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On Convergence of Inverse Functions of Operators*

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We study sequences $\{A_n\}$ of self-adjoint operators on a Hilbert space $\mathscr{H}$. We give a sufficient condition on a function f and on the $\{A_n\}$ such that $f(|A_n|) \to f(|A|)$ ensures $A_n \to A$. © 1988 Academic Press, Inc.

Let $\{A_n\}_{n=1}^{\infty}$ denote a sequence of self-adjoint operators on a Hilbert space $\mathscr{H}$ and let F be a real-valued function on $\mathbb{R}$. We study here the question of whether $T_n = F(A_n)$ converging to a self-adjoint limit T ensures that A_n converges to a self-adjoint limit A. We are especially interested in the case for which F is not injective. In that case, we also require some additional condition on the convergence of $\{A_n\}$ in some weak sense, from which we conclude the convergence of $\{A_n\}$ in a stronger sense. More specifically, we introduce the notion of heat kernel regularization: this is obtained by regularizing the bilinear form A_n with a heat kernel $K_n = \exp(-T_n)$. The regularized operator is $K_n A_n K_n$. A norm convergent heat kernel regularization leads to a norm convergence of the resolvents of A_n.

Let us start with a simple, motivating example. Let $\mathscr{H} = \mathscr{K} \oplus \mathscr{K}$ be a direct sum of isomorphic subspaces, and let A_k have the off-diagonal form

$$A_k = \begin{pmatrix} O & Q_k^* \\ Q_k & O \end{pmatrix}$$

with respect to this decomposition. The function $F(t) = t^2$ gives rise to the positive operator

$$A_n^2 = \begin{pmatrix} Q_n^* Q_n & O \\ O & Q_n Q_n^* \end{pmatrix}.$$

It is clear that while A_n is a square root of A_n^2, it is not a square root defined by the spectral theorem, which is diagonal with respect to $\mathscr{H} =$

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$\mathscr{H} \oplus \mathscr{H}$. The convergence $A_n^2 \to A^2$ does not in general yield $A_n \to A$, even if $\mathscr{H}$ is finite-dimensional. This concrete example [2] led us to the more general question.

We are concerned here with norm-resolvent convergence. Let $R_n(z) = (A_n - z)^{-1}$ be the resolvent of A_n.

DEFINITION 1. The sequence of self-adjoint operators $\{A_n\}_{n=1}^{\infty}$ has a norm-resolvent limit A if for all z with $\operatorname{Im} z \neq 0$, the operators $R_n(z)$ converge in norm to the resolvent $R(z)$ of a self-adjoint operator A.

Let us denote the norm-resolvent limit A of A_n by n.r. $\lim_{n \to \infty} A_n$. The two following results are well known.

PROPOSITION 2 [1]. *If $R_n(\pm i)$ converge in norm to operators $R_{\pm}$ with densely defined inverses, then $R_{\pm}$ is the resolvent of a self-adjoint operator A and* n.r. $\lim A_n = A$.

PROPOSITION 3 (Theorem VIII.20 of [3]). *Let g be a continuous function which vanishes at ∞. Then* n.r. $\lim A_n = A$ *ensures that as $n \to \infty$,*

$$\|g(A_n) - g(A)\| = o(1).$$

In this note we look for a partial converse to Proposition 3. We study functions $g(t) = f(|t|)$, where f is well behaved:

DEFINITION 4. A continuous function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is well behaved if f is monotonic and $\ln t \leqslant f(t)$ for t sufficiently large.

For any well-behaved f we define a function from self-adjoint operator A to "heat kernel" contraction semigroups K^{β} on $\mathscr{H}$ by the map

$$A \mapsto K(A)^{\beta} = \exp(-\beta f(|A|)), \qquad \beta \geqslant 0.$$

Let $K_n = K(A_n)$ be the semigroup for A_n. Since f increases monotonically to ∞, f has a continuous inverse. Also $A_n K_n$ is a bounded operator, with $\|A_n K_n\|$ bounded uniformly in n. In particular, if $\delta A_{nm} = (A_n - A_m)^{-}$ is the closure of $A_n - A_m$, then $K_n\, \delta A_{nm}\, K_m$ is bounded for all n, m. Note that with our assumptions on f, the sequence $\{K_n A_n K_n\}$ is norm convergent if and only if $\|K_n\, \delta A_{nm}\, K_m\| = o(1)$ as $n, m \to \infty$.

DEFINITION 5. The sequence $\{A_n\}$ has a convergent heat kernel regularization with respect to f if

$$\|K_n\, \delta A_{nm}\, K_m\| = o(1),$$

as $n, m \to \infty$.

Our main result is the following:

THEOREM 6. *Let f be well behaved and let $\{A_n\}$ be a sequence of operators with a convergent heat kernel regularization with respect to f. Let $f(|A_n|)$ have a norm-resolvent limit. Then there exists a self-adjoint operator A such that*

$$T = \text{n.r.}\lim f(|A_n|) = f(|A|),$$

and

$$\text{n.r.}\lim A_n = A.$$

LEMMA 7. *Let $S_n = (A_n + i)^{-1}$. Let $\partial_n = \partial(|A_n|)$, where $\partial\colon [0, \infty) \to [1, \infty)$ is a monotonic, continuous function tending to ∞, and satisfying $\partial(\lambda) = O(\lambda)$. Then $\partial_n S_n$ is bounded uniformly in n.*

Proof. By the spectral theorem, $|A_n|$ commutes with S_n, so the magnitude of the spectrum of $\partial_n S_n$ is

$$|(\lambda + i)^{-1}|\, \partial(|\lambda|) \tag{1}$$

for $\lambda \in \text{spectrum}\, A_n$. Since $\partial(|\lambda|) = O(|\lambda|)$, it follows that (1) is bounded. The bound is independent of n, and ensures $\|\partial_n S_n\| \leqslant \text{const}$.

Proof of the Theorem. The resolvent identity on the range of K_n is

$$\delta S = S_n - S_m = -S_m\, \delta A_{nm}\, S_n.$$

Let $E_n(\lambda)$ denote the spectral projection of $f(|A_n|)$ onto the subspace $f(|A_n|) \leqslant \lambda$. We study

$$\delta S = \delta S\, E_n + \delta S(I - E_n).$$

Using the lemma, as $\lambda \to \infty$, and letting ∂_n^{-1} denote the operator inverse of ∂_n (rather than the inverse function),

$$\begin{aligned} \|S_n(I - E_n)\| = \|S_n \partial_n \partial_n^{-1}(I - E_n)\| &= O(1)\, \|\partial_n^{-1}(I - E_n)\| \\ &= o(1), \end{aligned} \tag{2}$$

uniformly in n. Also

$$\begin{aligned} S_m(I - E_n) &= S_m \partial_m \partial_m^{-1}(I - E_n) \\ &= S_m \partial_m(\partial_m^{-1} - \partial_n^{-1})(I - E_n) + S_m \partial_m \partial_n^{-1}(I - E_n). \end{aligned}$$

Since $f(|A_n|)$ converges in the norm resolvent sense by hypothesis, it follows that $h(|A_n|)$ is norm-convergent where h is any bounded function which vanishes at ∞. In particular, $\|\partial_n^{-1} - \partial_m^{-1}\| = o(1)$ as $n, m \to \infty$. Thus

$$\|S_m(I - E_n)\| = o(1) + O(1)\, \|\partial_n^{-1}(I - E_n)\|$$
$$= o(1) + o(1). \tag{3}$$

From (2) to (3) we infer that given $\varepsilon > 0$, we can choose λ_0, N such that for $\lambda > \lambda_0$ and $n, m > N$,

$$\|\partial S(I - E_n)\| \leqslant \varepsilon. \tag{4}$$

Likewise, taking the adjoint, and exchanging n, m,

$$\|(I - E_m)\, \delta S\| \leqslant \varepsilon. \tag{5}$$

We have

$$\delta S\, E_n = E_m\, \delta S\, E_n + (I - E_m)\, \delta S\, E_n,$$

so from (4) to (5) we conclude

$$\|\delta S\| \leqslant 2\varepsilon + \|E_m\, \delta S\, E_n\|. \tag{6}$$

Now

$$E_m\, \delta S\, E_n = -S_m E_m\, \delta A_{nm}\, E_n S_n$$
$$= -S_m K_m^{-1} E_m K_m\, \delta A_{nm}\, K_n E_n K_n^{-1} S_n$$

and

$$\|E_m\, \delta S\, E_n\| \leqslant e^{2\lambda}\, \|K_m\, \delta A_{nm}\, K_n\|.$$

Using the hypothesis, we choose n, m sufficiently large that

$$\|E_m\, \delta S\, E_n\| \leqslant \varepsilon, \tag{7}$$

By (6) and (7)

$$\|\delta S\| \leqslant 3\varepsilon,$$

and hence there exists S such that $\|S_n - S\| \to 0$. We can repeat the same estimates with S_n^* to conclude that $S_n^* \to S^*$.

We wish to conclude that S is the resolvent of a self-adjoint operator A. It is sufficient to show that S and S^* have densely defined inverses, and to use Proposition 2. Let $g(\cdot) = (f^{-1}(\cdot)^2 + I)^{-1}$. Since $f \nearrow \infty$ and f is invertible, we infer that $f^{-1} \nearrow \infty$. Thus $g(\cdot) \searrow 0$ at infinity. By

Proposition 3, $g(f(|A_n|)) = (A_n^2 + I)^{-1}$ has a limit $(f^{-1}(T)^2 + I)^{-1}$, where $T = \text{n.r.}\lim f(|A_n|)$. But $S_n^* S_n = (A_n^2 + I)^{-1} = S_n S_n^*$. Thus

$$\lim_{n \to \infty} S_n^* S_n = \lim_{n \to \infty} S_n S_n^* = (f^{-1}(T)^2 + I)^{-1} = S^* S = S S^*$$

has a dense range and a densely defined inverse (with domain $D(f^{-1}(T)^2)$). Hence Range(S) = Domain(S^{-1}) and Domain(S^{*-1}) are dense. This completes the proof of the theorem.

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