K-Theory 2: 675–682, 1989

On Super-KMS Functionals and Entire Cyclic Cohomology*

ARTHUR JAFFE, ANDRZEJ LESNIEWSKI, and KONRAD OSTERWALDER**
Harvard University, Cambridge, MA 02138, U.S.A.

(Received: 5 December 1988)

Abstract. We formulate the super-KMS condition suggested by Connes and Kastler, in the context of entire cyclic cohomology of quantum algebras. We show that the Chern character of Jaffe, Lesniewski, and Osterwalder – associated by Kastler to a super-KMS functional – satisfies the entire growth condition. Hence, a super-KMS functional defines a cocycle for the entire cyclic cohomology of quantum algebras.

Key words. Cyclic cohomology, $\mathbb{C}^*$-algebras, KMS condition.

1. Introduction

The purpose of this note is to clarify the relation between the super-KMS property and entire cyclic cohomology of quantum algebras. Our interest in super-KMS functionals was inspired by work of Kastler [4] and by private conversations with Alain Connes. This generalization is the natural framework for entire cyclic cohomology in the case that the Laplace operator has continuous spectrum. Such situations can arise if the cohomology is based on a noncompact manifold. In intuitive terms, one would like to formulate this in terms of a noncompact, 'noncommutative manifold'.

Just as in statistical mechanics where a KMS state generalizes the notion of a Gibbs state, a super-KMS functional generalizes the positive temperature supertrace functional. This allows us to deal with situations which occur in examples, such as supersymmetric field theory on a noncompact manifold: the Laplace–Beltrami operator on loop space (the Hamiltonian of such a theory) is expected to have continuum spectrum, so the heat kernel it generates will not be trace class.

This is characteristic of many examples. Besides, from a conceptual point of view, it is irrelevant whether the heat kernel is trace class; this assumption can be replaced by the super-KMS property. Such functionals are not necessarily positive and, hence, are not states, but estimates can be proved by expressing ω as a linear combination of states.

The usual KMS property relates the cyclicity of a state ω to the analytic continuation of a group α_t of automorphisms. The super-KMS property also involves the super

*Supported in part by the National Science Foundation.
**Permanent address: ETH Zentrum, 8092 Zürich, Switzerland.

derivation d whose square generates the automorphism group α_t. Translation invariance of ω is a consequence of the super-KMS property.

2. Quantum Algebras and Cyclic Cohomology

Our present definition of a quantum algebra is more general than in [3]. We define it as a quadruple $(\mathscr{A}, \Gamma, \alpha_t, d)$ with the following properties:

(i) *Algebra.* The algebra $\mathscr{A} = \mathscr{A}_+ \oplus \mathscr{A}_-$ is a $\mathbb{Z}_2$-graded, unital C^*-algebra. Let $a = a_+ + a_-$ denote the decomposition of $a \in \mathscr{A}$ into homogeneous components. The grading induces an automorphism Γ of $\mathscr{A}$, defined by

$$a \mapsto a^\Gamma = a_+ - a_-. \tag{1}$$

(ii) *Group Action.* The family $t \to \alpha_t$, $t \in \mathbb{R}$, is a continuous, one-parameter group of $*$-automorphisms of $\mathscr{A}$, even under the grading. Thus

$$\alpha_t(a)^* = \alpha_t(a^*), \qquad \alpha_t(a)^\Gamma = \alpha_t(a^\Gamma). \tag{2}$$

By standard smoothing arguments, see Section 8.12 of [5], there exists a norm-dense $*$-subalgebra $\mathscr{A}_\alpha$ of $\mathscr{A}$ such that

$$t \to \alpha_t(a), \quad a \in \mathscr{A}_\alpha \tag{3}$$

extends to an entire, $\mathscr{A}$-valued function, which we denote by $\alpha_z(a)$.

(iii) *Even Derivation.* The infinitesimal generator of α_t

$$D = -i\frac{d}{dt}\alpha_t|_{t=0} \tag{4}$$

is an even derivation of $\mathscr{A}$ with domain including $\mathscr{A}_\alpha$. Since α_t is a $*$-automorphism, it follows that $\|\alpha_t(a)\| = \|a\|$. Hence α_t is uniquely determined by the action of D on $\mathscr{A}_\alpha$, where one can construct α_t by a convergent power series. It is therefore no loss of generality to assume that $\mathscr{A}_\alpha$ is the domain of D.

The derivation D is even by virtue of (2).

$$(Da)^\Gamma = Da^\Gamma, \qquad D(ab) = (Da)b + aDb. \tag{5a}$$

We define an adjoint derivation D^* of an even derivation D by

$$D^*a = -(Da^*)^*. \tag{5b}$$

Since $*$ and Γ commute, the adjoint D^* is also even. In our case, since α_t is a $*$-automorphism,

$$D^* = D, \tag{5c}$$

or equivalently

$$(Da)^* = -Da^*. \tag{5d}$$

(iv) *Super (Odd) Derivation.* We assume that d is a super (odd) derivation of $\mathscr{A}$ with a dense domain $D(d)$. Assume that $\mathscr{A}_\alpha$ is a core for d and that $d: \mathscr{A}_\alpha \to \mathscr{A}_\alpha$. A super-derivation satisfies

$$(da)^\Gamma = -da^\Gamma, \qquad d(ab) = (da)b + a^\Gamma\, db. \tag{6a}$$

The natural adjoint of a super-derivation d is the super-derivation d^+ defined by

$$d^+ a = (da^*)^{*\Gamma}. \tag{6b}$$

(v) *Square Root Property.* The square of any super derivation is an even derivation,

$$(d^2 a)^\Gamma = d^2 a^\Gamma, \qquad d^2(ab) = (d^2 a)b + a(d^2 b). \tag{7a}$$

The fundamental assumption which makes the present structure useful is that d is a square root of D, namely

$$D = d^2. \tag{7b}$$

We call this the square root property of the quantum algebra: the generator of α_t has a (super) square root.

REMARKS. 1. It can be checked from (7) that

$$\alpha_t \circ d = d \circ \alpha_t. \tag{8a}$$

This can be verified by power series expansion of α_t in t on the domain $\mathscr{A}_\alpha$, and extension to the domain of the closure of d by continuity.

2. We claim that if $d: \mathscr{A}_\alpha \to \mathscr{A}_\alpha$ is a super-derivation, then

$$(d^+)^2 = (d^2)^*. \tag{9a}$$

In fact

$$(d^+)^2 a = (d(d^+ a)^*)^{*\Gamma} = (d((da^*)^{*\Gamma})^*)^{*\Gamma} = (d(da^*)^\Gamma)^{*\Gamma} = -(d^2 a^*)^* = (d^2)^* a.$$

In particular, since $d^2 = D = D^*$, we have

$$(d^+)^2 = d^2 = D. \tag{8b}$$

It follows as in Remark 1 that

$$\alpha_t \circ d^+ = d^+ \circ \alpha_t. \tag{9b}$$

3. As a consequence of Remark 2, we see that: if $(\mathscr{A}, \Gamma, \alpha_t, d)$ is a quantum algebra, then $(\mathscr{A}, \Gamma, \alpha_t, d^+)$ is also a quantum algebra.

4. In general $d^+ \neq d$, even though their squares are equal. In case the graded derivation d is given by the graded commutator with an operator Q, then d^+ is given by the graded commutator with Q^*. Thus, the condition $d^+ = d$ is equivalent to the hermiticity of Q. In this case, $d^2 = \mathrm{Ad}(Q^2)$ and $(d^+)^2 = \mathrm{Ad}(Q^{*2})$. Thus $D^* = D$, which we assume,

ensures that $H = Q^2$ is symmetric – while $d^+ = d$, which we do not assume, is associated with a spectral condition (positivity of Q^2).

In order to set up the analytic framework, we introduce the Sobolev norm

$$\|a\|_* = \|a\| + \|da\|. \tag{10}$$

Let $\mathscr{C}^n(\mathscr{A})$ denote the space of $(n+1)$-linear functionals on $\mathscr{A}$ which are continuous with respect to the norm $\|\cdot\|_*$. Let $\|f_n\|_*$ denote the norm of $f_n \in \mathscr{C}^n(\mathscr{A})$ with respect to the norm $\|\cdot\|_*$. Define $\mathscr{C}(\mathscr{A})$ as the space of sequences $f = (f_0, f_1, \dots)$, where $f_n \in \mathscr{C}^n(\mathscr{A})$ and which satisfy the entire analyticity condition [1],

$$n^{1/2} \|f_n\|_*^{1/n} \to 0. \tag{11}$$

The space of cochains $\mathscr{C}(\mathscr{A})$ can be decomposed into components which are even or odd under the action of Γ,

$$\mathscr{C}(\mathscr{A}) = \mathscr{C}_+(\mathscr{A}) \oplus \mathscr{C}_-(\mathscr{A}). \tag{12}$$

We use the standard coboundary operators b and B, defined separately on $\mathscr{C}_+(\mathscr{A})$ and on $\mathscr{C}_-(\mathscr{A})$, and which extend linearly to $\mathscr{C}(\mathscr{A})$ [1]. These operators define the creation-annihilation complex

$$b\colon \mathscr{C}^n(\mathscr{A}) \to \mathscr{C}^{n+1}(\mathscr{A}), \qquad B\colon \mathscr{C}^{n+1}(\mathscr{A}) \to \mathscr{C}^n(\mathscr{A}) \tag{13}$$

and satisfy

$$b^2 = 0, \qquad B^2 = 0, \qquad Bb + bB = 0. \tag{14}$$

The coboundary operator $\partial = b + B$ is used to define entire cyclic cohomology. Explicitly

$$\begin{aligned}(Bf_n)(a_0, \dots, a_{n-1}) = \sum_{j=0}^{n-1} (-1)^{(n-1)j} (f_n(\mathbf{1}, a_{n-j}^\gamma, \dots, a_{n-1}^\gamma, a_0, \dots, a_{n-j-1}) +& \\ + (-1)^{n-1} f_n(a_{n-j}^\gamma, \dots, a_{n-1}^\gamma, a_0, \dots, a_{n-j-1}, \mathbf{1})),&\end{aligned} \tag{15}$$

and

$$\begin{aligned}(bf_n)(a_0, \dots, a_{n+1}) = \sum_{j=0}^{n} (-1)^j f_n(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) +& \\ + (-1)^{n+1} f_n(a_{n+1}^\gamma a_0, a_1, \dots, a_n).&\end{aligned} \tag{16}$$

Here

$$a^\gamma = \begin{cases} a^\Gamma, & \text{if } f_n \in \mathscr{C}_+^n(\mathscr{A}), \\ a, & \text{if } f_n \in \mathscr{C}_-^n(\mathscr{A}). \end{cases} \tag{17}$$

3. Super-KMS Functionals

We consider a continuous linear functional ω on $\mathscr{A}$. It is natural to restrict attention to the subalgebra $\mathscr{A}_\alpha$ on which d is defined and α_t is entire.

DEFINITION 3.1. A continuous linear functional ω on $\mathscr{A}$ has the sKMS property with respect to the quantum algebra $(\mathscr{A}, \Gamma, \alpha_t, d)$, if for all $a, b \in \mathscr{A}_\alpha$ and for all $z \in \mathbb{R}$,

$$\omega(\alpha_z(a)b) = \omega(b^\Gamma \alpha_{z+i}(a)) \tag{18}$$

and

$$\omega \circ d = 0, \quad \text{on } D(d). \tag{19}$$

REMARKS. 1. The assumption (18) extends to all complex $z \in \mathbb{C}$ by analytic continuation.

2. One could introduce a positive parameter $0 < \beta$ and define a β-sKMS property

$$\omega^\beta(\alpha_z(a)b) = \omega^\beta(b^\Gamma \alpha_{z+i\beta}(a)).$$

Replacing α_t by $\alpha_{t/\beta}$ reduces β-sKMS to sKMS.

3. We also use the notation, consistent with [3] defining

$$a(t) = \alpha_{it}(a), \quad a \in \mathscr{A}_\alpha. \tag{20}$$

4. Clearly, $\omega^\Gamma = \omega$, namely $\omega(a^\Gamma) = \omega(a)$, as a consequence of (18).

THEOREM 3.2. *An sKMS functional on a quantum algebra $(\mathscr{A}, \Gamma, \alpha_t, d)$ satisfies*

$$\omega \circ \alpha_t = \omega \text{ on } \mathscr{A}, \quad \text{for } t \in \mathbb{R}, \tag{21}$$

$$\omega \circ \alpha_z = \omega \text{ on } \mathscr{A}_\alpha, \quad \text{for } z \in \mathbb{C}, \tag{22}$$

and for $a, b \in \mathscr{A}_\alpha, z \in \mathbb{C}$,

$$\omega(adb(z)) = \omega((db)^\Gamma a(1-z)). \tag{23}$$

Proof. Since $d^2 = D$ is the generator of α_t, convergent power series on $\mathscr{A}_\alpha$ show that for $z \in \mathbb{C}$, $\omega \circ \alpha_z = \omega$ on the subalgebra $\mathscr{A}_\alpha$. This extends to $\mathscr{A}$ by continuity for all $z \in \mathbb{R}$, and hence (21)–(22) follow. Equation (23) is a straightforward consequence of Definition 3.1 and (22). □

Let us finish this section with an example which provides motivation for the definition of sKMS functionals. Assume that $\mathscr{A}$ is represented as an algebra of operators on a Hilbert space $\mathscr{H}$. We assume that α_t is spatial, so there exists a self-adjoint operator H on $\mathscr{H}$ which generates α_t,

$$\alpha_t(a) = \exp(itH)a\exp(-itH).$$

Thus $D = \mathrm{Ad}(H)$ is the generator of the automorphism. We also assume that there is a self-adjoint operator Q which is a square root of H, $H = Q^2$, and which is odd:

$Q^{\Gamma} = \Gamma Q \Gamma = -Q$. We can define $da = (Qa_+ - a_+ Q) + (Qa_- + a_- Q)$ as the graded derivation. We also assume that $\exp(-H)$ is a trace class.

The super-trace functional provides an elementary example of an sKMS functional. Define

$$\omega(\cdot) = \mathrm{Str}(\cdot e^{-H}) = \mathrm{Tr}(\Gamma \cdot e^{-H}). \tag{27}$$

It is easy to verify that (27) has the sKMS property, and the graded derivation satisfies $d = d^+$, yielding $0 \leqslant H$. This formula is very similar to the standard example of a KMS state

$$\omega_{\mathrm{KMS}}(\cdot) = \frac{\mathrm{Tr}(\cdot e^{-H})}{\mathrm{Tr}(e^{-H})}. \tag{28}$$

(Note that (28) does not satisfy the sKMS property.) In general, a KMS functional cannot be expressed in the form (28), but often one obtains a KMS state as a limit of states of the form (28). Likewise, an sKMS functional can often be obtained as a limit of functionals of the form (27), though it may not have such a representation. Furthermore, note that (27) is not normalized; in fact it may not be positive. For the example (27), if we define $Q_+ = \frac{1}{4}(1 - \Gamma)Q(1 + \Gamma)$, then the index of Q_+ is given by

$$\omega(\mathbf{1}) = \mathrm{Ind}(Q_+), \tag{29}$$

which in different examples may be positive, negative or zero.

4. The Chern Character

Let ω denote an sKMS functional on a quantum algebra. For $a_0, \ldots, a_n \in \mathscr{A}_\alpha$, we define

$$\begin{aligned} &\tau_n(a_0, a_1, \ldots, a_n) \\ &= i^{\varepsilon_n} \int_{\sigma_n} \omega(a_0 \alpha_{is_1}(da_1^{\Gamma}) \alpha_{is_2}(da_2) \alpha_{is_3}(da_3^{\Gamma}) \ldots \alpha_{is_n}(da_n^{\Gamma^n})) ds_1 \ldots ds_n, \end{aligned} \tag{30}$$

where $\varepsilon_n = n \bmod 2$. Here σ_n denotes the simplex

$$\{s \in \mathbb{R}^n : 0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n \leqslant 1\}. \tag{31}$$

We claim that τ is an entire cyclic cocycle:

THEOREM 4.1. *The cochain* $\tau = (\tau_0, \tau_1, \ldots, \tau_n, \ldots,)$ *is an element of* $\mathscr{C}(\mathscr{A})$ *and satisfies the cocycle condition*

$$\partial \tau = 0. \tag{32}$$

Moreover,

$$|\tau_n(a_0, \ldots, a_n)| \leqslant \frac{|\omega|(\mathbf{1})}{n!} \prod_{j=0}^{n} \|a_j\|_*. \tag{33}$$

REMARK. An sKMS functional is not necessarily positive. Hence for doing estimates we have to replace ω by the positive functional $|\omega|$, which according to the general theory, see e.g. [5], is defined as follows. For ω there is a unique decomposition

$$\omega = (\omega_{1+} - \omega_{1-}) + i(\omega_{2+} - \omega_{2-}) \tag{34}$$

with $\omega_{k\pm} \geqslant 0$ and $\omega_{k+} \perp \omega_{k-}$. Then

$$|\omega| = \omega_{1+} + \omega_{1-} + \omega_{2+} + \omega_{2-} \tag{35}$$

is positive and satisfies

$$|\omega(a)| \leqslant |\omega|(\mathbf{1})\,\|a\|. \tag{36}$$

Notice that in general $\omega_{k\pm}$ are not sKMS functionals.

Proof. The identity (32) can be established using the identities of Section 4 in a fashion similar to the proof of Theorem V.4 of [3] or of [4]. We need only prove that τ is entire and in fact satisfies (33).

We claim that

$$\left|\omega\left(a_0 \prod_{j=1}^{n} \alpha_{it_j}(da_j)\right)\right| \leqslant |\omega|(\mathbf{1})\,\|a_0\| \prod_{j=1}^{n} \|da_j\|, \tag{37}$$

from which we conclude that (33) holds. To prove (37), we use induction on n. We call the inequality (37) (i_n). Clearly (i_0) holds as

$$|\omega(a_0)| \leqslant |\omega|(\mathbf{1})\,\|a_0\|.$$

To prove (i_{n+1}) we use the Phragmén–Lindelöf theorem. The function

$$f(z) = \omega(a_0\alpha_{it_1}(da_1)\ldots\alpha_{it_n}(da_n)\alpha_z(da_{n+1})) \tag{38}$$

is holomorphic for

$$z \in \Omega_{t_n} := \{z \in \mathbb{C} : t_n < \operatorname{Im} z < 1\} \tag{39}$$

and continuous and bounded for $z \in \bar{\Omega}_{t_n}$. In fact using (i_n),

$$\begin{aligned} |f(s+it_{n+1})| &= |\omega(a_0\alpha_{it_1}(da_1)\ldots\alpha_{it_n}(da_n\alpha_{s+i(t_{n+1}-t_n)}(da_{n+1}))| \\ &\leqslant |\omega|(\mathbf{1})\,\|a_0\|\,\|da_1\|\ldots\|da_n\alpha_{s+i(t_{n+1}-t_n)}(da_{n+1})\| \\ &\leqslant |\omega|(\mathbf{1})\left(\|a_0\| \prod_{j=1}^{n} \|da_j\|\right)\|\alpha_{i(t_{n+1}-t_n)}(da_{n+1})\|, \end{aligned}$$

which is bounded, since $0 \leqslant t_{n+1} - t_n \leqslant 1$. In particular,

$$|f(s+it_n)| \leqslant |\omega|(\mathbf{1})\,\|a_0\| \prod_{j=1}^{n+1} \|da_j\|.$$

Furthermore, using the sKMS property (23) for $z = 1$, as well as (i_n),

$$\begin{aligned}|f(s+1)| &= |\omega(a_0\alpha_{it_1}(da_1)\dots\alpha_{it_n}(da_n)\alpha_{s+1}(da_{n+1}))| \\ &= |\omega(\alpha_s(da^{\Gamma}_{n+1})a_0\alpha_{it_1}(da_1)\dots\alpha_{it_n}(da_n))| \\ &\leqslant |\omega|(\mathbf{1})\,\|\alpha_s(da^{\Gamma}_{n+1})a_0\|\prod_{j=1}^{n}\|da_j\| \\ &\leqslant |\omega|(\mathbf{1})\,\|a_0\|\prod_{j=1}^{n+1}\|da_j\|.\end{aligned}$$

Therefore, by the Phragmén–Lindelöf theorem,

$$|f(z)| \leqslant |\omega|(\mathbf{1})\,\|a_0\|\prod_{j=1}^{n+1}\|da_j\|$$

for $z \in \bar{\Omega}_{t_n}$, and the proof of (i_{n+1}) is complete.

References

1. Connes, A.: Entire cyclic cohomology of Banach algebras and characters of θ-summable Fredholm modules, *K-Theory* **1** (1988), 519–548.
2. Ernst, K., Feng, P., Jaffe, A., and Lesniewski, A.: Quantum K-theory II. Homotopy invariance of the Chern character, to appear in *J. Funct. Anal.*
3. Jaffe, A., Lesniewski, A. and Osterwalder, K.: Quantum K-theory I. The Chern character, *Commun. Math. Phys.* **118** (1988), 1–14.
4. Kastler, D.: Cyclic cocycles from graded KMS functionals, *Commun. Math. Phys.*, to appear.
5. Pedersen, G.: *C*-Algebras and their Automorphism Groups*, Academic Press, New York (1979).