$\{WZ^*Y\} = \{YZ^*W\} \tag{3.45}$$

and

$$\{ZY^*\{UV^*W\}\} - \{UV^*\{ZY^*W\}\} = \{\{ZY^*U\}V^*W\} - \{U\{YZ^*V\}^*W\}. \tag{3.46}$$

Furthermore, the reduction of $\{WZ^*Y\}$ *to* D_n *coincides with the ordinary Jordan triple product.*

The proof of this theorem is straightforward and we omit the details.

Associated with $\{WZ^*Y\}$ is the super-Bergman operator $B(Z,W)$ which is defined by the exponential of the adjoint action of the vector field Q_W on the constant vector field Y, evaluated at the point Z (Ref. 10)

$$B(Z,W)Y := \exp \operatorname{ad}_{Q_W}(Y)|_Z. \tag{3.47}$$

Using Eq. (3.46), we find

$$B(Z,W)Y = Y - 2\{ZW^*Y\} + \{Z\{WY^*W\}^*Z\}. \tag{3.48}$$

From Eq. (3.44) we obtain the explicit expression

$$B(Z,W)Y = Y - 2(W^*SY)Z - 2(W^*SZ)Y + 2(Z^TSY)\bar{W} + (\overline{W^TSW})(Z^TSZ)Y - 2(\overline{W^TSW}) \times (Y^TSZ)Z - 2(W^*SY)(Z^TSZ)\bar{W} + 4(W^*SY)(W^*SZ)Z. \tag{3.49}$$

The significance of the Bergman operator for ordinary domains lies in its transformation properties. We will demonstrate in the next section that the super-Bergman operator has analogous properties.

IV. PROPERTIES OF THE SUPER-BERGMAN OPERATOR

In this section we study the properties of the super-Bergman operator. In particular, we compute its Berezinian and find that it has the form $N(Z,\bar{W})^{n-q}$, where $N(Z,W)$, the "supertriple determinant," is a polynomial in Z and $\bar{W}$. We can thus associate to a Cartan superdomain its genus p which turns out to be the difference of the genus of the underlying Cartan domain ($p_0=n$) and the "fermionic genus" ($p_1=q$).

We start with the following computational result.

Theorem IV.1: *If* $\gamma\in \mathrm{SO}(n|q,2)$, *then*

$$\operatorname{Ber}\gamma'(Z)=[(1,i)(C\xi_1(Z)+D\xi_2(Z))]^{-(n-q)}. \tag{4.1}$$

Proof: We compute the matrix of left derivatives at $Z=0$ as follows. We have

$$\frac{\partial}{\partial Z_l}(AZ)_k=A^T{}_{lk} \tag{4.2}$$

and

$$\frac{\partial}{\partial Z_l}(1,i)CZ\left[B\begin{pmatrix}1\\ i\end{pmatrix}\right]_k=(-1)^l[(1,i)C]_l\left[B\begin{pmatrix}1\\ i\end{pmatrix}\right]_k=\left[B\begin{pmatrix}1\\ i\end{pmatrix}(1,i)C\right]^T_{lk}. \tag{4.3}$$

Thus the derivative of $\gamma(Z)$ at zero is

$$\gamma'(0)=\left[\frac{A}{h}-\frac{1}{2h^2}B\begin{pmatrix}1\\ -i\end{pmatrix}(1,i)C\right]^T,$$

with h defined as in Eq. (3.38).

Since the Berezinian is invariant under the supertranspose, for $\operatorname{Ber}\gamma'(0)$ we have

$$\begin{aligned}\operatorname{Ber}\gamma'(0)=\operatorname{Ber}\left[\frac{A}{h}-\frac{1}{2h^2}B\begin{pmatrix}1\\ -i\end{pmatrix}(1,i)C\right]&=h^{n-q}\operatorname{Ber}A\operatorname{Ber}\left[I_{n|q}-\frac{1}{2h}A^{-1}B\begin{pmatrix}1\\ -i\end{pmatrix}(1,i)C\right]\\ &=h^{n-q}\operatorname{Ber}A\left[1-\frac{1}{2h}(1,i)CA^{-1}B\begin{pmatrix}1\\ -i\end{pmatrix}\right]\\ &=\frac{1}{2}h^{n-q-1}\operatorname{Ber}A(1,i)(D-CA^{-1}B)\begin{pmatrix}1\\ -i\end{pmatrix}.\end{aligned} \tag{4.4}$$

From the conditions (3.19) we extract the fact that

$$A^TS(A-BD^{-1}C)=S, \tag{4.5}$$

which implies that $\operatorname{Ber}A=\operatorname{Ber}(A-BD^{-1}C)^{-1}=\det D$. We also see directly from Eq. (3.12) that

$$D^TC=B^TSA. \tag{4.6}$$

Using this together with Eq. (3.19), we find that

$$D-CA^{-1}B=(D^T)^{-1}. \tag{4.7}$$

Returning to Eq. (4.4), we can now write

$$\mathrm{Ber}\ \gamma'(0)=\tfrac{1}{2}h^{n-q-1}\det D(1,i)(D^T)^{-1}\begin{pmatrix}1\\-i\end{pmatrix}. \tag{4.8}$$

Since D is just a 2×2 matrix, it is easy to check by direct computation that

$$(1,i)(D^T)^{-1}\begin{pmatrix}1\\-i\end{pmatrix}=\frac{2h}{\det D}, \tag{4.9}$$

which yields

$$\mathrm{Ber}\ \gamma'(0)=h^{n-q}. \tag{4.10}$$

To study the Berezinian at a point $W\neq 0$, we decompose $\gamma=\gamma_2\circ\gamma_1$, where

$$\gamma_1(Z)=0,\quad \gamma_2(0)=W. \tag{4.11}$$

We write the corresponding matrix blocks as

$$\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix},\quad \gamma_i=\begin{pmatrix}A_i & B_i\\ C_i & D_i\end{pmatrix},\quad i=1,2 \tag{4.12}$$

so that

$$\gamma=\begin{pmatrix}A_2A_1+B_2C_1 & A_2B_1+B_2D_1\\ C_2A_1+D_2C_1 & C_2B_1+D_2D_1\end{pmatrix}. \tag{4.13}$$

Note that Eq. (3.32) and the first condition of Eq. (4.11) imply

$$A_1\xi_1(Z)+B_1\xi_2(Z)=0,$$

$$C_1\xi_1(Z)+D_1\xi_2(Z)=\frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix}(1,i)[C_1\xi_1(Z)+D_1\xi_2(Z)]. \tag{4.14}$$

Using Eqs. (4.14) and (4.13) and applying the result (4.10), we can evaluate

$$[(1,i)(C\xi_1(Z)+D\xi_2(Z))]^{n-q}=[(1,i)((C_2A_1+D_2C_1)\xi_1(Z)+(C_2B_1+D_2D_1)\xi_2(Z))]^{n-q}$$

$$=\left[\frac{1}{2}(1,i)D_2\begin{pmatrix}1\\-i\end{pmatrix}(1,i)(C_1\xi_1(Z)+D_1\xi_2(Z))\right]^{n-q}$$

$$=\mathrm{Ber}\ \gamma_2'(0)[(1,i)(D_1-C_1A_1^{-1}B_1)\xi_2(Z)]^{n-q}. \tag{4.15}$$

The lower right matrix block of γ_1^{-1}, which we denote by $\widetilde{D}_1$, is given by $(D_1-C_1A_1^{-1}B_1)^{-1}$. Because $\gamma_1^{-1}(0)=Z$, we see from Eq. (3.32) that

$$\xi_2(Z)=\frac{\widetilde{D}_1\begin{pmatrix}1\\-i\end{pmatrix}}{(1,i)\widetilde{D}_1\begin{pmatrix}1\\-i\end{pmatrix}}. \tag{4.16}$$

By applying $(1,i)\tilde{D}_1^{-1}$ to this relation we find that

$$(1,i)(D_1-C_1A_1^{-1}B_1)\xi_2(Z)=\left[\frac{1}{2}(1,i)\tilde{D}_1\begin{pmatrix}1\\-i\end{pmatrix}\right]^{-1}. \tag{4.17}$$

Applying Eq. (4.10) again we reduce Eq. (4.15) to

$$\frac{\operatorname{Ber}\gamma_2'(0)}{\operatorname{Ber}(\gamma_1^{-1})'(0)} \tag{4.18}$$

and since by the chain rule $\gamma_1'(Z)(\gamma_1^{-1})'(0) = I$, we conclude that

$$[(1,i)(C\xi_1(Z)+D\xi_2(Z))]^{n-q}=\operatorname{Ber}\gamma_1'(Z)\gamma_2'(0)=\operatorname{Ber}\gamma'(Z). \tag{4.19}$$

□

We conclude from the preceding theorem that the superholomorphic polynomial

$$a_\gamma(Z):=(1,i)[C\xi_1(Z)+D\xi_2(Z)] \tag{4.20}$$

satisfies

$$\operatorname{Ber}\gamma'(Z)=a_\gamma(Z)^{-(n-q)}. \tag{4.21}$$

We now define the supertriple determinant

$$N(Z,W):=1-2W^*SZ+\overline{W^TSW}Z^TSZ. \tag{4.22}$$

Lemma IV.2: We have the equality

$$N(Z,W)=-2\xi(W)^*M\xi(Z), \tag{4.23}$$

where $\xi(Z)$ is defined in Eq. (3.29).

Proof: We must check that

$$\tfrac{1}{2}N(Z,W)=\xi_2(W)^*\xi_2(Z)-\xi_1(W)^*S\xi_1(Z). \tag{4.24}$$

To this end we compute

$$\begin{aligned}\xi_2(W)^*\xi_2(Z)&=\tfrac{1}{4}\{(Z^TSZ+1)(\overline{W^TSW}+1)+(Z^TSZ-1)(\overline{W^TSW}-1)\}\\&=\tfrac{1}{2}(1+Z^TSZ\overline{W^TSW}).\end{aligned} \tag{4.25}$$

□

Proposition IV.3: For the type IV superdomains

$$N(\gamma(Z),\gamma(W))=a_\gamma(Z)N(Z,W)\overline{a_\gamma(W)}. \tag{4.26}$$

Proof: From Lemma IV.2, Eq. (3.32), and the fact that

$$\gamma^*M\gamma=M, \tag{4.27}$$

we see that

$$N(Z',W')=\frac{N(Z,W)}{(1,i)[C\xi_1(Z)+D\xi_2(Z)](1,i)[C\xi_1(W)+D\xi_2(W)]}. \tag{4.28}$$

The proposition then follows from Theorem IV.1. □

Theorem IV.4: *Let $\gamma_Z \in Aut(\mathscr{D}_{n|q})$ be such that $\gamma_Z(0)=Z$. Then*

$$B(Z,Z)=\gamma_z{}'(0)S\gamma_z{}'(0)^*S. \tag{4.29}$$

Proof: If we write

$$\gamma_z=\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.30}$$

then γ will send 0 to Z provided that

$$\frac{1}{2h}B\begin{pmatrix} 1 \\ -i \end{pmatrix}=Z, \tag{4.31}$$

where h is defined in Eq. (3.38). We start by proving that

$$|h|^{-2}=1-2Z^*SZ+\overline{Z^TSZ}Z^TSZ. \tag{4.32}$$

Letting

$$D=\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \tag{4.33}$$

we have

$$|h|^2=\frac{1}{2}\det D+\frac{1}{4}\sum_{1<j<4} d_j^2 \tag{4.34}$$

since D is real. Using the relation from (3.19)

$$B^TSB=D^TD-I_2 \tag{4.35}$$

and (4.31), we see that

$$Z^TSZ=\frac{1}{4h^2}(1,-i)B^TSB\begin{pmatrix} 1 \\ -i \end{pmatrix}=\frac{1}{4h^2}[d_1^2+d_3^2-d_2^2-d_4^2-2i(d_1d_2+d_3d_4)]. \tag{4.36}$$

We thus obtain

$$\begin{aligned} Z^TSZ\overline{Z^TSZ}&=\frac{1}{16|h|^4}[(d_1^2+d_3^2-d_2^2-d_4^2)^2+4(d_1d_2+d_3d_4)^2] \\ &=\frac{1}{16|h|^4}\left[\left(\sum_j d_j^2\right)^2-4\det D^2\right] \\ &=\frac{1}{4|h|^2}\left[\sum_j d_j^2-2\det D\right]. \end{aligned} \tag{4.37}$$

Similarly, we have

$$Z^*SZ=\frac{1}{4|h|^2}(1,i)B^TSB\begin{pmatrix}1\\-i\end{pmatrix}=\frac{1}{4|h|^2}\left[-2+\sum_j d_j^2\right]. \tag{4.38}$$

Therefore,

$$Z^TSZ\overline{Z^TSZ}-2Z^*SZ=\frac{1}{4|h|^2}\left[\sum_j d_j^2-2\det D+4-2\sum_j d_j^2\right]=\frac{1}{|h|^2}-1, \tag{4.39}$$

which proves the assertion.

Now for the theorem we have

$$\gamma_z'(0)S\gamma_z'(0)^*S=\left[\frac{A}{h}-\frac{1}{2h^2}B\begin{pmatrix}1\\-i\end{pmatrix}(1,i)C\right]S\left[\frac{A^*}{\bar{h}}-\frac{1}{2\bar{h}^2}C^*\begin{pmatrix}1\\-i\end{pmatrix}(1,i)B^*\right]S. \tag{4.40}$$

Recall here that $\bar{C}=(\bar{c},-\bar{\delta})$, $\bar{B}=\binom{\bar{b}}{\bar{\beta}}$, $B^T=(b^t,\beta^t)$, $C^T=\binom{c^t}{-\delta^t}$. From the relation (3.12) we find

$$ASA^*=S+BB^*,\quad CSC^*=DD^*-I_2,\quad CSA^*=DB^*. \tag{4.41}$$

Hence we can rewrite Eq. (4.40) as

$$\frac{1}{|h|^2}(I_{n|q}+BB^TS)-\frac{1}{4|h|^4}B(U+U^*)B^TS+\frac{1}{4|h|^4}B\begin{pmatrix}1\\-i\end{pmatrix}(1,i)(DD^T-I_2)\begin{pmatrix}1\\-i\end{pmatrix}(1,i)B^TS, \tag{4.42}$$

where

$$U:=2\begin{pmatrix}1\\-i\end{pmatrix}\bar{h}(1,i)D. \tag{4.43}$$

Observe now that

$$(1,i)DD^T\begin{pmatrix}1\\-i\end{pmatrix}=\mathrm{tr}(DD^T)=\mathrm{tr}(D^TD)=(1,i)D^TD\begin{pmatrix}1\\-i\end{pmatrix}. \tag{4.44}$$

We can thus use $D^TD-B^TSB=I_2$ to reduce Eq. (4.40) to

$$\frac{1}{|h|^2}I_{n|q}+\frac{1}{4|h|^4}[B(4|h|^2I_2-U-U^*)B^TS]+\frac{1}{4|h|^4}B\begin{pmatrix}1\\-i\end{pmatrix}(1,i)(B^TSB)\begin{pmatrix}1\\-i\end{pmatrix}(1,i)B^TS. \tag{4.45}$$

Using the definitions of h and U, we easily compute that

$$4|h|^2I_2-(U+U^*)=4|h|^2\begin{pmatrix}0&-i\\i&0\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix}(1,i)D^TD\begin{pmatrix}1\\i\end{pmatrix}(1,-i)$$
$$-\frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix}(1,-i)D^TD\begin{pmatrix}1\\-i\end{pmatrix}(1,i). \tag{4.46}$$

This can be rewritten as

$$2|h|^2\left[\begin{pmatrix}1\\ i\end{pmatrix}(1,-i)-\begin{pmatrix}1\\ -i\end{pmatrix}(1,i)\right]-\frac{1}{2}\begin{pmatrix}1\\ -i\end{pmatrix}(1,i)B^TSB\begin{pmatrix}1\\ i\end{pmatrix}(1,-i)$$

$$-\frac{1}{2}\begin{pmatrix}1\\ i\end{pmatrix}(1,-i)B^TSB\begin{pmatrix}1\\ -i\end{pmatrix}(1,i). \tag{4.47}$$

We can apply Eq. (4.47) to express Eq. (4.45) solely in terms of B and h. After some minor simplification this yields

$$\gamma_z'(0)S\gamma_z'(0)^*S=\frac{1}{|h|^2}I_{n|q}+\frac{1}{2|h|^2}\left[B\begin{pmatrix}1\\ i\end{pmatrix}(1,-i)B^T-B\begin{pmatrix}1\\ -i\end{pmatrix}(1,i)B^TS\right]-\frac{1}{8|h|^4}$$

$$\times B\left[\begin{pmatrix}1\\ i\end{pmatrix}(1,-i)B^TSB\begin{pmatrix}1\\ -i\end{pmatrix}(1,i)+\begin{pmatrix}1\\ -i\end{pmatrix}(1,i)B^TSB\begin{pmatrix}1\\ i\end{pmatrix}(1,-i)\right]B^TS$$

$$+\frac{1}{4|h|^4}B\begin{pmatrix}1\\ -i\end{pmatrix}(1,i)B^TSB\begin{pmatrix}1\\ -i\end{pmatrix}(1,i)B^TS. \tag{4.48}$$

From Eq. (4.31) we have

$$B\begin{pmatrix}1\\ -i\end{pmatrix}=2hZ,\quad B\begin{pmatrix}1\\ i\end{pmatrix}=2\overline{hZ} \tag{4.49}$$

and thus, using this together with Eq. (4.32), we can rewrite Eq. (4.48) solely in terms of Z

$$\gamma_z'(0)S\gamma_z'(0)^*S=(1-2Z^*SZ+\overline{Z^TSZ}Z^TSZ)I_{n|q}+2\bar{Z}Z^TS-2ZZ^*S-2\bar{Z}(Z^TSZ)Z^*S$$

$$-2Z(\overline{Z^TSZ})Z^TS+4Z(Z^*SZ)Z^*S$$

$$=B(Z,Z). \tag{4.50}$$

□

Using Theorems IV.1 and IV.4 we can immediately conclude the following.

Theorem IV.5: *For type IV superdomains*

$$\text{Ber } B(Z,W)=N(Z,W)^{n-q}. \tag{4.51}$$

The above results allow us to apply the results of Sec. V of Ref. 3 to construct an invariant super-Kähler structure for the type IV superdomain. We state the result below and refer the reader to Ref. 3 for a full treatment.

Theorem IV.6: *The form*

$$\omega:=\sum_{k,l}(-1)^{\epsilon_k+1}d\bar{Z}_k\wedge dZ_l\frac{\partial^2}{\partial Z_l\partial\bar{Z}_k}\log N(Z,Z) \tag{4.52}$$

is an $\text{Aut}(\mathscr{D}_{n|q})$*-invariant super-Kähler form on* $\mathscr{D}_{n|q}$.

In this final subsection we exhibit an explicit form for γ_z. Define the $n|q\times 2$-dimensional matrix

$$X:=\frac{1}{1-\overline{Z^TSZ}Z^TSZ}((1-\overline{Z^TSZ})Z+(1-Z^TSZ)\bar{Z},\quad i(1+\overline{Z^TSZ})Z-i(1+Z^TSZ)\bar{Z}) \tag{4.53}$$

and the 2×2 matrix

$$D:=\frac{1}{2N(Z,Z)^{1/2}}\begin{pmatrix}-i(Z^TSZ-\overline{Z^TSZ}) & Z^TSZ+\overline{Z^TSZ}+2\\ Z^TSZ+\overline{Z^TSZ}-2 & i(Z^TSZ-\overline{Z^TSZ})\end{pmatrix}. \tag{4.54}$$

Choose a matrix A such that $A^c=A$ and

$$ASA^T=(S-SXX^TS)^{-1} \tag{4.55}$$

and set

$$\gamma_z=\begin{pmatrix}A & XD\\ X^TSA & D\end{pmatrix}. \tag{4.56}$$

It is easy to verify that $\gamma_z\in\mathrm{SO}(n|q,2)$ once we note that

$$XD=\frac{1}{N(Z,Z)^{1/2}}(-i(Z-\bar{Z}),Z+\bar{Z}) \tag{4.57}$$

and use the identities

$$\overline{Z^TSZ}=Z^*S\bar{Z},\quad Z^TS\bar{Z}=Z^*SZ. \tag{4.58}$$

The action of γ_z is given by

$$\gamma_z(W)=\frac{Z-(W^TSW)\bar{Z}+iN(Z,Z)^{1/2}AW}{1-\overline{Z^TSZ}W^TSW+iN(Z,Z)^{1/2}(1,i)X^TSAW} \tag{4.59}$$

so that in particular $\gamma_z(0)=Z$.

V. ODD SUPERDOMAINS OF TYPE IV

The definition of the type IV superdomain in Sec. III is not unique. While some of the choices made in the definition simply amount to different conventions and lead to equivalent spaces, there is a fundamental distinction between domains, based on whether complex conjugation is an involution of the first or second kind.[6] We have been using an involution of the first kind thus far in this article, defining the even Cartan superdomains of type IV. For such an involution the fermionic generators obey $\bar{\bar{\theta}}=\theta$. An involution of the second kind satisfies $\bar{\bar{\theta}}=-\theta$, i.e., its square is equal to the grading homomorphism Γ. Odd Cartan superdomains of type IV are superdomains with involutions of this type.

To make clear the various choices which are involved in defining $\mathscr{D}_{n|q}$, in this section we will study the situation in general and review the possibilities.

There are two conditions on group elements which define the group $\mathrm{SO}(n|q,2)$. These are the orthogonality property and the reality condition

$$\gamma^T M\gamma = M, \quad \gamma^c = N\gamma N^{-1}. \tag{5.1}$$

In order to obtain the correct base group, the matrices M and N must have the form

$$M = \begin{pmatrix} I_n & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & -I_2 \end{pmatrix}, \quad N = \begin{pmatrix} I_n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \tag{5.2}$$

where m and n are $q\times q$ matrices. The fundamental choice here is whether $NN^c = I$ or Γ. The former choice defines the even Lie superspheres, and the latter the odd. For the bulk of this article we have set $N=I$.

In terms of the submatrices of N, the choice of an involution of the first kind or of the second corresponds to the specification that $n\bar{n} = I_q$ (first kind) or $n\bar{n} = -I_q$ (second kind). The remaining choices for m and n are essentially trivial. That is, they produce spaces which are isomorphic under a linear change of variables. The only requirements on m and n are the the following consistency conditions:

$$m^t = -m, \quad m = -n^t \bar{m} n, \tag{5.3}$$

which are determined by Eq. (5.1). The first condition is obvious, and the second follows from taking the conjugate of the first relation of Eq. (5.1), observing that $(\gamma^T)^c = \Gamma\gamma^*\Gamma$, and then applying the second relation of Eq. (5.1).

The constructions of this article still apply, with only minor technical changes, for the construction of odd Lie superspheres. We can determine the possible forms of $N(Z,W)$ very simply. We see from the proof of Proposition IV.3 that $N(Z,W) = -2\xi(W)^* K\xi(Z)$, where K is the matrix appearing in the relation $\gamma^* K\gamma = K$. We see from the discussion above that $K = (N^T)^{-1}M$. Thus we have

$$N(Z,W) = (1 - 2z^*z + |z^tz|^2) - 2\theta^*(n^t)^{-1}m\theta + \overline{z^tz}(\theta^t m\theta) + z^tz(\overline{\theta^t m\theta}) + (\overline{\theta^t m\theta})(\theta^t m\theta). \tag{5.4}$$

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[1] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces* (Academic, New York/London, 1978).
[2] D. Borthwick, A. Lesniewski, and H. Upmeier, J. Funct. Anal. **112**, 153 (1993).
[3] D. Borthwick, S. Klimek, A. Lesniewski, and M. Rinaldi, *Matrix Cartan superdomains, super Toeplitz operators, and quantization*, Preprint 1993.
[4] O. Loos, *Bounded Symmetric Domains and Jordan Pairs* (University of California, Irvine, 1977).
[5] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, Lecture Notes in Mathematics, Vol. 570 (Springer-Verlag, Berlin, 1977).
[6] F. A. Berezin, *Introduction to Superanalysis* (Reidel, Dordrecht, 1987).
[7] Yu. Manin, *Gauge Field Theory and Complex Geometry* (Springer-Verlag, Berlin, 1988).
[8] D. Hernandez Ruiperez and J. Munoz Masque, J. Math. Pures Appl. **63**, 283 (1984).
[9] J. A. Wolf, in *Geometry and Analysis of Symmetric Spaces* (Marcel Dekker, New York, 1972).
[10] H. Upmeier, *Jordan C*-algebras and Symmetric Banach Manifolds* (North-Holland, Amsterdam, 1985).