On Convergence of Inverse Functions of Operators*

ARTHUR JAFFE, ANDRZEJ LESNIEWSKI, AND KONRAD OSTERWALDER

Harvard University, Cambridge, Massachusetts 02138

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We study sequences \( \{A_n\} \) of self-adjoint operators on a Hilbert space \( \mathcal{H} \). We give a sufficient condition on a function \( f \) and on the \( \{A_n\} \) such that \( f(|A_n|) \to f(|A|) \) ensures \( A_n \to A \).

Let \( \{A_n\}_{n=1}^{\infty} \) denote a sequence of self-adjoint operators on a Hilbert space \( \mathcal{H} \) and let \( F \) be a real-valued function on \( \mathbb{R} \). We study here the question of whether \( T_n = F(A_n) \) converging to a self-adjoint limit \( T \) ensures that \( A_n \) converges to a self-adjoint limit \( A \). We are especially interested in the case for which \( F \) is not injective. In that case, we also require some additional condition on the convergence of \( \{A_n\} \) in some weak sense, from which we conclude the convergence of \( \{A_n\} \) in a stronger sense. More specifically, we introduce the notion of heat kernel regularization: this is obtained by regularizing the bilinear form \( A_n \) with a heat kernel \( K_n = \exp(-T_n) \). The regularized operator is \( K_n A_n K_n \). A norm convergent heat kernel regularization leads to a norm convergence of the resolvents of \( A_n \).

Let us start with a simple, motivating example. Let \( \mathcal{X} = \mathcal{X} \oplus \mathcal{X} \) be a direct sum of isomorphic subspaces, and let \( A_k \) have the off-diagonal form

\[
A_k = \begin{pmatrix}
0 & Q^* \\
Q_k & 0
\end{pmatrix}
\]

with respect to this decomposition. The function \( F(t) = t^2 \) gives rise to the positive operator

\[
A_n^2 = \begin{pmatrix}
Q_n^* Q_n & 0 \\
0 & Q_n Q_n^*
\end{pmatrix}.
\]

It is clear that while \( A_n \) is a square root of \( A_n^2 \), it is not a square root defined by the spectral theorem, which is diagonal with respect to \( \mathcal{H} = \mathcal{X} \oplus \mathcal{X} \).

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Permanent address: Mathematics Department, ETH Zentrum, Zürich.
The convergence $A_n^2 \to A^2$ does not in general yield $A_n \to A$, even if $\mathcal{H}$ is finite-dimensional. This concrete example [2] led us to the more general question.

We are concerned here with norm-resolvent convergence. Let $R_n(z) = (A_n - z)^{-1}$ be the resolvent of $A_n$.

**Definition 1.** The sequence of self-adjoint operators $\{A_n\}_{n=1}^\infty$ has a norm-resolvent limit $A$ if for all $z$ with $\text{Im } z \neq 0$, the operators $R_n(z)$ converge in norm to the resolvent $R(z)$ of a self-adjoint operator $A$.

Let us denote the norm-resolvent limit $A$ of $A_n$ by $\lim_{n \to \infty} A_n$. The two following results are well known.

**Proposition 2.** [1] If $R_{n}(\pm t)$ converge in norm to operators $R_{\pm}$ with densely defined inverses, then $R_{\pm}$ is the resolvent of a self-adjoint operator $A$ and $\lim A_n = A$.

**Proposition 3.** (Theorem VIII.20 of [3]). Let $g$ be a continuous function which vanishes at $\infty$. Then $\lim A_n = A$ ensures that as $n \to \infty$,

$$\|g(A_n) - g(A)\| = o(1).$$

In this note we look for a partial converse to Proposition 3. We study functions $g(t) = f(t)$, where $f$ is well behaved:

**Definition 4.** A continuous function $f: \mathbb{R}_+ \to \mathbb{R}_+$ is well behaved if $f$ is monotonic and $\ln t \leq f(t)$ for $t$ sufficiently large.

For any well-behaved $f$ we define a function from self-adjoint operator $A$ to "heat kernel" contraction semigroups $K^\beta$ on $\mathcal{H}$ by the map

$$A \mapsto K(A)^\beta = \exp(-\beta f(|A|)), \quad \beta \geq 0.$$

Let $K_n = K(A_n)$ be the semigroup for $A_n$. Since $f$ increases monotonically to $\infty$, $f$ has a continuous inverse. Also $A_n K_n$ is a bounded operator, with $\|A_n K_n\|$ bounded uniformly in $n$. In particular, if $\delta A_{nm} = (A_n - A_m)^{-1}$ is the closure of $A_n - A_m$, then $K_n \delta A_{nm} K_m$ is bounded for all $n, m$. Note that with our assumptions on $f$, the sequence $\{K_n A_n K_n\}$ is norm convergent if and only if $\|K_n K_m\| = o(1)$ as $n, m \to \infty$.

**Definition 5.** The sequence $\{A_n\}$ has a convergent heat kernel regularization with respect to $f$ if

$$\|K_n \delta A_{nm} K_m\| = o(1),$$

as $n, m \to \infty$. 


Our main result is the following:

**Theorem 6.** Let $f$ be well behaved and let $\{A_n\}$ be a sequence of operators with a convergent heat kernel regularization with respect to $f$. Let $f(|A_n|)$ have a norm-resolvent limit. Then there exists a self-adjoint operator $A$ such that

$$T = \text{n.r. lim } f(|A_n|) = f(|A|),$$

and

$$\text{n.r. lim } A_n = A.$$

**Lemma 7.** Let $S_n = (A_n + i)^{-1}$. Let $\hat{\partial}_n = \partial(|A_n|)$, where $\partial: [0, \infty) \to [1, \infty)$ is a monotonic, continuous function tending to $\infty$, and satisfying $\partial(\lambda) = O(\lambda)$. Then $\hat{\partial}_n S_n$ is bounded uniformly in $n$.

**Proof.** By the spectral theorem, $|A_n|$ commutes with $S_n$, so the magnitude of the spectrum of $\hat{\partial}_n S_n$ is

$$|\lambda + i| \partial(|\lambda|)$$

for $\lambda \in \text{spectum } A_n$. Since $\partial(|\lambda|) = O(|\lambda|)$, it follows that (1) is bounded. The bound is independent of $n$, and ensures $\|\hat{\partial}_n S_n\| \leq \text{const}$.

**Proof of the Theorem.** The resolvent identity on the range of $K_n$ is

$$\delta S = S_n - S_m = -S_m \delta A_{nm} S_n.$$ 

Let $E_n(\lambda)$ denote the spectral projection of $f(|A_n|)$ onto the subspace $f(|A_n|) \leq \lambda$. We study

$$\delta S = \delta S E_n + \delta S(I - E_n).$$

Using the lemma, as $\lambda \to \infty$, and letting $\hat{\partial}_n^{-1}$ denote the operator inverse of $\hat{\partial}_n$ (rather than the inverse function),

$$\|S_n(I - E_n)\| = \|S_n \hat{\partial}_n \hat{\partial}_n^{-1}(I - E_n)\| = O(1) \|\hat{\partial}_n^{-1}(I - E_n)\| = o(1).$$

uniformly in $n$. Also

$$S_m(I - E_n) = S_m \hat{\partial}_m \hat{\partial}_m^{-1}(I - E_n)$$

$$= S_m \hat{\partial}_m (\hat{\partial}_m^{-1} - \hat{\partial}_m^{-1})(I - E_n) + S_m \hat{\partial}_m \hat{\partial}_m^{-1}(I - E_n).$$

Since $f(|A_n|)$ converges in the norm resolvent sense by hypothesis, it
follows that \( h(|A_n|) \) is norm-convergent where \( h \) is any bounded function which vanishes at \( \infty \). In particular, \( \|\partial_n^{-1} - \partial_m^{-1}\| = o(1) \) as \( n, m \to \infty \). Thus
\[
\|S_n(I - E_n)\| = o(1) + O(1) \|\partial_n^{-1}(I - E_n)\|
\]
\[
= o(1) + o(1).
\]
(3)

From (2) to (3) we infer that given \( \varepsilon > 0 \), we can choose \( \lambda_0, N \) such that for \( i > \lambda_0 \) and \( n, m > N \),
\[
\|\delta(I - E_n)\| \leq \varepsilon.
\]
(4)

Likewise, taking the adjoint, and exchanging \( n, m \),
\[
\|(I - E_m)\delta S\| \leq \varepsilon.
\]
(5)

We have
\[
\delta S E_n = E_m \delta S E_n + (I - E_m) \delta S E_n,
\]
so from (4) to (5) we conclude
\[
\|\delta S\| \leq 2\varepsilon + \|E_m \delta S E_n\|.
\]
(6)

Now
\[
E_m \delta S E_n = -S_m E_m \delta A_{nm} E_n S_n
\]
\[
= -S_m K_m E_m K_m \delta A_{nm} K_n E_n K_n^{-1} S_n
\]
and
\[
\|E_m \delta S E_n\| \leq e^{2\varepsilon} \|K_m \delta A_{nm} K_n\|.
\]

Using the hypothesis, we choose \( n, m \) sufficiently large that
\[
\|E_m \delta S E_n\| \leq \varepsilon,
\]
(7)

By (6) and (7)
\[
\|\delta S\| \leq 3\varepsilon.
\]

and hence there exists \( S \) such that \( \|S_n - S\| \to 0 \). We can repeat the same estimates with \( S_n^* \) to conclude that \( S_n^* \to S^* \).

We wish to conclude that \( S \) is the resolvent of a self-adjoint operator \( A \). It is sufficient to show that \( S \) and \( S^* \) have densely defined inverses, and to use Proposition 2. Let \( g(\cdot) = (f^{-1}(\cdot)^2 + 1)^{-1} \). Since \( f \not\to \infty \) and \( f \) is invertible, we infer that \( f^{-1} \not\to \infty \). Thus \( g(\cdot) \not\to 0 \) at infinity. By
Proposition 3. \( g(f(|A_n|)) = (A_n^2 + I)^{-1} \) has a limit \( (f^{-1}(T^2 + I))^{-1} \), where \( T = \text{n.r. lim } f(|A_n|) \). But \( S_n^* S_n = (A_n^2 + I)^{-1} = S_n S_n^* \). Thus

\[
\lim_{n \to \infty} S_n^* S_n = \lim_{n \to \infty} S_n S_n^* = (f^{-1}(T^2 + I))^{-1} = S^* S = SS^*
\]

has a dense range and a densely defined inverse (with domain \( D(f^{-1}(T^2)) \)). Hence \( \text{Range}(S) = \text{Domain}(S^{-1}) \) and \( \text{Domain}(S^*-1) \) are dense. This completes the proof of the theorem.

**REFERENCES**