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Non-perturbative Deformation Quantization of Cartan Domains

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We construct families of non-commuting $\mathbb{C}^*$-algebras of "quantized functions" for bounded irreducible Hermitian symmetric spaces. For this procedure, we use algebras of Toeplitz operators defined with respect to a perturbation of the ordinary Bergman metric. We prove the deformation quantization conditions for these algebras.

1. INTRODUCTION

In non-commutative geometry, the algebra of continuous functions on a manifold is replaced by a non-commutative $\mathbb{C}^*$-algebra [11]. A natural scheme for constructing non-commutative spaces is deformation quantization. This framework involves introducing a family of algebras depending on a deformation parameter ("Planck's constant"), which approach the classical (commuting) algebra in a certain limit. One, in effect, studies the "semi-classical" limit of the non-commutative algebra. The key relation of this limit is that the commutator of two "quantized functions" approaches zero, with a first-order correction given by the Poisson bracket on the manifold. This scheme was originally proposed in the context of formal power series in the deformation parameter [2, 4]. Recently, it has been extended to the non-perturbative setup (see [23] for some recent results

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and references), fitting thus naturally into the framework of non-commutative geometry.

In [16], this non-perturbative scheme has been applied to the Poincaré disc. The approach of [16] is based on the ideas of [4–6] to use Toeplitz operators as quantization maps (Toeplitz operators were also used in [24] to quantize the sphere.). Among the results of [16], non-perturbative estimates on the rate at which quantized functions and their products approach the classical limits were established. A similar quantization procedure has also been applied to the n-dimensional complex vector space [10]. The most important ingredient in the proof [16] of the deformation quantization conditions was the transitive action of a group of biholomorphic automorphisms. In the case of [10], these automorphisms were translations of the complex vector space. Because all of the irreducible Hermitian symmetric spaces of the non-compact type [14] possess such groups of biholomorphisms, it is natural to try to extend the results of [16] to these spaces.

In this paper, we present the general deformation quantization for such symmetric spaces. Our approach uses the ideas of Berezin who first proposed a quantization scheme of these spaces [5], and speculated on the physical interpretation of this procedure in [7]. Other references devoted to this subject include [21, 22, 25, 27, 28]. We define the non-commutative $\mathbb{C}^*$-algebras, as in [16], by considering Toeplitz operators defined with respect to a perturbation of the Lebesgue measure. The parameter occurring in the perturbed measure is related to Planck's constant. The main result of this paper is the non-perturbative proof of the deformation quantization conditions for the algebras so constructed.

There exists an extensive literature on the subject of Toeplitz operators on Hermitian symmetric domains, chiefly concerning operators defined with respect to the Lebesgue measure, see, e.g., [1, 3, 7, 8, 11]. The monograph [30] will contain a general analysis of the structure of the $\mathbb{C}^*$-algebras generated by the Toeplitz operators with continuous symbols which are defined with respect to the perturbed measure (the case of the unit disc was discussed in [16]). See also [26], for the structure of $\mathbb{C}^*$-algebras generated by a similar (Wiener–Hopf) type of Toeplitz operators.

The paper is organized as follows. In Section 2, we describe the quantization procedure for a general irreducible Hermitian symmetric space of the non-compact type and we state the main results (the deformation conditions) in several theorems. In Section 3, we prove these results in the general case, under several assumptions which are stated in the form of lemmas. In Section 4, these lemmas are proven using the general theory of Jordan algebras, while Section 5 outlines a more elementary argument for the special case of the type I domain.

2. Deformation Quantization

Let D be an irreducible Hermitain symmetric space of the non-compact type [12] which is realized as a bounded symmetric (Cartan) domain in $\mathbb{C}^N$. Recall that the standard Hermitian structure on D is defined as follows. Let $K_D(\zeta, \eta)$ be the Bergman (or reproducing) kernel associated with the Lebesgue measure $d^{2N}\zeta$ on D. We choose to normalize so that $K_D(0, \eta) = 1$, for all $\eta \in D$. Then the Bergman metric on D is defined by

$$h_B(\zeta) := \sum_{1 \leq j, k \leq N} \frac{\partial^2}{\partial \zeta_j \, \partial \bar{\zeta}_k} \log K_D(\zeta, \zeta) \, d\zeta_j \otimes d\bar{\zeta}_k. \tag{2.1}$$

In fact, the associated (non-degenerate) two-form,

$$\omega_B(\zeta) := \frac{i}{2} \sum_{1 \leq j, k \leq N} \frac{\partial^2}{\partial \zeta_j \, \partial \bar{\zeta}_k} \log K_D(\zeta, \zeta) \, d\zeta_j \wedge d\bar{\zeta}_k, \tag{2.2}$$

is closed, and so (D, ω_B) is a Kähler manifold. In particular, (D, ω_B) is a symplectic manifold.

Let $\operatorname{Aut}(D)$ denote the Lie group of holomorphic automorphisms of D. Under $\gamma \in \operatorname{Aut}(D)$, the Bergman kernel transforms according to

$$K_D(\gamma(\zeta), \gamma(\eta)) = \{\det \gamma'(\zeta)\}^{-1} \{\overline{\det \gamma'(\eta)}\}^{-1} K_D(\zeta, \eta), \tag{2.3}$$

where γ' denotes the Jacobian of γ. As a consequence, h_B and ω_B are invariant under $\operatorname{Aut}(D)$. Furthermore, the invariant measure on D is given by

$$d\mu_D(\zeta) := K_D(\zeta, \zeta) d^{2N}\zeta. \tag{2.4}$$

For each $\zeta \in D$, let $\gamma_\zeta \in \operatorname{Aut}(D)$ be chosen so that $\gamma_\zeta(0) = \zeta$ (the existence of γ_ζ follows easily from the properties of symmetric domains [14]). As a consequence of (2.3),

$$K_D(\zeta, \zeta) = |\det \gamma'_\zeta(0)|^{-2}. \tag{2.5}$$

It is easy to express the Poisson bracket $\{f, g\}(\zeta)$, $f, g \in C^\infty(D)$, associated to ω_B, in terms of γ_ζ. Indeed, since ω_B is $\operatorname{Aut}(D)$-invariant, $\{f, g\}(\zeta) = \{f \circ \gamma_\zeta, g \circ \gamma_\zeta\}(0)$. From the definition (2.2) of ω_B, we see that

$$\{f, g\}(0) = \sum_{1 \leq j \leq N} \beta_j (\partial_j f(0) \bar{\partial}_j g(0) - \partial_j g(0) \bar{\partial}_j f(0)), \tag{2.6}$$

where

$$\beta_j^{-1} = -\frac{\partial^2}{\partial \zeta_j \, \partial \bar{\zeta}_j} \log K_D(\zeta, \zeta)|_{\zeta = 0}. \tag{2.7}$$

Thus we have

$$\{f, g\}(\zeta) = \sum_{1 \leqslant j, k \leqslant N} \pi_{jk}(\zeta)(\partial_j f(\zeta)\bar{\partial}_k g(\zeta) - \partial_j g(\zeta)\bar{\partial}_k f(\zeta)), \tag{2.8}$$

where the matrix $\pi(\zeta)$ is given by

$$\pi(\zeta)_{jk} = \sum_{1 \leqslant l \leqslant N} \beta_l \gamma'_\zeta(0)_{jl} \gamma'_\zeta(0)^*_{lk}. \tag{2.9}$$

The aim of this section is to construct a quantum deformation of D. More specifically, we will define a family of $\mathbb{C}^*$-algebras $\mathscr{A}_r(D)$, for $r \geqslant r(D) > 0$, and a linear continuous map $T_r: C_b(D) \to \mathscr{A}_r(D)$ (the quantization map), where $C_b(D)$ denotes the space of continuous bounded functions on D. As $r \to \infty$, $T_r(f)$ approaches its classical value f in such a way that it reproduces the Poisson structure on D given by the Poisson bracket (2.7). We formulate these properties precisely at the end of this section.

We denote the deformation parameter by $r \in \mathbb{R}$ and set

$$\begin{aligned} d\mu_r(\zeta) &:= \Lambda_r K_D(\zeta, \zeta)^{-r} \, d\mu_D(\zeta), \\ K^r_D(\zeta, \eta) &:= K_D(\zeta, \eta)^r, \end{aligned} \tag{2.10}$$

where $\Lambda_r > 0$ will be chosen momentarily. Let S denote the set of those real numbers r for which $\int_D d\mu_r(\zeta) < \infty$. It is proven in [5, 13] that if D is a Cartan domain, then S contains the inverval $[r(D), \infty)$, for some $r(D) > 0$. We now choose Λ_r so that $d\mu_r$ is a probability measure. Let $\mathscr{H}_r(D)$ denote the closed subspace of $L^2(D, d\mu_r)$ consisting of holomorphic functions. Then $K^r_D(\zeta, \eta)$ is the integral kernel of the orthogonal projection $P: L^2(D, d\mu_r) \to \mathscr{H}_r(D)$, i.e., $K^r_D(\zeta, \eta)$ is the Bergman kernel for D associated with $d\mu_r$.

For $\gamma \in \mathrm{Aut}(D)$, we set

$$U(\gamma)\phi(\zeta) := \{\det(\gamma^{-1})'(\zeta)\}^r \phi(\gamma^{-1}(\zeta)), \qquad \phi \in \mathscr{H}_r(D), \tag{2.11}$$

where $\{\det(\gamma^{-1})'(\zeta)\}^r$ is defined as $\exp\{r \log \det(\gamma^{-1})'(\zeta)\}$, and where log is a fixed branch of the logarithm (for concreteness: $\log z = \log |z| + i \arg z$, where $-\pi < \arg z \leqslant \pi$). Then, as a consequence of (2.3), $U(\gamma)$ is unitary. Furthermore, for $\gamma_1, \gamma_2 \in \mathrm{Aut}(D)$,

$$U(\gamma_1 \gamma_2) = \sigma(\gamma_1, \gamma_2) U(\gamma_1) U(\gamma_2), \tag{2.12}$$

where

$$\sigma(\gamma_1, \gamma_2) := \frac{\{\det((\gamma_1\gamma_2)^{-1})'(\zeta)\}^r}{\{\det(\gamma_1^{-1})'(\zeta)\}^r \{\det(\gamma_2^{-1})'(\gamma_1^{-1}(\zeta))\}^r}. \tag{2.13}$$

We verify easily that the right hand side of (2.13) is, indeed, independent of ζ, and, furthermore,

$$|\sigma(\gamma_1, \gamma_2)| = 1,$$

$$\sigma(\gamma_2, \gamma_3)\sigma(\gamma_1\gamma_2, \gamma_3)^{-1}\sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_1, \gamma_2)^{-1} = 1,$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{Aut}(D)$. As a consequence, $\gamma \to U(\gamma)$ is a projective unitary representation of $\mathrm{Aut}(D)$ on $\mathcal{H}_r(D)$.

For $f \in C_b(D)$, we define a linear operator $T_r(f)\colon \mathcal{H}_r(D) \to \mathcal{H}_r(D)$ by

$$T_r(f)\phi = PM(f)\phi, \tag{2.14}$$

where $M(f)$ is pointwise multiplication by f. The action of $T_r(f)$ thus consists of multiplication by f followed by projection back into $\mathcal{H}_r(D)$. Clearly, $T_r(f)$ is a bounded linear operator on $\mathcal{H}_r(D)$. It is called a Toeplitz operator with symbol f. Explicitly,

$$T_r(f)\phi(\zeta) = \int_D K_D^r(\zeta, \eta) f(\eta)\phi(\eta)\, d\mu_r(\eta). \tag{2.15}$$

Let $\mathcal{A}_r(D)$ denote the $\mathbb{C}^*$-algebra generated by all such Toeplitz operators.

From now on, we assume that D is an irreducible bounded symmetric domain. The main result of this paper is to show that the mapping $f \to T_r(f)$ is a quantum deformation of D. The precise meaning of this statement is given by the theorems stated below, which will be proven in subsequent sections of this paper.

When unambiguous, we will denote the operator norm on $\mathcal{H}_r(D)$ by $\|\cdot\|$, and not state explicitly the dependence on r. We also denote by $\|f\|_\infty$ the sup-norm of $f \in C_b(D)$. Let $C_b^p(D)$ denote the space of functions in $C_b(D)$ with continuous and bounded derivatives out to order p. On this space we define the norm

$$\|f\|_{p,\infty} := \sum_{k=0}^{p} \sum_{j_1, \ldots, j_k} \|\partial_{j_1} \cdots \partial_{j_k} f\|_\infty. \tag{2.16}$$

THEOREM 2.1. *For any* $f \in C_b(D)$,

$$\lim_{r \to \infty} \|T_r(f)\| = \|f\|_\infty. \tag{2.17}$$

The two theorems below state that, in a suitable sense, the product $T_r(f)T_r(g)$ is a deformation of the ordinary pointwise product of functions fg.

THEOREM 2.2. *Let* $f, g \in C_b(D)$, *with* g *having support in some compact set* $K \subset D$. *Then*

$$\lim_{r \to \infty} \|T_r(f)T_r(g) - T_r(fg)\| = 0. \tag{2.18}$$

This theorem can be amplified, if we assume additionally that f and g are sufficiently smooth.

THEOREM 2.3. *There exists an integer m_0 such that for all $m \geqslant m_0$ and for $f, g \in C_b^m(D)$, with f having support in some compact set $K \subset D$, we can find a constant C_K (depending on K), such that*

$$\| T_r(f) T_r(g) - T_r(fg) + r^{-1} T_r(\pi_{jk}(\partial_j f)(\bar{\partial}_k g)) \| \leqslant C_K r^{-2} \|f\|_{m,\infty} \|g\|_{m,\infty}, \tag{2.19}$$

for sufficiently large r, where the matrix $\pi(\zeta)$ was defined in (2.9).

As an immediate consequence of Theorem 2.3 and the form of the Poisson bracket (2.8) we conclude that $\mathscr{A}_r(D)$ is a quantum deformation of the Poisson algebra of smooth functions on D. The ratio $1/r$ plays the role of Planck's constant.

THEOREM 2.4. *Under the assumptions of Theorem* 2.3,

$$\| r[T_r(f), T_r(g)] + T_r(\{f, g\}) \| \leqslant C_K r^{-1} \|f\|_{m,\infty} \|g\|_{m,\infty}, \tag{2.20}$$

for r sufficiently large.

3. Proof of Theorems 2.1, 2.2, and 2.3

In this section, we prove the main theorems stated in Section 2 under the assumption that the lemmas stated below hold.

LEMMA 3.1. *There is an $r_0 = r_0(D)$ such that the measures $d\mu_r$ satisfy the following conditions*:

(i) *There are constants $C_1, C_2 > 0$ such that the normalization constant Λ_r in* (2.10) *obeys*

$$C_1 r^N \leqslant \Lambda_r \leqslant C_2 r^N, \tag{3.1}$$

for all $r \geqslant r_0$, where N is the dimension of D;

(ii) *For any integer $q > 0$, there is a constant C such that*

$$\int |\eta|^{2q} \, d\mu_r(\eta) \leqslant C r^{-q}, \tag{3.2}$$

for all $r \geqslant r_0$.

LEMMA 3.2. *We can define the map* $D \ni \zeta \mapsto \gamma_\zeta \in \operatorname{Aut}(D)$, *with* $\gamma_\zeta(0) = \zeta$, *such that the following conditions are satisfied*:

(i) *The family* $\{\gamma_\zeta\}_{\zeta \in D}$ *is equicontinuous*;

(ii) *For* $\delta > 0$, $\sup_{|\zeta| \geqslant \delta} |\det \gamma'_\zeta(0)| < 1$;

(iii) *For any* k, *there are constants* C *and* $\rho, \rho' \geqslant 0$ *such that*

$$\sum_{i, j_1, \ldots, j_k} \left| \frac{\partial^k \gamma_\zeta(\eta)_i}{\partial \eta_{j_1} \cdots \partial \eta_{j_k}} \right| \leqslant C K_D(\eta, \eta)^\rho K_D(\zeta, \zeta)^{\rho'}. \tag{3.3}$$

Proof of Theorem 2.1. We note first that, for any r,

$$\|T_r(f)\| \leqslant \|f\|_\infty, \tag{3.4}$$

because the orthogonal projection onto $\mathscr{H}_r(D)$ has norm 1. We must therefore prove that

$$\|f\|_\infty \leqslant \lim_{r \to \infty} \|T_r(f)\|. \tag{3.5}$$

We can write

$$f(\zeta) = \int_D f(\gamma_\zeta(\eta))\, d\mu_r(\eta) + \left(f(\zeta) - \int_D f(\gamma_\zeta(\eta))\, d\mu_r(\eta) \right)$$

$$= (T_r(\gamma_\zeta^* f)\phi_0, \phi_0) + \int_D [f(\gamma_\zeta(0)) - f(\gamma_\zeta(\eta))]\, d\mu_r(\eta).$$

Because

$$T_r(\gamma^* f) = U(\gamma)^{-1} T_r(f)\, U(\gamma), \tag{3.6}$$

with $U(\gamma)$ unitary, we have $\|T_r(\gamma_\zeta^* f)\| = \|T_r(f\|$, and so

$$\|f\|_\infty \leqslant \|T_r(f)\| + \sup_\zeta \left| \int_D [f(\gamma_\zeta(0)) - f(\gamma_\zeta(\eta))]\, d\mu_r(\eta) \right|. \tag{3.7}$$

Suppose we are given an $\varepsilon > 0$. The function f is continuous and bounded, and Lemma 3.2(i) gives us that $\{\gamma_\zeta\}$ is equicontinuous. Therefore, we can choose $\delta > 0$ such that

$$\sup_\zeta |f(\gamma_\zeta(\eta)) - f(\gamma_\zeta(0))| < \varepsilon/2, \tag{3.8}$$

whenever $|\eta| < \delta$. The integral on the right-hand side of (3.7) can be broken into integration regions $\{|\eta| < \delta\}$ and $\{|\eta| \geqslant \delta\}$, which we label by I_1 and I_2. We have

$$\sup_\zeta |I_1| = \sup_\zeta \left| \int_{|\eta| < \delta} [f(\gamma_\zeta(0)) - f(\gamma_\zeta(\eta))]\, d\mu_r(\eta) \right|$$

$$\leqslant \sup_\zeta \sup_{|\eta| < \delta} |f(\gamma_\zeta(0)) - f(\gamma_\zeta(\eta))| \leqslant \varepsilon/2. \tag{3.9}$$

For the second half of the integral, we have

$$\begin{aligned}\sup_{\zeta} |I_2| &\leqslant 2\, \|f\|_{\infty} \int_{|\eta| \geqslant \delta} d\mu_r(\eta) \\ &= 2\, \|f\|_{\infty}\, A_r \int_{|\eta| \geqslant \delta} K_D(\eta, \eta)^{1-r}\, d^{2N}\eta \\ &\leqslant 2\, |D|\, \|f\|_{\infty}\, A_r \left\{ \sup_{|\eta| \geqslant \delta} K_D(\eta, \eta) \right\}^{1-r} \end{aligned} \tag{3.10}$$

where $|D| < \infty$ is the Lebesgue volume of D. Using Lemma 3.1(i), Lemma 3.2(ii), and Eq. (2.5), we thus have

$$\begin{aligned}\sup_{\zeta} |I_2| &\leqslant Cr^N \left\{ \sup_{|\eta| \geqslant \delta} |\det \gamma'_\eta(0)| \right\}^{2(r-1)} \\ &\leqslant C' r^N e^{-\alpha r}, \end{aligned} \tag{3.11}$$

with $C' > 0$, $\alpha > 0$. By choosing large enough r, we can make this bound smaller than $\varepsilon/2$. This proves (3.5) and thus establishes the theorem. ∎

Proof of Theorem 2.2. We start with the expression,

$$\begin{aligned}&(\psi, \{T_r(f)\, T_r(g) - T_r(fg)\}\, \phi)_r \\ &\quad = \int_{D \times D} K_D^r(\eta, \zeta)[f(\eta) - f(\zeta)]\, g(\zeta) \phi(\zeta) \overline{\psi(\eta)}\, d\mu_r(\zeta)\, d\mu_r(\eta), \end{aligned} \tag{3.12}$$

and make the substitution $\eta = \gamma_\zeta(\xi)$. Because of the transformation property (2.3) of K_D, we see that

$$\begin{aligned}K_D^r(\gamma_\zeta(\xi), \zeta)\, d\mu_r(\gamma_\zeta(\xi)) &= \frac{K_D^r(\gamma_\zeta(\xi), \gamma_\zeta(0))}{K_D^r(\gamma_\zeta(\xi), \gamma_\zeta(\xi))}\, d\mu_D(\xi) \\ &= \{\overline{\det \gamma'_\zeta(0)}\}^{-r} \{\overline{\det \gamma'_\zeta(\xi)}\}^{r}\, d\mu_r(\xi) \\ &= \frac{K_D^r(\zeta, \zeta)}{K_D^r(\zeta, \gamma_\zeta(\xi))}\, d\mu_r(\xi). \end{aligned} \tag{3.13}$$

We thus obtain

$$\begin{aligned}&(\psi, \{T_r(f)\, T_r(g) - T_r(fg)\}\, \phi)_r \\ &\quad = \int_{D \times D} [f(\gamma_\zeta(\xi)) - f(\gamma_\zeta(0))]\, g(\zeta) \phi(\zeta) \overline{\psi(\gamma_\zeta(\xi))} \\ &\qquad \times \frac{K_D^r(\zeta, \zeta)}{K_D^r(\zeta, \gamma_\zeta(\xi))}\, d\mu_r(\zeta)\, d\mu_r(\xi). \end{aligned} \tag{3.14}$$

Applying the Schwarz inequality to the ξ integration yields

$$\begin{aligned}
&|(\psi, \{T_r(f)T_r(g) - T_r(fg)\}\phi)_r| \\
&\leqslant \int_D |g(\zeta)\phi(\zeta)|\, K_D^r(\zeta,\zeta) \left\{\int_D \frac{|\psi(\gamma_\zeta(\xi))|^2}{|K_D^r(\zeta, \gamma_\zeta(\xi))|^2}\, d\mu_r(\xi)\right\}^{1/2} d\mu_r(\zeta) \\
&\quad\times \left\{\sup_\zeta \int_D |f(\gamma_\zeta(\xi)) - f(\gamma_\zeta(0))|^2\, d\mu_r(\xi)\right\}^{1/2}.
\end{aligned} \tag{3.15}$$

It follows from (2.3) and (2.11) that

$$\begin{aligned}
\int_D \frac{|\psi(\gamma_\zeta(\xi))|^2}{|K_D^r(\zeta, \gamma_\zeta(\xi))|^2}\, d\mu_r(\xi) &= K_D^r(\zeta,\zeta)^{-1}\, \|U(\gamma_\zeta)\psi\|^2 \\
&= K_D^r(\zeta,\zeta)^{-1}\, \|\psi\|^2.
\end{aligned} \tag{3.16}$$

Thus we obtain, using the restriction on the support of g,

$$\begin{aligned}
&|(\psi, \{T_r(f T_r(g) - T_r(fg)\}\phi)_r| \\
&\leqslant \|g\|_\infty\, \|\psi\| \int_K |\phi(\zeta)|\, K_D^r(\zeta,\zeta)^{1/2}\, d\mu_r(\zeta) \\
&\quad\times \left\{\sup_\zeta \int_D |f(\gamma_\zeta(\xi)) - f(\gamma_\zeta(0))|^2\, d\mu_r(\xi)\right\}^{1/2} \\
&\leqslant \|g\|_\infty\, \|\psi\|\, \|\phi\| \int_K K_D^r(\zeta,\zeta)\, d\mu_r(\zeta) \\
&\quad\times \left\{\sup_\zeta \int_D |f(\gamma_\zeta(\xi)) - f(\gamma_\zeta(0))|^2\, d\mu_r(\xi)\right\}^{1/2}.
\end{aligned} \tag{3.17}$$

The integral over the compact set K can be bounded, uniformly in r, by a finite constant, and we saw in the proof of Theorem 2.1 that

$$\lim_{r\to\infty} \left\{\sup_\zeta \int_D |f(\gamma_\zeta(\xi)) - f(\gamma_\zeta(0))|^2\, d\mu_r(\xi)\right\} = 0. \tag{3.18}$$

This concludes the proof of Theorem 2.2. ∎

Proof of Theorem 2.3. Let m be an even number such that $m - N \geqslant 4$, where N is the dimension of D. Following [16], our technique will be to evaluate

$$(\psi, T_r(f)T_r(g)\phi) = \int_{D\times D} K(\zeta,\eta) f(\zeta)\, g(\eta)\phi(\eta)\overline{\psi(\zeta)}\, d\mu_r(\zeta)\, d\mu_r(\eta). \tag{3.19}$$

We can change variables to $\eta = \gamma_\zeta(\xi)$, as in the proof of Theorem 2.2, to get

$$(\psi, T_r(f)\,T_r(g)\phi) = \int_{D\times D} f(\zeta)\, g(\gamma_\zeta(\xi))\phi(\gamma_\zeta(\xi))\,\overline{\psi(\zeta)} \times \frac{K_D^r(\zeta,\zeta)}{K_D^r(\gamma_\zeta(\xi),\zeta)}\, d\mu_r(\zeta)\, d\mu_r(\xi). \tag{3.20}$$

The next step will be to perform a Taylor expansion on $g(\gamma_\zeta(\xi))$, out to order m. We will group the resulting terms according to the powers of ξ and $\bar{\xi}$, denoting by $I_{p,q}$ the contribution to the integral (3.20) from the term in the expansion with p powers of ξ and q powers of $\bar{\xi}$. We will make use of the following two facts. For any holomorphic function χ,

$$\begin{aligned}&\int_D \chi(\eta)\, d\mu_r(\eta) = \chi(0),\\ &\int_D \chi(\eta)\bar{\eta}_l\, d\mu_r(\eta) = \partial_l \chi(0) \int_D |\eta_l|^2\, d\mu_r(\eta).\end{aligned} \tag{3.21}$$

This can be immediately seen for polynomials, because each domain has the circular symmetry $\eta \to e^{i\alpha}\eta$, α real, and the result can be extended to all holomorphic functions through the Runge approximation theorem.

Using the circular symmetry as in (3.21), one can quickly check that $I_{p,q} = 0$ if $p > q$. We obtain the first term in the expansion of (3.20),

$$\begin{aligned} I_{0,0} &= \int_{D\times D} f(\zeta)\, g(\zeta)\phi(\gamma_\zeta(\xi))\overline{\psi(\zeta)}\, \frac{K_D^r(\zeta,\zeta)}{K_D^r(\gamma_\zeta(\xi),\zeta)}\, d\mu_r(\zeta)\, d\mu_r(\xi)\\ &= \int_{D\times D} f(\zeta)\, g(\zeta)\phi(\zeta)\overline{\psi(\zeta)}\, d\mu_r(\zeta)\\ &= (\psi, T_r(fg)\phi). \end{aligned} \tag{3.22}$$

The next non-zero term is (in the following formulas we use the summation convention where each repeated index is summed over from 1 to N)

$$I_{0,1} = \int_{D\times D} f(\zeta)\bar{\partial}_k\, g(\zeta)\overline{\gamma'_\zeta(0)}_{kl}\bar{\xi}_l(\phi(\gamma_\zeta(\xi))\overline{\psi(\zeta)} \times \frac{K_D^r(\zeta,\zeta)}{K_D^r(\gamma_\zeta(\xi),\zeta)}\, d\mu_r(\zeta)\, d\mu_r(\xi). \tag{3.23}$$

Using (3.21), we find

$$I_{0,1} = c_l^2 \int_D f(\zeta)\bar{\partial}_k\, g(\zeta)\,\overline{\gamma'_\zeta(0)}_{kl}\overline{\psi(\zeta)}\, K_D^r(\zeta,\zeta)\, \frac{\partial}{\partial \xi_l} \times \left\{\frac{\phi(\gamma_\zeta(\xi))}{K_D^r(\gamma_\zeta(\xi),\zeta)}\right\}_{\xi=0} d\mu_r(\zeta), \tag{3.24}$$

where

$$c_l^2 = \int_D |\xi_l|^2 \, d\mu_r(\xi). \tag{3.25}$$

From the standard definition of the Bergman kernel as a sum over an orthonormal basis, one obtains the expansion

$$K_D^r(\xi, \xi) = 1 + \sum_{l=1}^{N} c_l^{-2} |\xi_l|^2 + \cdots. \tag{3.26}$$

This implies that

$$K_D(\xi, \xi) = 1 + \frac{1}{r} \sum_{l=1}^{N} c_l^{-2} |\xi_l|^2 + \cdots. \tag{3.27}$$

Comparing this to the definition (2.7) of β_l, we see that

$$c_l^2 = \beta_l r^{-1}. \tag{3.28}$$

We can interchange ξ and ζ derivatives as

$$\begin{aligned}\frac{\partial}{\partial \xi_l} \left\{ \frac{\phi(\gamma_\zeta(\xi))}{K_D^r(\gamma_\zeta(\xi), \zeta)} \right\}_{\xi=0} &= \gamma'_\zeta(0)_{jl} [K_D^r(\zeta, \zeta)^{-1} \partial_j \phi(\zeta) \\ &\quad - \phi(\zeta) K_D^r(\zeta, \zeta)^{-2} \partial_j K_D^r(\zeta, \zeta)] \\ &= \gamma'_\zeta(0)_{jl} \, \partial_j [K_D^r(\zeta, \zeta)^{-1} \phi(\zeta)].\end{aligned} \tag{3.29}$$

Substituting this into the expression (3.24) for $I_{0,1}$ and using (3.28), we get

$$\begin{aligned}I_{0,1} = r^{-1} \int_D & f(\zeta) \bar{\partial}_k \, g(\zeta) \pi_{jk}(\zeta) \overline{\psi(\zeta)} K_D^r(\zeta, \zeta) \partial_j \\ & \times [K_D^r(\zeta, \zeta)^{-1} \phi(\zeta)] \, d\mu_r(\zeta),\end{aligned} \tag{3.30}$$

which we can integrate by parts to give two terms,

$$\begin{aligned}I_{0,1} = & -r^{-1} \int_D \partial_j f(\zeta) \bar{\partial}_k \, g(\zeta) \pi_{jk}(\zeta) \phi(\zeta) \overline{\psi(\zeta)} \, d\mu_r(\zeta) \\ & - r^{-1} \int_D f(\zeta) \partial_j \bar{\partial}_k \, g(\zeta) \pi_{jk}(\zeta) \phi(\zeta) \overline{\psi(\zeta)} \, d\mu_r(\zeta)\end{aligned} \tag{3.31}$$

We now note that the next term in the expansion, $I_{1,1}$, is given by

$$\begin{aligned}I_{1,1} &= \int_{D \times D} f(\zeta) \partial_j \bar{\partial}_k \, g(\zeta) \gamma'_\zeta(0)_{jl} \overline{\gamma'_\zeta(0)}_{kl} \, |\xi_l|^2 \, \phi(\gamma_\zeta(\xi)) \overline{\psi(\zeta)} \\ &\qquad \times \frac{K_D^r(\zeta, \zeta)}{K_D^r(\gamma_\zeta(\xi), \zeta)} \\ &= r^{-1} \int_D f(\zeta) \partial_j \bar{\partial}_k \, g(\zeta) \pi_{jk}(\zeta) \phi(\zeta) \overline{\psi(\zeta)} \, d\mu_r(\zeta).\end{aligned} \tag{3.32}$$

Combining (3.31) with (3.32), we have

$$I_{0,1}+I_{1,1}=-r^{-1}(\psi, T_r(\pi_{jk}(\zeta)(\partial_j f)(\bar{\partial}_k g))\phi). \tag{3.33}$$

To complete the proof, all that remains is to bound the other terms as $r\to\infty$. For $1\leqslant p<q<m$, we want to rewrite the q powers of $\bar{\xi}$ coordinates as derivatives with respect to ξ coordinates, just as we did for the two simple cases in (3.21). In general, this results in a combination of derivatives with respect to ξ of order $\leqslant m$, evaluated at $\xi=0$, multiplied by integrals over the corresponding absolute values of ξ coordinates. The derivatives can be moved onto f and g by integration by parts, and the absolute values appearing in the integrals can all be bounded by $|\xi|^4$, since $q\geqslant 2$. Thus we have

$$|I_{p,q}|\leqslant C\,\|f\|_{m,\infty}\,\|g\|_{m,\infty}\,\|\phi\|\,\|\psi\|\int_D|\xi|^4\,d\mu_r(\xi). \tag{3.34}$$

By Lemma 3.1(ii), we can therefore bound

$$\sum_{1\leqslant p<q<m}|I_{p,q}|\leqslant Cr^{-2}\,\|f\|_{m,\infty}\,\|g\|_{m,\infty}\,\|\phi\|\,\|\psi\|. \tag{3.35}$$

The remainder term in the expansion of (3.20) is given by

$$R=\int_{D\times D}f(\zeta)G(\zeta,\xi)\phi(\gamma_\zeta(\xi))\overline{\psi(\zeta)}\,\frac{K_D^r(\zeta,\zeta)}{K_D^r(\gamma_\zeta(\xi),\zeta)}\,d\mu_r(\zeta)\,d\mu_r(\xi), \tag{3.36}$$

where, by Taylor's theorem,

$$G(\zeta,\xi)=\frac{1}{(m-1)!}\int_0^1 ds(1-s)^{m-1}\,\frac{d^m}{ds^m}\,g(\gamma_\zeta(s\xi)). \tag{3.37}$$

Each derivative with respect to s picks up a power of ξ, and by Lemma 3.2(iii), for every k there are constants C and ρ,ρ' such that

$$\sum_{i,j_1,\ldots,j_k}\left|\frac{\partial^k\gamma_\zeta(\xi)_i}{\partial\xi_{j_1}\cdots\partial\xi_{j_k}}\right|\leqslant CK_D(\xi,\xi)^\rho K_D(\zeta,\zeta)^{\rho'}. \tag{3.38}$$

Using these facts, it is straightforward to find constants C and σ,σ' such that

$$|G(\zeta,\xi)|\leqslant C\,\|g\|_{m,\infty}\,K_D(\xi,\xi)^\sigma K_D(\zeta,\zeta)^{\sigma'}\,|\xi|^m. \tag{3.39}$$

Because of this bound on G, we have

$$\begin{aligned}|R|\leqslant C\,\|g\|_{m,\infty}\int_{D\times D}&|f(\zeta)\psi(\zeta)|\,K_D(\zeta,\zeta)^{r+\sigma'}K_D(\xi,\xi)^\sigma\,|\xi|^m\\ &\times\frac{|\phi(\gamma_\zeta(\xi))|}{|K_D^r(\gamma_\zeta(\xi),\zeta)|}\,d\mu_r(\zeta)\,d\mu_r(\xi).\end{aligned} \tag{3.40}$$

Applying the Schwarz inequality to the ξ integration, and using the compact support of f, gives

$$|R| \leqslant C \|f\|_{m,\infty} \|g\|_{m,\infty} \left\{ \int_D K_D(\xi, \xi)^{2\sigma} |\xi|^{2m} \, d\mu_r(\xi) \right\}^{1/2}$$

$$\times \int_K |\psi(\zeta)| \, K_D(\zeta, \zeta)^{r+\sigma'} \left\{ \int_D \frac{|\phi(\gamma_\zeta(\xi))|^2}{|K_D^r(\gamma_\zeta(\xi), \zeta)|^2} \, d\mu_r(\xi) \right\}^{1/2} d\mu_r(\zeta). \quad (3.41)$$

Using Lemma 3.1(i)–(ii), we can show that

$$\int_D K_D(\xi, \xi)^{2\sigma} |\xi|^{2m} \, d\mu_r(\xi) = \frac{\Lambda_r}{\Lambda_{r-2\sigma}} \int_D |\xi|^{2m} \, d\mu_{(r-2\sigma)}(\xi)$$
$$\leqslant Cr^{-m}. \quad (3.42)$$

We also have, as we saw in the proof of Theorem 2.2,

$$\int_D \frac{|\phi(\gamma_\zeta(\xi))|^2}{|K_D^r(\gamma_\zeta(\xi), \zeta)|^2} \, d\mu_r(\xi) = \|\phi\|^2 \, K_D^r(\zeta, \zeta)^{-1}. \quad (3.43)$$

We thus obtain

$$|R| \leqslant Cr^{-m/2} \|f\|_{m,\infty} \|g\|_{m,\infty} \|\phi\| \int_K |\psi(\zeta)| \, K_D(\zeta, \zeta)^{r/2+\sigma'} \, d\mu_r(\zeta)$$

$$\leqslant Cr^{-m/2} \|f\|_{m,\infty} \|g\|_{m,\infty} \|\phi\| \, \|\psi\| \left\{ \int_K K_D(\zeta, \zeta)^{r+2\sigma'} \, d\mu_r(\zeta) \right\}^{1/2}.$$

The remaining integral is given by

$$\int_K K_D^r(\zeta, \zeta) \, d\mu_r(\zeta) = \Lambda_r \int_K K_D(\zeta, \zeta)^{2\sigma'} \, d\mu_D(\zeta).$$

This integral over K is independent of r, and can be absorbed into the constant C_K. Using Lemma 3.1(i), we obtain the bound

$$|R| \leqslant C_K r^{-(m-N)/2} \|f\|_{m,\infty} \|g\|_{m,\infty} \|\phi\| \, \|\psi\|. \quad (3.44)$$

Recall that we have chosen m so that $(m-N) \geqslant 4$. This concludes the proof of the theorem. ▮

4. Proof of Lemmas 3.1 and 3.2

In this section we complete the proof of Theorems 2.1–2.4 by proving Lemmas 3.1 and 3.2 for all irreducible bounded symmetric domains. It is well known [14] that these domains can be classified into four infinite series (the "classical" domains) and two "exceptional" domains of dimensions 16 and 27, respectively. While Berezin [5] studied only the classical domains on a case-by-case basis, it is more satisfying to have a uniform treatment applying also to the two exceptional domains, whose holomorphic automorphism groups are closely related to the exceptional Lie groups. Thus our proof of Lemmas 3.1 and 3.2 does not depend on the classification of symmetric domains but uses the theory of Jordan algebras and triple systems [20, 29]. In Section 5, a direct and elementary proof is sketched for the basic example of type I domains, which also serve as an illustration of the Jordan theoretic concepts.

Let $Z \cong \mathbb{C}^N$ be a finite-dimensional complex vector space. A *Jordan triple product* [19, 20] is a ternary operation

$$\xi, \eta, \zeta \mapsto \{\xi\eta^*\zeta\} \in Z \tag{4.1}$$

on Z which is complex bilinear and symmetric in $\xi, \zeta \in Z$, conjugate linear in $\eta \in Z$ (indicated by the *-symbol), and satisfies the so-called "Jordan triple identity." Let Z be endowed with a (positive hermitian) Jordan triple product [20], and define *triple idempotents* $e \in Z$ by the condition

$$\{ee^*e\} = e. \tag{4.2}$$

To $\eta \in Z$, we can associate the positive operator which takes $\zeta \mapsto \{\eta\eta^*\zeta\}$. By the spectral theorem, we obtain a decomposition

$$\eta = \sum_{1 \leqslant i \leqslant n} \eta_i e_i, \tag{4.3}$$

where $e_1, ..., e_n \in Z$ are (minimal, orthogonal) triple idempotents, and the "singular numbers," $\eta_1 \geqslant \cdots \geqslant \eta_n \geqslant 0$, are uniquely determined by η. The number n is called the rank of Z. One can show that

$$|\eta| := \eta_1 \tag{4.4}$$

defines a norm on Z, whose open unit ball

$$D := \{\eta \in Z : |\eta| < 1\} \tag{4.5}$$

is a bounded symmetric domain. Conversely, every such domain can be realized in this way [20, 29]. The group

$$K := \{\gamma \in \mathrm{Aut}(D) : \gamma(0) = 0\} \tag{4.6}$$

consists of linear transformations which leave the norm (4.4) invariant. Assume from now on that D is irreducible. Let

$$Z := \bigoplus_{0 \leqslant i \leqslant j \leqslant n} Z_{ij} \tag{4.7}$$

be the Peirce decomposition [20, Theorem 3.14], where

$$Z_{ij} := \left\{ \zeta \in Z \colon \{e_k e_k^* \zeta\} = \frac{\delta_{ik} + \delta_{jk}}{2} \zeta, \forall 1 \leqslant k \leqslant n \right\}. \tag{4.8}$$

Here $e_1, ..., e_n$ are as in (4.3). Then $Z_{00} = \{0\}$, $Z_{ii} = \mathbb{C} e_i$ (for $1 \leqslant i \leqslant n$), and the numbers

$$a := \dim_{\mathbb{C}} Z_{ij} \qquad (1 \leqslant i < j \leqslant n) \tag{4.9}$$

and

$$b := \dim_{\mathbb{C}} Z_{0_j} \quad (1 \leqslant j \leqslant n) \tag{4.10}$$

are independent of i, j and of the choice of $e_1, ..., e_n$. Applying (4.7), we obtain

$$N = n + \frac{a}{2} n(n-1) + nb. \tag{4.11}$$

We now introduce an important class of linear operators on Z, namely the *Bergman operators*

$$B(\xi, \eta)\zeta := \zeta - 2\{\xi \eta^* \zeta\} + \{\xi \{\eta \zeta^* \eta\}^* \xi\}, \tag{4.12}$$

depending on $\xi, \eta \in Z$. Note that (4.12) is complex linear in ζ and "sesqui-holomorphic" in (ξ, η). By [19, 27] the determinant

$$\det B(\xi, \eta) = \Delta(\xi, \eta)^p \tag{4.13}$$

is the pth power of a sesqui-polynomial function $\Delta(\xi, \eta)$ (a polynomial in ξ, a conjugate polynomial in η), which is called the *Jordan triple determinant*. The integer power p is given by

$$p = 2 + a(n-1) + b \tag{4.14}$$

and is called the *genus*. Moreover, the Bergman kernel of D satisfies

$$K_D(\zeta, \eta) = \det B(\zeta, \eta)^{-1} = \Delta(\zeta, \eta)^{-p}. \tag{4.15}$$

In terms of singular values (4.3), we have

$$\Delta(\eta, \eta) = \prod_{1 \leqslant i \leqslant n} (1 - \eta_i^2). \tag{4.16}$$

For $r > 1 - 1/p$, the measure

$$d\mu_r(\zeta) = \pi^{-N} \frac{\Gamma_D(pr)}{\Gamma_D(pr - N/n)} \Delta(\zeta, \zeta)^{p(r-1)} d^{2N}\zeta \tag{4.17}$$

is a probability measure on D [17, 30]. Here $d^{2N}\zeta$ is the Lebesgue measure for the K-invariant inner product $(\zeta|\eta)$ on Z which satisfies $(e_i|e_i)=1$ for all i, and

$$\Gamma_D(\lambda) := (2\pi)^{(N-n)/2} \prod_{1\leqslant j\leqslant n} \Gamma\left(\lambda - \frac{a}{2}(j-1)\right) \tag{4.18}$$

is the Koecher–Gindikin Γ-function associated with (the "radial part" of) D. Comparing with (2.10) and (4.15), we see that

$$\Lambda_r = \pi^{-N} \frac{\Gamma_D(pr)}{\Gamma_D(pr - N/n)}. \tag{4.19}$$

Applying Stirling's formula to each factor of (4.18), we obtain

$$\Lambda_r \sim \frac{(pr)^N}{\pi^N} \tag{4.20}$$

as $r\to\infty$, where $\sim$ means that the quotient of both sides tends to 1. This proves Lemma 3.1(i). Since $d\mu_r(\eta)$ and $|\eta|$ are K-invariant, the integral in Lemma 3.1(ii) is given in polar coordinates [18, Proposition 3.2.7] as

$$\begin{aligned}\int_D |\eta|^{2q}\, d\mu_r(\eta) = \Lambda_r \int_{1\geqslant \eta_1\geqslant \cdots \geqslant \eta_n \geqslant 0} \eta_1^{2q} \prod_{1\leqslant i\leqslant n} (1-\eta_i^2)^{p(r-1)}\\ \times \prod_{1\leqslant i<j\leqslant n} (\eta_i^2-\eta_j^2)^a \prod_{1\leqslant i\leqslant n} \eta_i^{2b+1}\, d\eta_1\cdots d\eta_n.\end{aligned} \tag{4.21}$$

Here a and b are the characteristic multiplicities defined in (4.9) and (4.10), and we use (4.4) and (4.16). Putting $y_i := \eta_i^2$, the integral becomes

$$\begin{aligned}\int_D |\eta|^{2q}\, d\mu_r(\eta) &= 2^{-n}\Lambda_r \int_{1\geqslant y_1\geqslant \cdots \geqslant y_n\geqslant 0} y_1^q \prod_{1\leqslant i\leqslant n} (1-y_i)^{p(r-1)}\\ &\quad \times \prod_{1\leqslant i<j\leqslant n} (y_i-y_j)^a \prod_{1\leqslant i\leqslant n} y_i^b\, dy_1\cdots dy_n\\ &\leqslant 2^{-n}\Lambda_r \int_{0\leqslant y_1,\ldots,y_n\leqslant 1} y_1^q\\ &\quad \times \prod_{1\leqslant i\leqslant n} (1-y_i)^{p(r-1)} y_i^{a(n-i)+b}\, dy_1\cdots dy_n\\ &= 2^{-n}\Lambda_r B(p(r-1)+1, q+a(n-1)+b+1)\\ &\quad \times \prod_{2\leqslant i\leqslant n} B(p(r-1)+1, a(n-i)+b+1),\end{aligned} \tag{4.22}$$

where B is the beta function. Applying Stirling's formula again, and using (4.20) and (4.11), we see that this upper bound has the asymptotic value

$$2^{-n}\frac{(pr)^N}{\pi^N}\frac{\Gamma(q+a(n-1)+b+1)}{[p(r-1)]^{q+a(n-1)+b+1}}\prod_{2\leqslant i\leqslant n}\frac{\Gamma(a(n-i)+b+1)}{[p(r-1)]^{a(n-i)+b+1}}$$
$$\sim Cr^{\{N-[q+a(n-1)+b+1]-\sum_{2\leqslant i\leqslant n}[a(n-i)+b+1]\}}=Cr^{-q}, \tag{4.23}$$

as $r\to\infty$. This proves Lemma 3.1(ii).

For every $\zeta\in D$, the transformation [20, Proposition 9.8]

$$\gamma_\zeta(\eta):=\zeta+B(\zeta,\zeta)^{1/2}\,(\eta^{-\zeta}) \tag{4.24}$$

defines an element in $\mathrm{Aut}(D)$ with $\gamma_\zeta(0)=\zeta$. Here

$$\eta^{-\zeta}:=B(\eta,-\zeta)^{-1}\,(\eta+\{\eta\zeta^*\eta\}) \tag{4.25}$$

denotes the so-called "quasi-inverse." If $\eta\in D$, $\xi\in Z$, and $t\in\mathbb{C}$ is small, we have the "addition formula" [20, A3]

$$(\eta+t\xi)^{-\zeta}=\eta^{-\zeta}+B(\eta,-\zeta)^{-1}\,([t\xi]^{(-\zeta^{\eta})}). \tag{4.26}$$

This implies that the complex derivative

$$\gamma'_\zeta(\eta)=B(\zeta,\zeta)^{1/2}\,B(\eta,-\zeta)^{-1}, \tag{4.27}$$

since for every $\vartheta\in Z$,

$$\frac{(t\xi)^\vartheta}{t}=B(t\xi,\vartheta)^{-1}\,(\xi-t\{\xi\vartheta^*\xi\})\to\xi \tag{4.28}$$

as $t\to 0$. It follows that

$$\begin{aligned}|\gamma_\zeta(\eta)-\gamma_\zeta(0)|&=|B(\zeta,\zeta)^{1/2}\eta^{-\zeta}|\\&=|\gamma'_\zeta(\eta)(\eta+\{\eta\zeta^*\eta\})|\\&\leqslant|\gamma'_\zeta(\eta)|\cdot|\eta+\{\eta\zeta^*\eta\}|\\&\leqslant\frac{1}{1-|\eta|}\cdot C\,|\eta|,\end{aligned} \tag{4.29}$$

since Cauchy's inequality [29, Corollary 1.13] implies

$$|\gamma'_\zeta(\eta)|\leqslant\frac{1}{1-|\eta|}\sup_{|\xi|\leqslant 1}|\gamma_\zeta(\xi)|=\frac{1}{1-|\eta|}. \tag{4.30}$$

This proves Lemma 3.2(i). By (4.27), we have

$$\gamma'_\zeta(0)=B(\zeta,\zeta)^{1/2}, \tag{4.31}$$

so that (4.13) implies

$$\det \gamma'_\zeta(0) = \det B(\zeta, \zeta)^{1/2} = \Delta(\zeta, \zeta)^{p/2}$$

$$= \prod_{1 \leqslant i \leqslant n} (1 - \zeta_i^2)^{p/2} \leqslant (1 - |\zeta|^2)^{p/2} \leqslant 1 - \varepsilon, \tag{4.32}$$

whenever $|\zeta| \geqslant \delta \geqslant 0$. Here ζ_i are the singular values of ζ. This proves Lemma 3.2(ii).

Finally, for each k, the kth complex derivative of $\gamma_\zeta(\eta)$ can be estimated by

$$|\gamma_\zeta^{(k)}(\eta)| \leqslant C\, |B(\eta, -\zeta)^{-1}|^{k+1}, \tag{4.33}$$

as follows from (4.27). Another application of Cauchy's inequality shows

$$|B(\eta, \eta)^{1/2} B(\eta, -\zeta)^{-1} B(\zeta, \zeta)^{1/2}| = |\gamma'_\eta(\zeta)\gamma'_\zeta(0)| = |(\gamma_\eta \circ \gamma_\zeta)'\,(0)| \leqslant 1. \tag{4.34}$$

It follows that

$$|B(\eta, -\zeta)^{-1}| \leqslant |B(\eta, \eta)^{-1/2}| \cdot |B(\zeta, \zeta)^{-1/2}|. \tag{4.35}$$

Now, [20, Corollary 3.15] implies that $B(\eta, \eta)$ is a diagonal operator with respect to the Peirce decomposition (4.7),

$$B(\eta, \eta) z_{ij} = (1 - \eta_i^2)(1 - \eta_j^2) z_{ij}, \tag{4.36}$$

where $z_{ij} \in Z_{ij}$. Since

$$(1 - \eta_i^2)^{-1/2} (1 - \eta_j^2)^{-1/2} \leqslant \prod_{1 \leqslant k \leqslant n} (1 - \eta_k^2)^{-1} = \Delta(\eta, \eta)^{-1}, \tag{4.37}$$

for all i, j (even for $i = j$), we conclude

$$|B(\eta, \eta)^{-1/2}| \leqslant \Delta(\eta, \eta)^{-1} = K_D(\eta, \eta)^{1/p}, \tag{4.38}$$

and therefore

$$|B(\eta, -\zeta)^{-1}| \leqslant K_D(\eta, \eta)^{1/p} K_D(\zeta, \zeta)^{1/p}. \tag{4.39}$$

This proves Lemma 3.2(iii).

5. Type I Domains

In this section, we consider the important special case of the "type I" domains and give direct proofs of Lemmas 3.1 and 3.2, thereby illustrating the abstract Jordan theoretic concepts used in Section 4. The vector space

$Z = \mathbb{C}^{m \times n}$ of all complex $m \times n$-matrices has dimension $N = mn$, and the Jordan triple product

$$\{\xi\eta^*\zeta\} = \tfrac{1}{2}(\xi\eta^*\zeta + \zeta\eta^*\xi), \tag{5.1}$$

for all $\xi, \eta, \zeta \in \mathbb{C}^{m \times n}$. Note that this "anti-commutator" product makes sense even for non-square matrices. The triple idempotent condition (4.2) becomes $ee^*e = e$ and characterizes the partial isometries. Thus (4.3) becomes the familiar singular value decomposition for matrices and the norm (4.4) is the operator norm of the matrix η, regarded as a Hilbert space operator from $\mathbb{C}^n$ to $\mathbb{C}^m$. The open unit ball (4.5) becomes the domain,

$$D_{m,n} = \{\zeta \in \mathbb{C}^{m \times n} : I_n - \zeta^*\zeta > 0\}, \tag{5.2}$$

of type I [14]. For $1 \leqslant i \leqslant \min(m, n)$, the $m \times n$-matrices e_i with 1 at the i, ith place,

$$[e_i]_{jk} := \delta_{ij}\delta_{ik}, \tag{5.3}$$

where $1 \leqslant j \leqslant m$, $1 \leqslant k \leqslant n$, form a maximal system of orthogonal triple idempotents. The corresponding Peirce decomposition (4.7) is a symmetrized version of the usual matrix units, and we have $a = 2$ and $b = |m - n|$, so that $b = 0$ for square matrices.

Th Bergman operators (4.12) have the form

$$B(\xi, \eta)\zeta := (I_m - \xi\eta^*)\zeta(I_n - \eta^*\xi), \tag{5.4}$$

and satisfy

$$\det B(\xi, \eta) = \det(I_n - \eta^*\xi)^{m+n}. \tag{5.5}$$

Thus $\Delta(\xi, \eta) := \det(I_n - \eta^*\xi)$ is the triple determinant and $p = m + n$. The Bergman kernel is

$$K_{D_{m,n}}(\zeta, \eta) = \det(I_n - \eta^*\zeta)^{-(m+n)}. \tag{5.6}$$

The invariant measure on $D_{m,n}$ is

$$d\mu_{D_{m,n}}(\zeta) = \det(I_n - \zeta^*\zeta)^{-(m+n)} d^{2mn}\zeta, \tag{5.7}$$

and thus according to the prescription of Section 2, we define

$$\begin{aligned} d\mu_r(\zeta) &:= \Lambda_r \det(I_n - \zeta^*\zeta)^{(m+n)(r-1)} d^{2mn}\zeta, \\ K^r_{D_{m,n}}(\zeta, \eta) &:= \det(I_n - \eta^*\zeta)^{-(m+n)r}. \end{aligned} \tag{5.8}$$

The normalization factor, A_r, is given by

$$A_r = \pi^{-mn} \prod_{k=1}^{n} \frac{\Gamma(\sigma+m+k)}{\Gamma(\sigma+k)}, \tag{5.9}$$

where $\sigma = (m+n)(r-1)$.

The group $U(m, n)$ acts on $D_{m,n}$ by holomorphic automorphisms. Let γ be an element of $U(m, n)$, i.e.,

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.10}$$

where the submatrices A, B, C, and D have dimensions $m \times m$, $m \times n$, $n \times m$, and $n \times n$, respectively, and satisfy

$$A^*A - C^*C = I_m,$$
$$A^*B = C^*D,$$
$$D^*D - B^*B = I_n.$$

The corresponding element of $\mathrm{Aut}(D_{m,n})$ is

$$\gamma\colon \eta \mapsto (A\eta + B)(C\eta + D)^{-1}. \tag{5.11}$$

For $\zeta \in D_{m,n}$, we define γ_ζ by

$$\gamma_\zeta := \begin{pmatrix} (I_m - \zeta\zeta^*)^{-1/2} & (I_m - \zeta\zeta^*)^{-1/2}\zeta \\ (I_n - \zeta^*\zeta)^{-1/2}\zeta^* & (I_n - \zeta^*\zeta)^{-1/2} \end{pmatrix}. \tag{5.12}$$

As an automophism of $D_{m,n}$, γ_ζ is given by

$$\begin{aligned} \gamma_\zeta(\eta) &= (I_m - \zeta\zeta^*)^{-1/2} (\eta + \zeta)(I_n + \zeta^*\eta)^{-1} (I_n - \zeta^*\zeta)^{1/2} \\ &= \zeta + (I_m - \zeta\zeta^*)^{1/2} \eta (I_n + \zeta^*\eta)^{-1} (I_n - \zeta^*\zeta)^{1/2}. \end{aligned} \tag{5.13}$$

Proof of Lemma 3.1 *for* $D_{m,n}$. For condition (i), we see immediately from (5.9) that $A_r \sim r^{mn}$ as $r \to \infty$. Condition (ii) is proven as follows. Let q_j be the jth column of η, and let Z_j be the $m \times (j-1)$ matrix formed from the first $(j-1)$ columns of η. Then define $w_j = (I_m - Z_j Z_j^*)^{-1/2} q_j$. The change of variables from η to the w_j's gives (see [15])

$$\int_{D_{m,n}} d\mu_r(\eta) = A_r \prod_{j=1}^{n} \left\{ \int_{|w_j|^2 < 1} (1 - |w_j|^2)^{(m+n)(r-1)+n-j} d^{2m} w_j \right\}. \tag{5.14}$$

We clearly have

$$|\eta|^2 = \sum_{j=1}^{n} |q_j|^2 \leqslant \sum_{j=1}^{n} |w_j|^2. \tag{5.15}$$

Therefore, by changing variables as above, we get the inequality,

$$\int_{D_{m,n}} |\eta|^{2q}\, d\mu_r(\eta) \leqslant CA_r \int d^{2m}w_1 \cdots d^{2m}w_n \left(\sum_{j=1}^{n} |w_j|^2\right)^q$$
$$\times \prod_{l=1}^{n} (1-|w_j|^2)^{\sigma+l-1}, \tag{5.16}$$

where $\sigma=(m+n)(r-1)$. Changing variables again to $u_j=|w_j|^2$ yields

$$\int_{D_{m,n}} |\eta|^{2q}\, d\mu_r(\eta) \leqslant CA_r \int_0^1 u_1^{m-1}\, du_1 \cdots u_n^{m-1}\, du_n \left(\sum_{j=1}^{n} u_j\right)^q$$
$$\times \prod_{l=1}^{n} (1-u_j)^{(m+n)(r-1)+n-l}. \tag{5.17}$$

This multiple integral can be broken up into a product of one-dimensional integrals. These all have the form of beta functions,

$$\int_0^1 (1-u)^\rho u^{k+m-1}\, du = B(\rho+1, k+m), \tag{5.18}$$

which we can bound by $C\rho^{-k-m}$ as $\rho\to\infty$. Using this bound in the expression (5.17) and counting up the powers of r gives us

$$\int_{D_{m,n}} |\eta|^{2q}\, d\mu_r(\eta) \leqslant Cr^{-q}. \quad \blacksquare \tag{5.19}$$

Proof of Lemma 3.2(i) *for* $D_{m,n}$. We show that there exist constants C_1, C_2, and δ such that

$$|\gamma_\zeta(\eta)-\gamma_\zeta(0)| \leqslant \frac{C_1\,|\eta|}{1-C_2\,|\eta|}, \tag{5.20}$$

provided $|\eta|<\delta$. Note first that if A and B are $l\times m$ and $m\times n$ matrices, respectively, then

$$|AB| \leqslant lmn\,|A|\,|B|. \tag{5.21}$$

Thus for

$$|\gamma_\zeta(\eta)-\gamma_\zeta(0)| = |(I_m-\zeta\zeta^*)^{1/2}\eta(I_n+\zeta^*\eta)^{-1}(I_n-\zeta^*\zeta)^{1/2}|, \tag{5.22}$$

we obtain a bound,

$$|\gamma_\zeta(\eta)-\gamma_\zeta(0)| \leqslant C\,|\eta|\,|(I_n+\zeta^*\eta)^{-1}|. \tag{5.23}$$

To complete the proof, we observe that

$$\begin{aligned} |(I_n+\zeta^*\eta)^{-1}|^2 &= \sum_{k,l=0}^{\infty} \operatorname{tr}[(\zeta^*\eta)^k (\eta^*\zeta)^l] \\ &\leqslant n \sum_{k,l=0}^{\infty} (C\,|\eta|)^{k+l} \\ &= \frac{n}{(1-C\,|\eta|)^2}. \quad \blacksquare \end{aligned} \tag{5.24}$$

Proof of Lemma 3.2(ii) *for* $D_{m,n}$. First, note that

$$|\det \gamma'_\zeta(0)|^2 = \det(I_n - \zeta^*\zeta)^{(m+n)}. \tag{5.25}$$

Let λ be the largest eigenvalue of $\zeta^*\zeta$. The condition that $|\zeta| \geqslant \delta$ implies that $\lambda > \mu\delta^2$, with some μ depending only on n. Thus we have

$$|\det \gamma'_\zeta(0)|^2 \leqslant (1-\mu\delta^2)^{(m+n)}, \tag{5.26}$$

for δ suffiiently small and all ζ. ∎

Proof of Lemma 3.2(iii) *for* $D_{m,n}$. The only potentially unbounded factor in $\gamma_\zeta(\eta)$ or its derivatives is the $(I_n+\zeta^*\xi)$ appearing in the denominator. Since each derivative will give a term with an extra such factor in the denominator, we see that we can bound

$$\sup_\zeta \sum_{i,j_1,\ldots,j_k} \left| \frac{\partial^k \gamma_\zeta(\xi)_i}{\partial\xi_{j_1}\cdots\partial\xi_{j_k}} \right| \leqslant C\,|(I_n+\zeta^*\xi)^{-1}|^{k+1}. \tag{5.27}$$

Assume for the moment that $m \geqslant n$. Let σ be an eigenvalue of $\zeta^*\xi$, and let λ^2 be the largest eigenvalue of $\xi^*\xi$. We can argue that $|\sigma|^2 \leqslant \lambda^2$, as follows. Let x be an eigenvector associated to σ, then

$$|\sigma|^2 = (\zeta^*\xi x, \zeta^*\xi x) = (\xi x, \zeta\zeta^*\xi x) \leqslant (\xi x, \xi x) \leqslant \lambda^2. \tag{5.28}$$

Using this fact, we obtain

$$\begin{aligned} \sup_\zeta |(I_n+\zeta^*\xi)^{-1}| &\leqslant \frac{n^{1/2}}{1-\lambda} \\ &\leqslant \frac{2n^{1/2}}{1-\lambda^2}. \end{aligned} \tag{5.29}$$

Noting also that

$$\det(I_n - \xi^*\xi) \geqslant (1-\lambda^2)^n, \tag{5.30}$$

we have

$$\sup_{\zeta} |(I_n + \zeta^*\xi)^{-1}| \leqslant 2n^{1/2} \det(I_n - \xi^*\xi)^{-1/n}. \tag{5.31}$$

If $n \geqslant m$, then n is simply replaced by m in this formula. We complete the proof by combining (5.31) with (5.27). ∎

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