Letters in Mathematical Physics **39:** 367–378, 1997.

Periodic Motion of Atoms Near a Charged Wire

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(Received: 26 February 1996)

Abstract. We study the classical motion of an atom in the vicinity of an infinite straight wire which carries an oscillating uniform charge. This system has been proposed as a mechanism for trapping cold neutral atoms. The parameters of the problem are the magnitude Q and frequency of oscillation ω of the charge, the mass M and polarizability α of the atom, and the angular momentum L of the atom about the wire. For $\omega \neq 0$ and $2\alpha MQ^2$ greater than, but close to, L^2, we prove that the atom's radial motion is periodic (with period $2\pi/\omega$), and that the atom moves in a helical path around the wire. For $2\alpha MQ^2 \leqslant L^2$ we prove that the atom must either collide with the wire or else escape to infinity in the radial direction.

Mathematics Subject Classifications (1991): 34C25, 70K20, 81Q05.

Key words: oscillating charge, periodic motion.

1. Introduction

Recently there has been much interest in systems which can be used to trap cold atoms. The Paul trap [5] is one example where electromagnetic fields provide the trapping mechanism. The effect of the Paul trap is represented by a very simple Hamiltonian, with a time dependent harmonic oscillator potential, and both the classical and quantum solutions can be found explicitly (see [1] and references therein). In this Letter, we will be concerned with another proposal for an electromagnetic trap, discussed in [2], involving the interaction of a neutral atom with a charged wire. The Hamiltonian describing this trap is also quite simple to write down (see Section 2 below), but in general it cannot be exactly solved. One case where it can be solved is when the wire carries a uniform time independent charge. In this case the classical motion of the atom is unstable, and the atom either hits the wire or escapes to infinity in finite time. Depending on the parameters of the problem, the quantum Hamiltonian either has no bound states or is not well-defined (see [4], pp. 1665–1667).

In the paper [2], Hau *et al.* studied the case where the charge is uniformly distributed along the wire and varies sinusoidally in time. They showed that an

⋆ Supported in part by a Northeastern University Research grant.
⋆⋆ Supported in part by the National Science Foundation under grant DMS-9424344.

oscillating charge on the wire allows the possibility of bound states for both the classical and quantum problems. Their conclusions rest on two analyses: they study a time-independent approximation, obtained by a quantum mechanical version of the Kapitza method [3], and they perform numerical simulations of the full model. For both the classical and quantum problems they predict a range of parameters for which bounded motion occurs.

In this Letter, we begin a rigorous analysis of the classical solutions of the oscillating-charge model of [2]. We prove that for one range of parameters all solutions of the equations of motion either hit the wire or escape to infinity. That is, there is no bounded motion in this regime. For another range of parameters we prove that there are solutions which remain within a finite, non-zero distance of the wire, corresponding to the bound motion discovered by [2]. It is worth noting that the frequency of oscillation of the charge is an irrelevant parameter in this classical model, and can be scaled out.

There are many further questions suggested by the results in this Letter. We can show that the bounded solutions described above are marginally stable under small perturbations. It would be very interesting to know if they are truly stable. Secondly, it would be very interesting to know whether there are bound states of the quantum system corresponding to these classical periodic solutions, and if so, how these states behave in the crossover from the quantum to the classical regime.

The Letter is organized as follows. In Section 2, we define the Hamiltonian and prove our result about the absence of bound solutions. In Section 3, we prove that bound motion occurs in a certain range of parameters. In Section 4 we show that these solutions are marginally stable under small perturbations.

2. Absence of Bound Motion for the Classical Equation

We will follow the notation used in [2] for the motion of a neutral atom in the vicinity of a rigid, straight wire carrying a uniformly distributed time-dependent charge $q(t) = Q\cos(\omega t/2)$. This produces the potential $V = -2\alpha q^2/r^2$, where r is the distance from the wire, and α is the atom's polarizability. The atom moves freely in the direction parallel to the wire, and so we consider only the perpendicular motion. The atom's radial motion, at fixed angular momentum L, is described by the following time-dependent Hamiltonian:

$$H(t) = \frac{p_r^2}{2M} + \frac{1}{r^2}\left[\frac{L^2}{2M} - \alpha Q^2\right] - \frac{\alpha Q^2}{r^2}\cos\omega t. \tag{2.1}$$

Here p_r is the radial conjugate momentum, and M is the atom's mass. The radial equation of motion follows now from Hamilton's equations:

$$\frac{M}{2}\frac{d^2r}{dt^2} = \frac{1}{r^3}\left[\frac{L^2}{2M} - \alpha Q^2\right] - \frac{\alpha Q^2}{r^3}\cos\omega t. \tag{2.2}$$

It is convenient to change variables, and define

$$\rho(t) = r(t)^2. \tag{2.3}$$

Denoting a time derivative by a dot, (2.2) becomes

$$\tfrac{1}{2}\rho\ddot{\rho} - \tfrac{1}{4}\dot{\rho}^2 + A + B\cos\omega t = 0, \tag{2.4}$$

where

$$A = \frac{2\alpha Q^2}{M} - \frac{L^2}{M^2}, \qquad B = \frac{2\alpha Q^2}{M}. \tag{2.5}$$

The equation of motion (2.2) can be solved when $\omega = 0$. Unless $L^2 = 4\alpha Q^2 M$ all solutions either escape to infinity or reach $r = 0$ in finite time; the circular orbit at the value $L^2 = 4\alpha Q^2 M$ is unstable to small perturbations of the velocity. As explained in the introduction, it has been argued that for $\omega \neq 0$, the equation should have stable periodic solutions over some range of parameters. We will first prove a result which shows that there can be no bounded solutions unless Q is sufficiently large. We will call a solution *singular* if $r(t) = 0$ for some t.

THEOREM 2.1. *Suppose* $\omega \neq 0$ *and* $0 < \alpha Q^2 \leqslant L^2/2M$. *Then* (2.2) *has no bounded nonsingular solution.*

Proof. Suppose $\rho(t)$ is a bounded nonsingular solution of (2.4), so that for some constant $K, 0 < \rho(t) \leqslant K$ for all t. Then by integrating (2.4) over the interval $[0,T]$ and noting that $\mathrm{d}/\mathrm{d}t(\rho\dot{\rho}) = \rho\ddot{\rho} + \dot{\rho}^2$, we deduce that

$$\tfrac{1}{2}\rho(T)\dot{\rho}(T) = \tfrac{1}{2}\rho(0)\dot{\rho}(0) - AT + \frac{B}{\omega}\sin(\omega T) + \frac{3}{4}\int_0^T \dot{\rho}(t)^2\,\mathrm{d}t. \tag{2.6}$$

Since $A \leqslant 0$, and $0 < \rho(T) \leqslant K$, we deduce

$$\dot{\rho}(T) \geqslant \frac{\rho(0)\dot{\rho}(0)}{K} - \frac{2B}{\omega K} + \frac{3}{2K}\int_0^T \dot{\rho}(t)^2\,\mathrm{d}t. \tag{2.7}$$

If $\int_0^\infty \dot{\rho}(t)^2\,\mathrm{d}t = \infty$, then (2.7) implies that $\dot{\rho}(T) \to \infty$ as $T \to \infty$, which in turn implies that $\rho(T) \to \infty$. But this contradicts our assumption that $\rho(T) \leqslant K$.

If $\int_0^\infty \dot{\rho}(t)^2\,\mathrm{d}t < \infty$, then by (2.6), $\frac{1}{2}\rho(T)\dot{\rho}(T) + AT - (B/\omega)\sin(\omega T)$ approaches a limit, say γ, as $T \to \infty$. If $A \neq 0$, this implies that $\dot{\rho}(T) \to \infty$ as $T \to \infty$, which is a contradiction. If $A = 0$, then a simple argument shows that for T sufficiently large there are infinitely many intervals of length $\pi/2\omega$ where the following estimate holds

$$|\dot{\rho}(T)| \geqslant \frac{|\gamma| + B\sqrt{2}/|\omega|}{K}.$$

This again is a contradiction. □

3. Periodic Solution of the Classical Equation

Following the results of Section 2, we shall assume henceforth that $A > 0$, and look for periodic solutions of (2.2). For convenience, we rescale variables and define

$$\sigma(t) = \sqrt{\frac{2A}{3}}\ \frac{\omega}{B}\ \rho\left(\frac{t}{\omega}\right). \tag{3.1}$$

We also introduce the following small parameter for the problem

$$\beta = \frac{2A}{3B}. \tag{3.2}$$

Then the equation of motion (2.4) becomes

$$\sigma\ddot{\sigma} - \tfrac{1}{2}\dot{\sigma}^2 + 3\beta^2 + 2\beta\cos t = 0. \tag{3.3}$$

Our main result in this section will be that for $|\beta|$ sufficiently small, (3.3) has a positive periodic solution. This will, in turn, imply that (2.2) has a periodic solution $r(t)$ provided that the ratio $L^2/2M\alpha Q^2$ is less than, but sufficiently close to, 1. The scaling (3.1) shows that this solution moves to infinity as $L^2/2M\alpha Q^2$ approaches 1. Note that the condition for existence is independent of the frequency of oscillation ω, and also that this solution is unrelated to the classical unstable solution described in Section 2. The motion parallel to the wire is constant, so the particle's trajectory is a perturbed helix with axis along the wire.

THEOREM 3.1. *There is a positive number β_0 such that for all $|\beta| < \beta_0$, (3.3) has a solution $\sigma(t)$ satisfying $\sigma(t) = \sigma(t + 2\pi) > 0$ for all t.*

In order to prove Theorem 3.1 we will rewrite (3.3) as an integral equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sigma\dot{\sigma}) = \tfrac{3}{2}\dot{\sigma}^2 - 3\beta^2 - 2\beta\cos t. \tag{3.4}$$

Although our goal is to find a periodic solution of (3.4), we will first look for solutions on the interval $[0, 2\pi]$. The integral of (3.4) gives

$$\sigma(t)\dot{\sigma}(t) = \sigma(0)\dot{\sigma}(0) + \int_0^t (\tfrac{3}{2}\dot{\sigma}^2 - 3\beta^2 - 2\beta\cos s)\,\mathrm{d}s. \tag{3.5}$$

If we look for a solution of (3.3) as a series in β, the first few terms can be readily calculated: they are

$$\begin{aligned}\sigma(t) \ =\ & 1 + 2\beta\cos t + \beta^2[-\tfrac{1}{4}\cos 2t + \tfrac{11}{16}] + \\ & +\beta^3[\tfrac{1}{12}\cos 3t + \tfrac{3}{8}\cos t] + \cdots\end{aligned} \tag{3.6}$$

This suggests looking for a solution $\sigma(t)$ of the following form:

$$\sigma(t) = 1 + \beta u(t). \tag{3.7}$$

If we substitute (3.7) into (3.5), we can write the equation entirely in terms of the function $\dot{u}(t)$ and the initial values $u(0)$ and $\dot{u}(0)$:

$$\begin{aligned}&\left(1 + \beta u(0) + \beta \int_0^t \dot{u}(s)\,\mathrm{d}s\right) \dot{u}(t)\\ &\quad = (1 + \beta u(0))\dot{u}(0) + \int_0^t (\tfrac{3}{2}\beta \dot{u}^2 - 3\beta - 2\cos s)\,\mathrm{d}s.\end{aligned} \tag{3.8}$$

Clearly, $u(0)$ and $\dot{u}(0)$ are free parameters in this integral equation. We will exploit this freedom to find a periodic solution. Let $\xi = (\beta, x, y)$ be any triplet of real numbers. For any continuous function $f(t)$ on $[0, 2\pi]$, we define the following functions:

$$A_\xi[f](t) = 1 + x + \beta \int_0^t f(s)\,\mathrm{d}s, \tag{3.9}$$

$$B_\xi[f](t) = (1 + x)y - 3\beta t - 2\sin t + \tfrac{3}{2}\beta \int_0^t f(s)^2\,\mathrm{d}s. \tag{3.10}$$

It is clear that both $A_\xi[f]$ and $B_\xi[f]$ are differentiable functions on $[0, 2\pi]$. Furthermore we have the following result.

LEMMA 3.2. *Suppose there exist* $\xi = (\beta, x, y)$, *and* $f(t)$ *continuous on* $[0, 2\pi]$, *satisfying the following four conditions:*

$$\begin{aligned}&A_\xi[f](t) > 0, \quad 0 \leqslant t \leqslant 2\pi,\\ &A_\xi[f](t)\, f(t) = B_\xi[f](t), \quad 0 \leqslant t \leqslant 2\pi,\\ &\int_0^{2\pi} f(t)\,\mathrm{d}t = 0, \qquad \int_0^{2\pi} f(t)^2\,\mathrm{d}t = 4\pi.\end{aligned} \tag{3.11}$$

Then $\sigma(t) = A_\xi[f](t)$ *is a positive, smooth, periodic solution of* (3.3).

Proof. It follows immediately from (3.8), (3.9), (3.10) and (3.11) that $\sigma(t)$ satisfies the differential equation. Positivity is guaranteed by the first condition. So the only issues are periodicity and smoothness. We must show that $\sigma^{(n)}(t)$ is continuous in $[0, 2\pi]$, and that $\sigma^{(n)}(0) = \sigma^{(n)}(2\pi)$, for all $n \geqslant 0$. The third condition in (3.11) gives $A_\xi[f](0) = A_\xi[f](2\pi)$, which implies that $\sigma(2\pi) = \sigma(0)$. Note that $\dot{\sigma}(t) = \beta f(t)$, so it is sufficient to prove that $f(t)$ and all its derivatives are continuous and agree at the endpoints. Now the second condition gives

$$f(t) = \frac{B_\xi[f](t)}{A_\xi[f](t)}. \tag{3.12}$$

The fourth condition guarantees that $B_\xi[f](0) = B_\xi[f](2\pi)$ which combined with (3.12), implies that $f(0) = f(2\pi)$. Higher-order derivatives of f are obtained from (3.12), which relates $f^{(n)}(t)$ on the left side to lower-order derivatives of f on the right side. So an inductive argument shows that all derivatives are continuous and agree at 0 and 2π. $\square$

Therefore, the proof of Theorem 3.1 reduces to showing that for $|\beta|$ sufficiently small, there are numbers x and y, and a continuous function f on $[0, 2\pi]$, satisfying the conditions of Lemma 3.2. Our strategy will be to show that, for $|\beta|$, x and y sufficiently small, there is a unique function f_ξ satisfying the first two conditions. Then we will show that, for $|\beta|$ sufficiently small, there are unique values of x and y for which f_ξ also satisfies the third and fourth conditions. The first part of this construction uses the contraction mapping theorem; the second part uses the inverse function theorem.

We write $||\cdot||$ to denote the sup norm on continuous functions on the interval $[0, 2\pi]$. For any $R > 0$, we define the Banach space

$$\mathcal{B}_R = \{f\colon [0, 2\pi] \to \mathbf{R} \mid f \text{ continuous, } ||f|| \leqslant R\}. \tag{3.13}$$

For a continuous function f on $[0, 2\pi]$, and a triplet $\xi = (\beta, x, y)$ we define the map

$$\Phi_\xi[f] = \frac{B_\xi[f]}{A_\xi[f]}. \tag{3.14}$$

LEMMA 3.3. *For any $R > 2$, there is a neighborhood $\mathcal{N}(R)$ of $(0, 0, 0)$ such that for any $\xi \in \mathcal{N}(R)$, the map Φ_ξ is a contraction on $\mathcal{B}_R$.*

Proof. We must show that there is α, satisfying $0 < \alpha < 1$, such that for any f, g in $\mathcal{B}_R$,

$$||\Phi_\xi[f]|| \leqslant R \tag{3.15}$$

and

$$||\Phi_\xi[f] - \Phi_\xi[g]|| \leqslant \alpha ||f - g||. \tag{3.16}$$

If we write

$$\varepsilon = |x| + 2\pi|\beta|R, \qquad \alpha = \frac{8\pi|\beta|R}{1 - \varepsilon},$$

then the inequalities are implied by the following three conditions:

$$\varepsilon < 1, \quad \alpha < 1, \quad (1 + |x|)|y| + 6\pi|\beta| + 2 + 3\pi|\beta|R^2 \leqslant (1 - \varepsilon)R. \tag{3.17}$$

For any $R > 2$, these can be satisfied by taking $|\beta|$, $|x|$ and $|y|$ sufficiently small. $\square$

Remark. The requirement that Φ_ξ be a contraction puts quite a severe restriction on the size of $|\beta|$. A rough estimate shows that $|\beta|$ cannot exceed $1/20\pi$. A convenient value to take is $R = 10$, in which case Φ_ξ is a contraction whenever $|\beta| \leqslant 10^{-3}$, $|x| \leqslant 10^{-1}$ and $|y| \leqslant 1$. We will write henceforth $\mathcal{B}$ in place of $\mathcal{B}_{10}$, and $\mathcal{N} = \mathcal{N}(10)$.

We now apply the contraction mapping theorem to conclude the following result.

LEMMA 3.4. *For every $\xi \in \mathcal{N}$, there is a unique $f_\xi \in \mathcal{B}$ satisfying the first two conditions in* (3.11). *Furthermore, there is C such that for every $\xi, \eta \in \mathcal{B}$,*

$$||f_\xi - f_\eta|| \leqslant C|\xi - \eta|. \tag{3.18}$$

Proof. The existence of a fixed point for Φ_ξ (which follows from Lemma 3.3) implies the second condition in (3.11). The first condition follows from (3.17), which is satisfied for the choices of R and $\mathcal{N}$ described above. The contraction mapping theorem also guarantees uniqueness of the fixed point.

To show the continuity of f_ξ, note first that for any $g \in \mathcal{B}$, using (3.9) and (3.10),

$$||A_\xi[g] - A_\eta[g]|| \leqslant |\xi - \eta|\,((1 + 2\pi||g||), \tag{3.19}$$

$$||B_\xi[g] - B_\eta[g]|| \leqslant |\xi - \eta|\,(1 + |\xi| + |\eta| + 6\pi + 3\pi||g||^2). \tag{3.20}$$

Using the notation of Lemma 3.3 we have $||A_\xi[g]|| > 1 - \varepsilon > 0$ for all $\xi \in \mathcal{N}$ and all $g \in \mathcal{B}$. Thus, it follows from (3.14) and (3.19), (3.20) that there is a constant C_1, independent of $g \in \mathcal{B}$, and of $\xi, \eta \in \mathcal{N}$, such that

$$||\Phi_\xi[g] - \Phi_\eta[g]|| \leqslant C_1\,|\xi - \eta|. \tag{3.21}$$

Since f_ξ is defined to be the fixed point of Φ, we have from (3.16) and (3.21)

$$\begin{aligned} ||f_\xi - f_\eta|| &= ||\Phi_\xi[f_\xi] - \Phi_\eta[f_\eta]|| \\ &\leqslant ||\Phi_\xi[f_\xi] - \Phi_\eta[f_\xi]|| + ||\Phi_\eta[f_\xi] - \Phi_\eta[f_\eta]|| \\ &\leqslant C_1|\xi - \eta| + \alpha||f_\xi - f_\eta||. \end{aligned} \tag{3.22}$$

Taking $C = C_1/(1 - \alpha)$ gives the stated result. $\square$

To complete the proof of Theorem 3.1, we will use the inverse function theorem. First we put $\beta = 0$ and calculate Φ. For any $g \in \mathcal{B}$,

$$\Phi_{(0,x,y)}[g](t) = (1 + x)^{-1}((1 + x)y - 2\sin t). \tag{3.23}$$

Therefore, the fixed point of $\Phi_{(0,x,y)}$ is

$$f_{(0,x,y)}(t) = y - \frac{2}{1+x}\sin t. \tag{3.24}$$

It is easy to check that the third and fourth conditions of (3.11) can be satisfied for (3.24) only if $x = y = 0$. Let us define the following function on $\mathbb{R}^3$:

$$\begin{aligned} &F\colon \mathbb{R}^3 \to \mathbb{R}^3, \\ &F(\beta, x, y) = \left(\beta, \int_0^{2\pi} f_\xi(t)\,\mathrm{d}t, \int_0^{2\pi} f_\xi(t)^2\,\mathrm{d}t\right). \end{aligned} \tag{3.25}$$

The proof of Theorem 3.1 follows from Lemma 3.2 and the next result.

LEMMA 3.5. *There is $\beta_0 > 0$, such that for all $|\beta| \leqslant \beta_0$, there is a unique $\xi \in \mathcal{N}$ for which $F(\xi) = (\beta, 0, 4\pi)$.*

Proof. Assuming the conditions of the inverse function theorem are satisfied, we deduce that F is a homeomorphism from a neighborhood of $(0,0,0)$ to a neighborhood of $(0,0,4\pi)$. The result follows.

To apply the inverse function theorem, we need to know that F is differentiable in a neighborhood of $(0,0,0)$ and that its derivative is invertible at this point. Recall how the derivative is defined for a map $G\colon X \to Y$ between Banach spaces X and Y [6]. For each $x \in X$, there is a linear map $DG_x\colon X \to Y$ such that for all $\xi \in X$, and θ sufficiently small,

$$||G(x + \theta\xi) - G(x) - \theta DG_x(\xi)|| = \mathrm{O}(\theta^2). \tag{3.26}$$

The map DG_x is the derivative of G at x. In fact, we will use this term 'derivative' only when the operator DG_x also depends continuously on x. The usual chain rule holds for a composition of differentiable maps.

Let us define the maps of Banach spaces $\psi\colon \mathbb{R}^3 \to \mathcal{B}$, and $G, H\colon \mathcal{B} \to \mathcal{B}$ by the following rules:

$$\psi(\xi) = f_\xi, \qquad G[f](t) = \int_0^t f(s)\,\mathrm{d}s, \qquad H[f](t) = \int_0^t f(s)^2\,\mathrm{d}s. \tag{3.27}$$

Using these definitions, we can write

$$F(\beta, x, y) = (\beta, G[\psi(\xi)](2\pi), H[\psi(\xi)](2\pi)).$$

The evaluation map $\mathcal{B} \to \mathbb{R}, g \mapsto g(2\pi)$ is differentiable. It is easy to check that both G and H are differentiable. Therefore, the differentiability of F will follow by the chain rule once we show that ψ is differentiable in a neighborhood of $(0,0,0)$.

Let $g \in \mathcal{B}$; with $\xi = (\beta, x, y)$, (3.14) gives

$$\Phi_\xi[g](t) \;=\; \left(1 + x + \beta \int_0^t g(s)\,\mathrm{d}s\right)^{-1} \times$$
$$\times \left[(1+x)y - 3\beta t - 2\sin t + \tfrac{3}{2}\beta \int_0^t g(s)^2\,\mathrm{d}s\right]. \tag{3.28}$$

The derivatives of (3.28) with respect to x, y and β, are easily calculated:

$$\frac{\partial}{\partial x}\Phi_\xi[g](t) = \left(1 + x + \beta \int_0^t g(s)\,\mathrm{d}s\right)^{-1} [-\Phi_{(\beta,x,y)}[g](t) + y],$$

$$\frac{\partial}{\partial y}\Phi_\xi[g](t) = \left(1 + x + \beta \int_0^t g(s)\,\mathrm{d}s\right)^{-1} (1+x),$$

$$\frac{\partial}{\partial \beta}\Phi_\xi[g](t)$$
$$= \left(1 + x + \beta \int_0^t g(s)\,\mathrm{d}s\right)^{-1} \times$$
$$\times \left[-\left(\int_0^t g(s)\,\mathrm{d}s\right)\Phi_{(\beta,x,y)}[g](t) - 3t + \frac{3}{2}\int_0^t g(s)^2\,\mathrm{d}s\right].$$

We will collect these maps together as a vector-valued function and write

$$\nabla\Phi_\xi[g] = \left(\frac{\partial}{\partial\beta}\Phi_\xi[g], \frac{\partial}{\partial x}\Phi_\xi[g], \frac{\partial}{\partial y}\Phi_\xi[g]\right). \tag{3.29}$$

By following the method used in the proof of Lemma 3.4, it is easy to show that $\nabla\Phi_\xi[g]$ is a continuous function of $\xi = (\beta, x, y)$, for all $\xi \in \mathcal{N}$. Furthermore, there is a constant C_3 such that for all $g, h \in \mathcal{B}$

$$||\nabla\Phi_\xi[g] - \nabla\Phi_\xi[h]|| \leqslant C_3\, ||g - h||. \tag{3.30}$$

The derivative of (3.28) with respect to g is also readily calculated. We denote it by $D\Phi_\xi[g]$; it is the linear map on $\mathcal{B}$ defined by

$$D\Phi_\xi[g](h)$$
$$= \left(1 + x + \beta \int_0^t g(s)\,\mathrm{d}s\right)^{-1} \times$$
$$\times \left[-\beta\left(\int_0^t h(s)\,\mathrm{d}s\right)\Phi_\xi[g](t) + 3\beta \int_0^t g(s)h(s)\,\mathrm{d}s\right]. \tag{3.31}$$

Again it is clear that $D\Phi_\xi[g]$ depends continuously on $\xi \in \mathcal{N}$ and $f \in \mathcal{B}$. Furthermore, with the notation of Lemma 3.3, we have

$$||D\Phi_\xi[g](h)|| \leqslant \alpha||h|| \tag{3.32}$$

and therefore $D\Phi_\xi[g]$ is also a contraction on $\mathcal{B}$ for every $\xi \in \mathcal{N}$ and $g \in \mathcal{B}$. Accordingly, for $\xi \in \mathcal{N}$ we define the three-component vector-valued function ∇f_ξ to be the unique solution of the equation

$$\nabla f_\xi = \nabla\Phi_\xi[f_\xi] + D\Phi_\xi[f_\xi](\nabla f_\xi). \tag{3.33}$$

We have abused notation a little; the last term on the right side of (3.33) is the three-component vector-valued function obtained by applying $D\Phi_\xi[f_\xi]$ to each component of ∇f_ξ. The definition (3.33) is motivated by taking the derivative of the fixed point equation $f_\xi = \Phi_\xi[f_\xi]$ with respect to ξ.

By repeating the arguments in the proof of Lemma 3.4, and using the bounds (3.30) and (3.32), it follows that ∇f_ξ depends continuously on ξ.

To finish the proof that ψ is differentiable, we will now show ∇f_ξ is the derivative of f_ξ. Let $a \in \mathbb{R}^3$, with $|a|$ small, and consider

$$\begin{aligned} &f_{\xi+a} - f_\xi - \langle a, \nabla f_\xi\rangle \\ &\quad = \Phi_{\xi+a}[f_{\xi+a}] - \Phi_\xi[f_\xi] - \langle a, \nabla\Phi_\xi[f_\xi]\rangle - \langle a, D\Phi_\xi[f_\xi](\nabla f_\xi)\rangle \\ &\quad = \langle a, (\nabla\Phi_\xi[f_{\xi+a}] - \nabla\Phi_\xi[f_\xi])\rangle + D\Phi_\xi[f_\xi](f_{\xi+a} - f_\xi) - \\ &\qquad - D\Phi_\xi[f_\xi](\langle a, \nabla f_\xi\rangle) + \mathrm{O}(a^2) + \mathrm{O}(||f_{\xi+a} - f_\xi||^2). \end{aligned} \tag{3.34}$$

We have used the definition (3.26) for the derivative, applied to the maps (3.29) and (3.31). Now using (3.30), (3.18), linearity of $D\Phi_\xi$, and (3.32), we deduce that

$$(1-\alpha)||f_{\xi+a} - f_\xi - \langle a, \nabla f_\xi\rangle|| \leqslant \mathrm{O}(a^2). \tag{3.35}$$

Hence, we conclude that ψ is differentiable in $\mathcal{N}$.

It only remains to compute the Jacobian of F at $(0,0,0)$. We denote the partial derivatives of ψ with respect to β, x and y as ψ_β, ψ_x and ψ_y, respectively. Also as a shorthand we write $(g,h) = \int_0^{2\pi} g(t)h(t)\,\mathrm{d}t$, we denote the constant function 1 by 1, and use 0 to denote the point $(0,0,0) \in \mathcal{N}$. Then F can be written as

$$F(\beta,x,y) = (\beta, (1,\psi(\xi)), (\psi(\xi),\psi(\xi))). \tag{3.36}$$

Hence, the Jacobian of F at 0 is the following:

$$J = 2|(1,\psi_x(0))(\psi(0),\psi_y(0)) - (1,\psi_y(0))(\psi(0),\psi_x(0))|. \tag{3.37}$$

We can calculate the derivative of ψ at 0 using the second equation of (3.11): the result is

$$\psi_x(0)[t] = 2\sin t, \qquad \psi_y(0)[t] = 1.$$

Recall that $\psi(0)[t] = -2\sin t$. Hence,

$$(1, \psi_x(0)) = 0, \qquad (1, \psi_y(0)) = 2\pi \quad \text{and} \quad (\psi(0), \psi_x(0)) = -4\pi.$$

So the Jacobian is $J = 16\pi^2$.

Since $J \neq 0$ we are done. □

4. Behavior of the Periodic Solution Under Small Perturbations

We will now examine the infinitesimal stability of the periodic solution which was derived in Section 3. That is, we will investigate how it behaves under a small perturbation of the initial conditions. Our result is that to leading order in $|\beta|$, the solution is *marginally stable*, and a small perturbation of size $\mathrm{O}(|\beta|)$ grows linearly over a time interval of length $\mathrm{O}(|\beta|^{-1})$.

We will view the motion of the atom over one period as a discrete dynamical system. That is, the solution $\sigma(t)$ of (3.3) over the interval $[0, 2\pi]$ maps the initial conditions $\sigma(0), \dot{\sigma}(0)$ into $\sigma(2\pi), \dot{\sigma}(2\pi)$. Suppose we choose $\sigma(0)$ to be close to 1, and $\dot{\sigma}(0)$ to be close to 0. If the motion is stable, then the iterates of these values under the mapping should remain close to these values. To test this hypothesis, we compute the derivative of this map at the periodic solution. Actually we expand the solution as a power series in β and compute only the first few terms, so our result below will be valid only for perturbations of the initial conditions which are themselves $\mathrm{O}(\beta)$. That is why we describe the result as *infinitesimal stability*; we do not address the question of stability under perturbations larger than $\mathrm{O}(\beta)$. First we note the following result.

LEMMA 4.1. *Let $\sigma(t)$ be a periodic solution of* (3.3). *Then for β sufficiently small,* $\dot{\sigma}(0) = 0$.

Proof. First note that $\sigma(-t)$ is also a periodic solution of (3.3). With the notation of Lemma 3.3 and the succeeding remark, if $|\beta|$ is sufficiently small, then there is a unique periodic solution for which $\xi \in \mathcal{N}$. Hence $\sigma(t) = \sigma(-t)$, so σ is even. Hence $\dot{\sigma}(0) = 0$. □

We now state our main result in the following lemma.

LEMMA 4.2. *Let $\sigma(t) = 1 + \beta u(t)$ be a solution of* (3.3). *Write $u(0) = q$ and $\dot{u}(0) = y$, and define the map $K\colon (q, y) \mapsto (u(2\pi), \dot{u}(2\pi))$. Let M be the*

2×2 *matrix of derivatives of* K *evaluated at the periodic solution. Then for* $|\beta|$ *sufficiently small, both eigenvalues of* M *are equal to* $1 + \mathrm{O}(\beta^2)$.

Proof. By solving (3.3) to $\mathrm{O}(\beta^2)$, we calculate the following values for the solution at $t = 2\pi$:

$$\begin{aligned} \sigma(2\pi) &= 1 + \beta q + 2\pi y\beta + \beta^2(\pi^2 y^2 - 12\pi y) + \mathrm{O}(\beta^3), \\ \dot{\sigma}(2\pi) &= \beta y + \beta^2 \pi y^2 + \mathrm{O}(\beta^3). \end{aligned} \tag{4.1}$$

This defines the following map of initial values (q, y):

$$\begin{aligned} q &\mapsto q + 2\pi y + \beta(\pi^2 y^2 - 12\pi y) + \mathrm{O}(\beta^2), \\ y &\mapsto y + \beta\pi y^2 + \mathrm{O}(\beta^2). \end{aligned} \tag{4.2}$$

The derivative of (4.2) is a 2×2 matrix; when evaluated at the periodic point, with $y = 0$, both of its eigenvalues are $1 + \mathrm{O}(\beta^2)$. □

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