JOURNAL OF FUNCTIONAL ANALYSIS **102**, 314–330 (1991)

Pfaffians on Banach Spaces*

SLAWOMIR KLIMEK AND ANDRZEJ LESNIEWSKI

Harvard University, Cambridge, Massachusetts 02138

Communicated by L. Gross

Received August 29, 1990

We present a theory of relative Pfaffians on infinite-dimensional Banach spaces. © 1991 Academic Press, Inc.

I. INTRODUCTION

In 1956, A. Grothendieck used the methods and ideas of his theory of tensor products of topological vector spaces to develop a theory of Fredholm determinants on Banach spaces. This work [G] attracted little attention, even though it is by far the most profound study in the subject of functional determinants. The theory of Fredholm determinants on Hilbert spaces is conceptually and technically easier than the Banach space theory, but it is also less natural and often insufficient for applications, see, e.g., [RS].

In this paper we are concerned with the theory of Pfaffians on Banach spaces. This extends the theory of infinite dimensional Pfaffians, previously developed in the context of Hilbert spaces [PS, JLW]. Recall that if $A = \{A_{jk}\}$ is a skew symmetric $n \times n$ matrix, then its Pfaffian is defined by

$$Pf(A) = \begin{cases} (2^k k!)^{-1} \sum\limits_{\pi \in S_{2k}} (-1)^\pi A_{\pi(1)\pi(2)} \dots A_{\pi(2k-1)\pi(2k)}, & \text{if } n = 2k, \\ 0, & \text{if } n = 2k+1, \end{cases} \tag{I.1}$$

where S_{2k} is the group of permutations of $2k$ elements. From the point of view of infinite dimensional analysis, the relative Pfaffian [JLW, PS] is a more appropriate object to study. Let A and B be two skew symmetric

* Supported in part by the National Science Foundation under Grant DMS/PHY 88–16214.

0022-1236/91 $3.00

$2k \times 2k$ matrices with A invertible. Then the relative Pfaffian $\mathrm{Pf}(A, B)$ can be represented as

$$\mathrm{Pf}(A, B) := \frac{\mathrm{Pf}(A^{-1} - B)}{\mathrm{Pf}(A^{-1})}. \tag{I.2}$$

The relative Pfaffian has the property that

$$\mathrm{Pf}(A, B)^2 = \det(I - AB). \tag{I.3}$$

Definition (I.2) motivates our general definition of the relative Pfaffian. As a typical example [JLW], we consider an unbounded, skew symmetric operator $\mathbf{Q}$ on a Hilbert space $\mathscr{H}$ which can be written as $Q = Q_0 - V$. Here Q_0 is unbounded and invertible, and V is a suitable bounded perturbation. The domain $E := D(|Q_0|^{1/2-\varepsilon})$, $0 < \varepsilon < 1/2$, can be given the topology of a Banach space. Then $(Q_0)^{-1}$ can be viewed as a bounded operator from E' to E, where E' is the topological dual of E. Assume now that $(Q_0)^{-1}\colon E' \to E$ and $V\colon E \to E'$ are nuclear mappings. Then the relative Pfaffian $\mathrm{Pf}(Q_0^{-1}, V)$ is defined.

The paper is organized as follows. In Section II we defined the relative Pfaffian and study its basic properties. Section III contains a number of useful algebraic identities. In Section IV we define and study the properties of the relative Pfaffian minor. In Section V we relate the present theory to the Hilbert space theory.

II. Relative Pfaffian

Let E be a complex Banach space, and let E' be its topological dual equipped with the usual structure of a Banach space. By $\langle \cdot, \cdot \rangle$, we denote the canonical pairing between E' and E. Let $\bigwedge^n(E)$ denote the nth exterior power of E equipped with the projective norm [G, S]. Explicitly, for $\omega \in \bigwedge^n(E)$, the projective norm is defined by

$$\|\omega\|_1 := \inf\left\{ \sum_{j=1}^{\infty} \|x_j^1\| \, \|x_j^2\| \cdots \|x_j^n\| \right\}, \tag{II.1}$$

where the infimum is taken over all possible representations of ω of the form

$$\omega = \sum_{j=1}^{\infty} x_j^1 \wedge x_j^2 \wedge \cdots \wedge x_j^n, \qquad x_j^k \in E. \tag{II.2}$$

Likewise, we define $\bigwedge^n(E')$, the nth projective exterior power of E'. We observe that the pairing $\langle \cdot, \cdot \rangle: E' \times E \to C$ induces a natural continuous pairing $\langle \cdot, \cdot \rangle: \bigwedge^n(E') \times \bigwedge^n(E) \to C$ given by

$$\langle e_1 \wedge \cdots \wedge e_n, x_1 \wedge \cdots \wedge x_n \rangle := \det\{\langle e_j, x_k \rangle\}. \tag{II.3}$$

As a consequence of Hadamard's inequality,

$$|\langle e_1 \wedge \cdots \wedge e_n, x_1 \wedge \cdots \wedge x_n \rangle| \leqslant n^{n/2} \sum_{j=1}^{n} \|e_j\| \, \|x_j\|, \tag{II.4}$$

and thus, for arbitrary $\eta \in \bigwedge^n(E')$, $\omega \in \bigwedge^n(E)$,

$$|\langle \eta, \omega \rangle| \leqslant n^{n/2} \|\eta\|_1 \, \|\omega\|_1. \tag{II.5}$$

Definition II.1. Let $A \in \bigwedge^2(E')$, $B \in \bigwedge^2(E)$. The relative Pfaffian, $\mathrm{Pf}(A, B)$, is a function $\bigwedge^2(E') \times \bigwedge^2(E) \to \mathbb{C}$ given by

$$\mathrm{Pf}(A, B) := \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \langle \bigwedge^n A, \bigwedge^n B \rangle. \tag{II.6}$$

Theorem II.2. *The above series converges absolutely and*

$$|\mathrm{Pf}(A, B)| \leqslant \sum_{n=0}^{\infty} \frac{1}{(n!)^2} |\langle \bigwedge^n A, \bigwedge^n B \rangle| \leqslant \exp\{2e \|A\|_1 \|B_1\}. \tag{II.7}$$

Proof. From (II.5),

$$\sum_{n=0}^{\infty} \frac{1}{(n!)^2} |\langle \bigwedge^n A, \bigwedge^n B \rangle| \leqslant \sum_{n=0}^{\infty} \frac{1}{(n!)^2} (2n)^n (\|A\|_1 \|B\|_1)^n.$$

Using the inequality

$$\frac{n^n}{n!} \leqslant e^{n-1}, \tag{II.8}$$

we can bound the above sum by

$$\sum_{n=0}^{\infty} \frac{1}{n!} (2e \|A\|_1 \|B\|_1)^n = \exp\{2e \|A\|_1 \|B\|_1\}. \quad \blacksquare$$

Next we establish Hölder continuity of the relative Pfaffian.

THEOREM II.3. *For* $A_1, A_2 \in \bigwedge^2 E'$, $B_1, B_2 \in \bigwedge^2 E$,

$$\begin{aligned} &|\mathrm{Pf}(A_1, B_1) - \mathrm{Pf}(A_2, B_2)| \\ &\quad \leqslant 2\{\|A_1 - A_2\|_1(\|B_1\|_1 + \|B_2\|_1) \\ &\quad\quad + \|B_1 - B_2\|_1(\|A_1\|_1 + \|A_2\|_1)\} \\ &\quad\quad \times \exp\{2e(\|A_1\|_1 + \|A_2\|_1)(\|B_1\|_1 + \|B_2\|_1\}. \end{aligned} \tag{II.9}$$

Proof. Define $A(s) := sA_1 + (1-s)A_2$, and $B(s) := sB_1 + (1-s)B_2$. Observe that for $A_j \in \bigwedge^2(E')$, $A_1 \wedge \cdots \wedge A_n$ is symmetric under permutation of $A_1, ..., A_n$. Thus by the fundamental theorem of calculus,

$$\begin{aligned} &\mathrm{Pf}(A_1, B_1) - \mathrm{Pf}(A_2, B_2) \\ &\quad = \int_0^1 \frac{d}{ds} \mathrm{Pf}(A(s), B(s))\, ds \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{(n!)^2} n \int_0^1 \{\langle (A_1 - A_2) \wedge \textstyle\bigwedge^{n-1} A(s), \bigwedge^n B(s)\rangle \\ &\quad\quad + \langle \textstyle\bigwedge^n A(s), (B_1 - B_2) \wedge \bigwedge^{n-1} B(s)\rangle\}\, ds. \end{aligned}$$

Using (II.5) and (II.8),

$$\begin{aligned} &|\mathrm{Pf}(A_1, B_1) - \mathrm{Pf}(A_2, B_2)| \\ &\quad \leqslant \sum_{n=1}^{\infty} \frac{1}{(n!)^2} (2n)^n \{\|A_1 - A_2\|_1(\|B_1\|_1 + \|B_2\|_1) \\ &\quad\quad + \|B_1 - B_2\|_1(\|A_1\|_1 + \|A_2\|_1)\}(\|A_1\|_1 + \|A_2\|_1)^{n-1}(\|B_1\|_1 + \|B_2\|_1)^{n-1} \\ &\quad \leqslant 2\{\|A_1 - A_2\|_1(\|B_1\|_1 + \|B_2\|_1) + \|B_1 - B_2\|_1(\|A_1\|_1 + \|A_2\|_1)\} \\ &\quad\quad \times \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \{2e(\|A_1\|_1 + \|A_2\|_1)^{n-1}(\|B_1\|_1 + \|B_2\|_1)\}^{n-1}, \end{aligned}$$

which is (II.9). ∎

Let $\bigwedge^n_{\mathrm{alg}}(E)$ denote the algebraic exterior power of the Banach space E. The projective exterior power $\bigwedge^n(E)$ is the completion of $\bigwedge^n_{\mathrm{alg}}(E)$ in the norm $\|\cdot\|_1$ of (II.1).

COROLLARY II.4. *Let* $A \in \bigwedge^2(E')$ *and* $B \in \bigwedge^2(E)$. *There exists sequences* $A_n \in \bigwedge^2_{\mathrm{alg}} E'$ *and* $B_n \in \bigwedge^2_{\mathrm{alg}} E$ *such that* $\|A - A_n\|_1 \to 0$, $\|B - B_n\|_1 \to 0$, *and*

$$\mathrm{Pf}(A, B) = \lim_{n \to \infty} \mathrm{Pf}(A_n, B_n). \tag{II.10}$$

Let G, E, and H be Banach spaces. There is a natural continuous pairing

$$\cdot : (G \otimes E') \times (E \otimes H) \to G \otimes H, \tag{II.11}$$

where $G \otimes E'$, $E \otimes H$, and $G \otimes H$ are equipped with the projective topology, such that

$$(g \otimes e) \cdot (x \otimes h) = \langle e, x \rangle \, g \otimes h, \tag{II.12}$$

for $g \in G$, $e \in E'$, $x \in E$, and $h \in H$. As a consequence of (II.12),

$$\|X \cdot Y\|_1 \leqslant \|X\|_1 \, \|Y\|_1, \tag{II.13}$$

for $X \in G \otimes E'$ and $Y \in E \otimes H$. The pairing $\cdot$ is associative in the sense that $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$, for $X \in G \otimes E'$, $Y \in E \otimes F'$, $Z \in F \otimes H$.

We will use (II.11) to define various pairings between exterior powers of Banach spaces. For example, since $\bigwedge^2(E)$ can be identified as a closed subspace of $E \otimes E$ via $x \wedge y \to x \otimes y - y \otimes x$, we have a pairing $\bigwedge^2(E') \times \bigwedge^2(E) \to E' \otimes E$ such that

$$\begin{aligned}(e \wedge f) \cdot (x \wedge y) = \langle f, x \rangle e \otimes y - \langle f, y \rangle e \otimes x \\ + \langle e, y \rangle f \otimes x - \langle e, x \rangle f \otimes y,\end{aligned} \tag{II.14}$$

for $e, f \in E'$, $x, y \in E$. Clearly, $\|A \cdot B\|_1 \leqslant 4 \|A\|_1 \|B\|_1$, for $A \in \bigwedge^2(E')$, $B \in \bigwedge^2(E)$. Since $E' \otimes E$ is naturally identified with the space of nuclear operators on E, this shows that $A \cdot B$ can be regarded as an operator on E.

The last pairing can also be interpreted as follows. We denote by $I_1^a(E, E')$ the space of skew symmetric nuclear operators from the Banach space E to E', and by $I_1^a(E', E)$ the space of skew symmetric nuclear operators from E' to E. We have natural isomorphisms $\phi : \bigwedge^2(E) \to I_1^a(E', E)$ and $\phi' : \bigwedge^2(E') \to I_1^a(E, E')$ given by

$$\begin{aligned}\langle e, \phi(B) f \rangle &= \langle f \wedge e, B \rangle, \qquad e, f \in E'. \\ \langle \phi'(A) x, y \rangle &= \langle A, x \wedge y \rangle, \qquad x, y \in E.\end{aligned} \tag{II.15}$$

Then $A \cdot B = \phi(B) \, \phi'(A)$.

For T nuclear, we let $\det(I - T)$ denote the Fredholm determinant as defined by Grothendieck [G].

THEOREM II.5. *For $A \in \bigwedge^2(E')$, $B \in \bigwedge^2(E)$,*

$$\mathrm{Pf}(A, B)^2 = \det(I - A \cdot B). \tag{II.16}$$

Proof. Because of the continuity of both sides of (II.16), we may assume that $B \in \bigwedge^2_{\mathrm{alg}}(E)$. Then, there are finitely many linearly independent vectors $x_1, ..., x_n \in E$ such that

$$B = \frac{1}{2} \sum_{1 \leqslant j,k \leqslant n} B_{jk} x_j \wedge x_k, \qquad B_{jk} = -B_{kj}. \tag{II.17}$$

Let $F \subset E$ be the vector subspace of E spanned by $x_1, ..., x_n$. Clearly, $A \cdot B$ is a finite rank operator whose image is contained in F. We let

$$A_{jk} := \langle A, x_j \wedge x_k \rangle, \qquad 1 \leqslant j, \quad k \leqslant n, \tag{II.18}$$

and observe that

$$\det(I - A \cdot B) = \det \left\{ \delta_{jk} - \sum_p A_{jp} B_{pk} \right\}. \tag{II.19}$$

We claim that the left-hand side of (II.16) can also be written in this form. We let $F^\perp := \{e \in E' \colon \langle e, F \rangle = 0\}$, and $F^0 := E'/F^\perp$. Then F^0 is dual to F with the duality given by $\langle \pi(e), x \rangle := \langle e, x \rangle$, where $\pi \colon E' \to F^0$ is the canonical projection. Let $\pi(A) \in \bigwedge^2(F^0)$ be the projection of A; explicitly, if $A = \sum_j e_j \wedge f_j$, then

$$\pi(A) = \sum_j \pi(e_j) \wedge \pi(f_j). \tag{II.20}$$

Then

$$\mathrm{Pf}(A, B) = \mathrm{Pf}(\pi(A), B). \tag{II.21}$$

Now let $x_j' \in F^0$, $1 \leqslant j \leqslant n$, be the basis for F^0, dual to x_j, $1 \leqslant j \leqslant n$. Then

$$\pi(A) = \frac{1}{2} \sum_{1 \leqslant j,k \leqslant n} A_{jk} x_j' \wedge x_k', \qquad A_{jk} = -A_{kj}, \tag{II.22}$$

with A_{jk} given by (II.18). But now,

$$\mathrm{Pf}(A, B)^2 = \mathrm{Pf}(\{A_{jk}\}, \{B_{jk}\})^2 = \det\left(\delta_{jk} - \sum_p A_{jp} B_{pk}\right), \tag{II.23}$$

by (I.3), and the proof is complete. ▌

III. Algebraic Identities

In this section we prove a number of algebraic identities involving the pairing $\langle \cdot, \cdot \rangle$ defined in Section II. The most interesting among them is (III.6), which will be used in the next section to compute Fréchet derivatives of the relative Pfaffian.

To formulate our first identity, we replace the dual pair (E, E') by (E', E''), and notice that an element $B \in \bigwedge^2(E)$ can be naturally identified as an element of $\bigwedge^2(E'')$, as $E \subsetneq E''$. Hence, if $B \in \bigwedge^2(E)$, and $A \in \bigwedge^2(E')$, then $\mathrm{Pf}(B, A)$ is defined. An immediate consequence of (II.6) is:

THEOREM III.1. *For* $(A, B) \in \bigwedge^2(E') \times \bigwedge^2(E)$,

$$\mathrm{Pf}(A, B) = \mathrm{Pf}(B, A). \tag{III.1}$$

Let $\mathscr{L}(E)$ denote the space of bounded linear operators on E. There is a natural action of $\mathscr{L}(E)$ on $\bigwedge^2(E)$; namely for $V \in \mathscr{L}(E)$ and $B = \sum_j x_j \wedge y_j \in \bigwedge^2(E)$ we set

$$(V \wedge V)B := \sum_j Vx_j \wedge Vy_j. \tag{III.2}$$

Likewise, for $A \in \sum_j e_j \wedge f_j \in \bigwedge^2(E')$ we set

$$(V' \wedge V')A := \sum_j V'e_j \wedge V'f_j, \tag{III.3}$$

where V' denotes the adjoint of V.

THEOREM III.2. *Let* $A_1, ..., A_n \in \bigwedge^2(E')$, $B_1, ..., B_n \in \bigwedge^2(E)$, *and* $V \in \mathscr{L}(E)$. *Then*

$$\left\langle \bigwedge_{j=1}^{n} (V' \wedge V') A_j, \bigwedge_{j=1}^{n} B_j \right\rangle = \left\langle \bigwedge_{j=1}^{n} A_j, \bigwedge_{j=1}^{n} (V \wedge V) B_j \right\rangle. \tag{III.4}$$

Proof. The identity is clear for $A_j = e_j \wedge f_j$ and $B_j = x_j \wedge y_j$, $j = 1, ..., n$. The general case follows by linearity and continuity. ∎

COROLLARY III.3. *For* $(A, B) \in \bigwedge^2(E') \times \bigwedge^2(E)$, $V \in \mathscr{L}(E)$,

$$\mathrm{Pf}((V' \wedge V')A, B) = \mathrm{Pf}(A, (V \wedge V)B). \tag{III.5}$$

Our next identity relates $\langle \bigwedge_{j=1}^{n+1} A_j, \bigwedge_{j=1}^{n+1} B_j \rangle$ to $\langle \bigwedge_{j=1}^{n} A_j, \bigwedge_{j=1}^{n} B_j \rangle$.

THEOREM III.4. *For* $A_0, A_1, ..., A_n \in \bigwedge^2(E')$, $B_0, B_1, ..., B_n \in \bigwedge^2(E)$,

$$\left\langle \bigwedge_{j=0}^{n} A_j, \bigwedge_{j=0}^{n} B_j \right\rangle$$

$$= \sum_{p=0}^{m} \langle A_p, B_0 \rangle \left\langle \bigwedge_{\substack{0 \leqslant j \leqslant n \\ j \neq p}} A_j, \bigwedge_{j=1}^{n} B_j \right\rangle$$

$$+ \frac{1}{2} \sum_{\substack{0 \leqslant p,q \leqslant n \\ p \neq q}} \left\langle (A_p \cdot B_0 \cdot A_q + A_q \cdot B_0 \cdot A_p) \wedge \bigwedge_{j \neq p} A_j, \bigwedge_{j=1}^{n} B_j \right\rangle. \tag{III.6}$$

Proof. For a Banach space X, we set $\bigwedge^0(X) = \mathbb{C}$, and let $\bigwedge(X) := \bigoplus_{n \geqslant 0} \bigwedge^n(X)$, where $\bigoplus$ denotes the algebraic direct sum. Let $\langle \cdot, \cdot \rangle: \bigwedge(E') \times \bigwedge(E) \to \mathbb{C}$ denote the natural pairing induced by the pairing between E' and E. For $e \in E'$, we let $b_e^*: \bigwedge^n(E') \to \bigwedge^{n+1}(E')$ be the operator defined by

$$b_e^*(e_1 \wedge \cdots \wedge e_n) := e \wedge e_1 \wedge \cdots \wedge e_n. \tag{III.7}$$

In the same manner we define the operator $c_x^*: \bigwedge^n(E) \to \bigwedge^{n+1}(E)$, for $x \in E$. These operators define the corresponding operators on $\bigwedge(E')$ and $\bigwedge(E)$, which we denote also by b_e^* and c_x^*, respectively. Let $b_x: \bigwedge^{n+1}(E') \to \bigwedge^n(E')$ denote the adjoint of c_x^*, and let $c_e: \bigwedge^{n+1}(E) \to \bigwedge^n(E)$ denote the adjoint of b_e^*. These operators obey the algebra

$$\{b_e^*, b_x\} = \langle e, x \rangle I, \qquad \{b_e^*, b_f^*\} = 0, \tag{III.8}$$

$$\{c_e, c_x^*\} = \langle e, x \rangle I, \qquad \{c_x^*, c_y^*\} = 0. \tag{III.9}$$

Here $\{b_e^*, b_x\} := b_e^* b_x + b_x b_e^*$, etc. Each $A = \sum_j e_j \wedge f_j \in \bigwedge^2(E')$ determines a linear operator (denoted by the same symbol) A on $\bigwedge(E')$, namely

$$A = \sum_j b_{e_j}^* b_{f_j}^*. \tag{III.10}$$

Likewise, to each $B = \sum_j x_j \wedge y_j \in \bigwedge^2(E)$ we assign the operator

$$B = \sum_j c_{x_j}^* c_{y_j}^* \tag{III.11}$$

on $\bigwedge(E)$. Let $\omega' := (1, 0, 0, ...) \in \bigwedge(E')$ and $\omega := (1, 0, 0, ...) \in \bigwedge(E)$. Then, for $A_1, ..., A_n \in \bigwedge(E')$, $B_1, ..., B_n \in \bigwedge(E)$,

$$\left\langle \bigwedge_{j=1}^{n} A_j, \bigwedge_{j=1}^{n} B_j \right\rangle = \left\langle \prod_{j=1}^{n} A_j \omega', \prod_{j=1}^{n} B_j \omega \right\rangle. \tag{III.12}$$

Now, since both sides of (III.6) are linear and continuous in the A's and the B's, it is sufficient to take $A_j = e_j \wedge f_j$, $B_j = x_j \wedge y_j$. Then, since $b_z \omega' = 0$,

$$\begin{aligned}
&\left\langle \bigwedge_{j=0}^{n} e_j \wedge f_j, \bigwedge_{j=0}^{n} x_j \wedge y_j \right\rangle \\
&= \left\langle \sum_{j=0}^{n} b_{e_j}^* b_{f_j}^* \omega', \prod_{j=0}^{n} c_{x_j}^* c_{y_j}^* \omega \right\rangle \\
&= \left\langle b_{y_0} b_{x_0} \prod_{j=0}^{n} b_{e_j}^* b_{f_j}^* \omega', \prod_{j=1}^{n} c_{x_j}^* c_{y_j}^* \omega \right\rangle \\
&= \sum_{p=0}^{n} \left\langle b_{y_0} [b_{x_0}, b_{e_p}^* b_{f_p}^*] \prod_{j \neq p} b_{e_j}^* b_{f_j}^* \omega', \prod_{j=1}^{n} c_{x_j}^* c_{y_j}^* \omega \right\rangle \\
&= \sum_{p=0}^{n} \left\langle \{b_{y_0}, [b_{x_0}, b_{e_p}^* b_{f_p}^*]\} \prod_{j \neq p} b_{e_j}^* b_{f_j}^* \omega', \prod_{j=1}^{n} c_{x_j}^* c_{y_j}^* \omega \right\rangle \\
&\quad - \sum_{\substack{0 \leqslant p,q \leqslant n \\ p \neq q}} \left\langle [b_{x_0}, b_{e_p}^* b_{f_p}^*][b_{y_0}, b_{e_q}^* b_{f_q}^*] \prod_{j \neq p,q} b_{e_j}^* b_{f_j}^* \omega', \prod_{j=1}^{n} c_{x_j}^* c_{y_j}^* \omega \right\rangle .
\end{aligned}$$

Using (III.8) we find that

$$[b_x, b_e^* b_f^*] = \langle e, x \rangle\, b_f^* - \langle f, x \rangle\, b_e^*, \tag{III.13}$$

and

$$\{b_y, [b_x, b_e^* b_f^*]\} = \langle e, x \rangle \langle f, y \rangle - \langle f, x \rangle \langle e, y \rangle = \langle e \wedge x, f \wedge y \rangle. \tag{III.14}$$

This gives the first term on the right-hand side of (III.6). Furthermore, comparing the identity

$$\begin{aligned}
&[b_x, b_{e_1}^* b_{f_1}^*][b_y, b_{e_2}^* b_{f_2}^*] + [b_x, b_{e_2}^* b_{f_2}^*][b_y, b_{e_1}^* b_{f_1}^*] \\
&\quad = \langle e_1 \wedge e_2, x \wedge y \rangle\, b_{f_1}^* b_{f_2}^* - \langle e_1 \wedge f_2, x \wedge y \rangle\, b_{f_1}^* b_{e_2}^* \\
&\qquad - \langle f_1 \wedge e_2, x \wedge y \rangle\, b_{e_1}^* b_{f_2}^* + \langle f_1 \wedge f_2, x \wedge y \rangle\, b_{e_1}^* b_{e_2}^*
\end{aligned}$$

with the identity

$$\begin{aligned}
&(e_1 \wedge f_1) \cdot (x \wedge y) \cdot (e_2 \wedge f_2) + (e_2 \wedge f_2) \cdot (x \wedge y) \cdot (e_1 \wedge f_1) \\
&\quad = -\langle e_1 \wedge e_2, x \wedge y \rangle f_1 \wedge f_2 + \langle e_1 \wedge f_2, x \wedge y \rangle f_1 \wedge e_2 \\
&\qquad + \langle f_1 \wedge e_2, x \wedge y \rangle\, e_1 \wedge f_2 - \langle f_1 \wedge f_2, x \wedge y \rangle\, e_1 \wedge e_2
\end{aligned}$$

yields the second term on the right-hand side of (III.6). ∎

COROLLARY III.5. *For* $A \in \bigwedge^2(E')$, $B_0, B_1, ..., B_n \in \bigwedge^2(E)$,

$$\frac{1}{n+1}\left\langle \bigwedge^{n+1} A, \bigwedge_{j=0}^{n} B_j \right\rangle = \langle A, B_0 \rangle \left\langle \bigwedge^{n} A, \bigwedge_{j=1}^{n} B_j \right\rangle$$

$$+ n \left\langle A \cdot B_0 \cdot A \wedge \bigwedge^{n-1} A, \bigwedge_{j=1}^{n} B_j \right\rangle. \quad \text{(III.15)}$$

IV. RELATIVE PFAFFIAN MINOR

In this section we define and study the properties of the relative Pfaffian minor. Its significance in the theory of Pfaffians is similar to the significance of Fredholm minors in the theory of Fredholm determinants [G].

Let ∇ denote Fréchet derivative on $\bigwedge^2(E)$. As a consequence of (II.7), the function

$$\mathbb{C} \ni z \to \mathrm{Pf}(A, B + zX) \in \mathbb{C} \quad \text{(IV.1)}$$

is entire and thus $\nabla^n \mathrm{Pf}(A, B)$ exists for $n \geqslant 0$.

DEFINITION IV.1. The nth relative Pfaffian minor of $(A, B) \in \bigwedge^2(E') \times \bigwedge^2(E)$ is

$$\mathrm{Pf}^{(n)}(A, B) := n! \, \nabla^n \mathrm{Pf}(A, B). \quad \text{(IV.2)}$$

THEOREM IV.2. *For* (A, B), $(V, X) \in \bigwedge^2(E') \times \bigwedge^2(E)$, *the function*

$$\mathbb{C} \ni z \to \mathrm{Pf}^{(n)}(A + zV, B + zX) \in (\bigwedge^{2n}(E))',$$

is entire. Furthermore,

$$\|\mathrm{Pf}^{(n)}(A, B)\|_1 \leqslant \exp\{2e \|A\|_1 (\|B\|_1 + n)\}. \quad \text{(IV.3)}$$

Proof. The first statement is a consequence of the definition and (II.7). To prove (IV.3), write for $X_1, ..., X_n \in \bigwedge^2(E)$

$$\langle \mathrm{Pf}^{(n)}(A, B), X_1 \wedge \cdots \wedge X_n \rangle = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \mathrm{Pf}\left(A, B + \sum_{j=1}^{n} z_j X_j\right)\Bigg|_{z=0}. \quad \text{(IV.4)}$$

Using (II.7) and Cauchy's bound on an n-disc of radii $\|X_j\|^{-1}$, $1 \leqslant j \leqslant n$, we obtain

$$|\langle \mathrm{Pf}^{(n)}(A, B), X_1 \wedge \cdots \wedge X_n \rangle| \leqslant \exp\{2e\|A\|_1(\|B\|_1 + n)\} \prod_{j=1}^{n} \|X_j\|_1.$$

This implies that

$$|\langle \mathrm{Pf}^{(n)}(A, B), \omega \rangle| \leqslant \exp\{2e\|A\|_1(\|B\|_1 + n)\} \|\omega\|_1,$$

for arbitrary $\omega \in \bigwedge^{2n}(E)$ and (IV.3) follows. ∎

Our next result shows that the Pfaffian minor is Hölder continuous as a map

$$\bigwedge^2(E') \times \bigwedge^2(E) \ni (A, B) \to \mathrm{Pf}^{(n)}(A, B) \in (\bigwedge^{2n}(E))'. \tag{IV.5}$$

THEOREM IV.3. *For* $(A_1, B_1), (A_2, B_2) \in \bigwedge^2(E') \times \bigwedge^2(E)$,

$$\begin{aligned} &\|\mathrm{Pf}^{(n)}(A_1, B_1) - \mathrm{Pf}^{(n)}(A_2, B_2)\|_1 \\ &\quad \leqslant (\|A_1 - A_2\|_1 + \|B_1 - B_2\|_1) \exp\{2e(\|A_2\|_1 + \|A_1 - A_2\|_1 + 1) \\ &\qquad \times (\|B_2\|_1 + \|B_1 - B_2\|_1 + n + 1)\}. \end{aligned} \tag{IV.6}$$

Proof. As in the proof of Theorem II.3,

$$\mathrm{Pf}^{(n)}(A_1, B_1) - \mathrm{Pf}^{(n)}(A_2, B_2) = \int_0^1 \frac{d}{ds} \mathrm{Pf}^{(n)}(A(s), B(s))\, ds. \tag{IV.7}$$

By Theorem IV.2, the function $\mathbb{C} \ni z \to \mathrm{Pf}^{(n)}(A + zV, B + zX) \in \bigwedge^{2n}(E')$ is entire. Therefore, Cauchy's bound on a circle of radius $(\|V\|_1 + \|X\|_1)^{-1}$ yields

$$\begin{aligned} &\left\| \frac{d}{ds} \mathrm{Pf}^{(n)}(A + sV, B + sX) \right\| \\ &\quad \leqslant (\|V\|_1 + \|X\|_1) \exp\{2e(\|A\|_1 + \|V\|_1 + 1) \\ &\qquad \times (\|B\|_1 + \|X\|_1 + n + 1)\}, \end{aligned} \tag{IV.8}$$

and (IV.6) follows. ∎

Remark. Estimate (IV.6) with $n = 0$ is weaker than Theorem II.13, because the method of proof using Cauchy's bound is cruder than the method used to prove (II.9).

THEOREM IV.4. *Assume that $(I-A\cdot B)^{-1}\in\mathscr{L}(E)$. Then*

$$\mathrm{Pf}^{(n)}(A,B)=\mathrm{Pf}(A,B)\bigwedge^n(I-A\cdot B)^{-1}A\in\bigwedge^{2n}(E'), \tag{IV.9}$$

where $(I-A\cdot B)^{-1}A\in\bigwedge^2(E')$ is defined by $\langle (I-A\cdot B)^{-1}A, x\wedge y\rangle :=$ $\langle A, (I-A\cdot B)^{-1}x\wedge y\rangle=\langle A, x\wedge(I-A\cdot B)^{-1}y\rangle$, for $x,y\in E$.

Proof. We use induction on n. Let $\mathbb{R}\ni s\to B(s)\in\bigwedge^2(E)$ be a continuously differentiable function. Then with $\dot{B}(s):=dB(s)/ds$,

$$\frac{d}{ds}\det(I-A\cdot B(s))=-\mathrm{tr}((I-A\cdot B(s))^{-1}A\cdot\dot{B}(s))\det(I-A\cdot B(s)), \tag{IV.10}$$

where the trace of $\sum_{j=1}^{\infty}e_j\otimes x_j\in E'\otimes E$ is defined by $\mathrm{tr}(\sum_{j=1}^{\infty}e_j\otimes x_j):=$ $\sum_{j=1}^{\infty}\langle e_j,x_j\rangle$. From (II.16) and (IV.10) we infer that

$$\begin{aligned}\frac{d}{ds}\mathrm{Pf}(A,B(s))&=-\frac{1}{2}\mathrm{tr}((I-A\cdot B(s))^{-1}A\cdot\dot{B}(s))\,\mathrm{Pf}(A,B(s))\\&=\langle (I-A\cdot B(s))^{-1}A,\dot{B}(s)\rangle\,\mathrm{Pf}(A,B(s)).\end{aligned} \tag{IV.11}$$

Choosing $B(s)=B+sX$ we obtain

$$\mathrm{Pf}^{(1)}(A,B)=\mathrm{Pf}(A,B)(I-A\cdot B)^{-1}A, \tag{IV.12}$$

which proves (IV.9) for $n=1$.

We claim that for any $n\geqslant 1$,

$$\begin{aligned}&\frac{\partial^n}{\partial s_1\cdots\partial s_n}\mathrm{Pf}\left(A,B+\sum_{j=1}^n s_jX_j\right)\\&\quad=\frac{1}{n!}\left\langle\bigwedge^n\left(I-A\cdot\left(B+\sum_{j=1}^n s_jX_j\right)\right)^{-1}A,\bigwedge_{j=1}^n X_j\right\rangle\\&\quad\quad\times\mathrm{Pf}\left(A,B+\sum_{j=1}^n s_jX_j\right).\end{aligned} \tag{IV.13}$$

Indeed, we compute

$$\begin{aligned}&\frac{\partial^{n+1}}{\partial s_1\cdots\partial s_{n+1}}\mathrm{Pf}\left(A,B+\sum_{j=1}^n s_jX_j\right)\\&\quad=\frac{\partial}{\partial s_{n+1}}\left\{\frac{1}{n!}\left\langle\bigwedge^n\left(I-A\cdot\left(B+\sum_{j=1}^{n+1} s_jX_j\right)\right)^{-1}A,\bigwedge_{j=1}^n X_j\right\rangle\right.\\&\quad\quad\left.\times\mathrm{Pf}\left(A,B+\sum_{j=1}^{n+1} s_jX_j\right)\right\}\end{aligned}$$

$$
\begin{aligned}
=\frac{1}{n!}\Bigg\{n\Bigg\langle&\Bigg(I-A\cdot\Bigg(B+\sum_{j=1}^{n+1}s_jX_j\Bigg)\Bigg)^{-1}A\cdot X_{n+1}\\
&\cdot\Bigg(I-A\cdot\Bigg(B+\sum_{j=1}^{n+1}s_jX_j\Bigg)\Bigg)^{-1}A\\
&\wedge\bigwedge^{n-1}\Bigg(I-A\cdot\Bigg(B+\sum_{j=1}^{n+1}s_jX_j\Bigg)\Bigg)^{-1}A,\bigwedge_{j=1}^{n}X_j\Bigg\rangle\\
&+\Bigg\langle\Bigg(I-A\cdot\Bigg(B+\sum_{j=1}^{n+1}s_jX_j\Bigg)\Bigg)^{-1}A,X_{n+1}\Bigg\rangle\\
&\times\Bigg\langle\bigwedge^{n}\Bigg(I-A\cdot\Bigg(B+\sum_{j=1}^{n+1}s_jX_j\Bigg)\Bigg)^{-1},\bigwedge_{j=1}^{n}X_j\Bigg\rangle\operatorname{Pf}\Bigg(A,B+\sum_{j=1}^{n+1}s_jX_j\Bigg)\Bigg\}.
\end{aligned}
$$

Using (III.15) we can rewrite this as

$$
\begin{aligned}
\frac{1}{(n+1)!}&\Bigg\langle\bigwedge^{n+1}\Bigg(I-A\cdot\Bigg(B+\sum_{j=1}^{n+1}s_jX_j\Bigg)\Bigg)^{-1}A,\bigwedge_{j=1}^{n+1}X_j\Bigg\rangle\\
&\times\operatorname{Pf}\Bigg(A,B+\sum_{j=1}^{n+1}s_jX_j\Bigg),
\end{aligned}
$$

which completes the inductive step of the proof of (IV.3). Evaluating (IV.13) at $s=0$ yields (IV.9). ∎

COROLLARY IV.5. *For all* $(A, B)\in\bigwedge^2(E')\times\bigwedge^2(E)$,

$$\operatorname{Pf}^{(n)}(A,B)\in\bigwedge^{2n}(E'). \tag{IV.14}$$

Our last theorem relates the Pfaffian minor to the Fredholm minor [G]. Let $M^{(n)}(T):=n!\,\nabla^n\det(I-T)$ be the nth Fredholm minor of $T\in E'\otimes E$.

THEOREM IV.6. *For* $(A, B)\in\bigwedge^2(E')\times\bigwedge^2(E)$ *and* $x_1, ..., x_{2n}\in E$,

$$\Bigg\langle\operatorname{Pf}^{(n)}(A,B),\bigwedge_{j=1}^{2n}x_j\Bigg\rangle^2=\Bigg\langle\bigwedge_{j=1}^{2n}\phi'(A)\,x_j,M^{(2n)}(A\cdot B)\bigwedge_{j=1}^{2n}x_j\Bigg\rangle, \tag{IV.15}$$

where ϕ' *is the naural isomorphism* $\bigwedge^2(E')\cong I_1^a(E,E')$.

Proof. It is sufficient to prove (IV.15) for $I-A\cdot B$ invertible (the general case follows by a perturbation argument). In this case

$$M^{(n)}(T)=\det(I-T)\bigwedge^n(I-T)^{-1}. \tag{IV.16}$$

But then from (IV.9),

$$\begin{aligned}\left\langle \mathrm{Pf}^{(n)}(A, B), \bigwedge_{j=1}^{2n} x_j \right\rangle^2 &= \mathrm{Pf}(A, B)^2 \left\langle \wedge^n (I - A\cdot B)^{-1} A, \bigwedge_{j=1}^{2n} x_j \right\rangle^2 \\ &= \det(I - A\cdot B)\, \mathrm{Pf}(\{\langle (I-A\cdot B)^{-1}A, x_j \wedge x_k\rangle\})^2 \\ &= \det(I - A\cdot B) \det(\{\langle (I-A\cdot B)^{-1}A, x_j \wedge x_k)\rangle\})^2.\end{aligned}$$

Furthermore, from (II.15),

$$\begin{aligned}\langle (I-A\cdot B)^{-1}A, x_j \wedge x_k\rangle &= \langle A, x_j \wedge (I-A\cdot B)^{-1}x_k\rangle \\ &= \langle \phi'(A)\, x_j, (I-A\cdot B)^{-1}x_k\rangle,\end{aligned}$$

and the claim follows. ▮

V. Pfaffians on Hilbert Spaces

In this section we assume that $E = \mathscr{H}$ is a separable Hilbert space, and let $\{x_j\}_{j=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$. By $\{x_j'\}_{j=1}^{\infty}$ we denote the basis for $\mathscr{H}'$ which is dual to $\{x_j\}_{j=1}^{\infty}$.

For $A \in \bigwedge^2(\mathscr{H}')$, $B \in \bigwedge^2(\mathscr{H})$ we set

$$A_{jk} := \langle A, x_j \wedge x_k\rangle, \qquad B_{jk} := \langle x_j' \wedge x_k', B\rangle. \tag{V.1}$$

Clearly,

$$A_{jk} = -A_{kj}, \qquad B_{jk} = -B_{kj}. \tag{V.2}$$

For each finite $S \subset \mathbb{N}$, let A_S and B_S denote the restrictions of A and B to the subspaces spanned by x_j and x_j', $j \in S$, respectively. Then A_S and B_S are finite dimensional, skew symmetric matrices. In the theorem below, we give an expression for $\mathrm{Pf}(A, B)$ in terms of $\mathrm{Pf}(A_S)$ and $\mathrm{Pf}(B_S)$. This expression coincides with the formula used in [JLW, PS] to define the relative Pfaffian on a Hilbert space.

Theorem V.1. *Let $A \in \bigwedge^2(\mathscr{H}')$, $B \in \bigwedge^2(\mathscr{H})$. Then*

$$\mathrm{Pf}(A, B) = \sum_{S \subset \mathbb{N}} \mathrm{Pf}(A_S)\, \mathrm{Pf}(B_S). \tag{V.3}$$

Proof. We will show that

$$\frac{1}{(n!)^2}\langle \wedge^n A, \wedge^n B\rangle = \sum_{\substack{S \subset \mathbb{N} \\ |S|=2n}} \mathrm{Pf}(A_S)\,\mathrm{Pf}(B_S). \tag{V.4}$$

In fact,

$$\begin{aligned}
&\frac{1}{(n!)^2}\langle \wedge^n A, \wedge^n B\rangle \\
&\quad = \frac{1}{(2^n n!)^2} \sum_{j_1, \ldots, j_{2n}} \sum_{k_1, \ldots, k_{2n}} \\
&\qquad \times A_{1_1 j_2} \cdots A_{j_{2n-1} j_{2n}} B_{k_1 k_2} \cdots B_{k_{2n-1} k_{2n}} \langle x'_{j_1} \wedge \cdots \wedge x'_{j_{2n}}, x_{k_1} \wedge \cdots \wedge x_{k_{2n}}\rangle.
\end{aligned}$$

Fix a set $S \subset \mathbb{N}$ of $2n$ elements: $S = \{s_1, \ldots, s_{2n}\}$ and sum over all j's such that $\{j_1, \ldots, j_{2n}\} = S$. The only nonvanishing contributions come from those k's for which also $\{k_1, \ldots, k_{2n}\} = S$. Therefore, denoting by S_{2n} the group of permutations of $2n$ elements, we can write the above expression as

$$\begin{aligned}
&\frac{1}{(2^n n!)^2} \sum_{\pi \in S_{2n}} \sum_{\rho \in S_{2n}} \\
&\qquad \times A_{s_{\pi(1)} s_{\pi(2)}} \cdots A_{s_{\pi(2n-1)} s_{\pi(2n)}} B_{s_{\rho(1)} s_{\rho(2)}} \cdots B_{s_{\rho(2n-1)} s_{\rho(2n)}} \\
&\qquad \times \langle x'_{s_{\pi(1)}} \wedge \cdots \wedge x'_{s_{\pi(2n)}}, x_{s_{\rho(1)}} \wedge \cdots \wedge x_{s_{\rho(2n)}}\rangle \\
&\quad = \left\{\frac{1}{2^n n!} \sum_{\pi \in S_{2n}} (-1)^\pi A_{s_{\pi(1)} s_{\pi(2)}} \cdots A_{s_{\pi(2n-1)} s_{\pi(2n)}}\right\} \\
&\qquad \times \left\{\frac{1}{2^n n!} \sum_{\pi \in S_{2n}} (-1)^\pi B_{s_{\pi(1)} s_{\pi(2)}} \cdots B_{s_{\pi(2n-1)} s_{\pi(2n)}}\right\} \\
&\quad = \mathrm{Pf}(A_S)\,\mathrm{Pf}(B_S). \quad \blacksquare
\end{aligned}$$

In fact, on a Hilbert space, it is possible to extend the definition of the relative Pfaffian to a larger class of operators [JLW, JKL]. Let $\wedge^2_{\mathrm{Hilb}}(\mathscr{H})$ and $\wedge^2_{\mathrm{Hilb}}(\mathscr{H}')$ denote the exterior powers of $\mathscr{H}$ and $\mathscr{H}'$, respectively, equipped with the usual topology of a Hilbert space, see, e.g., [RS]. Then formulas (II.15) define isomorphisms of Hilbert spaces $\phi': \wedge^2_{\mathrm{Hilb}}(\mathscr{H}') \to I_2^a(\mathscr{H}, \mathscr{H}')$ and $\phi: \wedge^2_{\mathrm{Hilb}}(\mathscr{H}) \to I_2^a(\mathscr{H}', \mathscr{H})$, where $I_2^a(\mathscr{H}, \mathscr{H}')$ denotes the space of all skew symmetric, Hilbert–Schmidt operators from $\mathscr{H}$ to $\mathscr{H}'$.

THEOREM V.2. *For* $A \in \wedge^2_{\mathrm{Hilb}}(\mathscr{H}')$, $B \in \wedge^2_{\mathrm{Hilb}}(\mathscr{H})$, *the series* (II.6) *converges absolutely. Furthermore, formula* (V.3) *holds and*

$$|\mathrm{Pf}(A, B)| \leqslant \sum_{n=0}^{\infty} \frac{1}{(n!)^2} |\langle \wedge^n(A), \wedge^n(B) \rangle|$$
$$\leqslant \exp\left\{\frac{1}{4}(\|\phi'(A)\|_2^2 + \|\phi(B)\|_2^2)\right\}. \tag{V.5}$$

Finally, let us remark that choosing a complex structure on $\mathscr{H}$, we have the isomorphisms $I_2^a(\mathscr{H}, \mathscr{H}') \cong I_2^a(\mathscr{H}', \mathscr{H}) \cong I_2^a(\mathscr{H})$. We thus recover the original structure of [JLW]. In fact, much sharper estimates on $\mathrm{Pf}(A, B)$ are obtained in [JKL].

As a specific example, we consider $\mathscr{H} = L^2(X, dx)$, where X is a separable, locally compact space, and where dx is a Borel measure on X. Note that $L^2(X, dx)$ carries a natural complex structure given by complex conjugation, and thus we can identify $\mathscr{H}'$ with $\mathscr{H}$. Clearly, $\wedge^2_{\mathrm{Hilb}}(\mathscr{H}) = L_a^2(X^k, \otimes_{j=1}^k dx_j)$, the space of square-integrable, skew symmetric functions on X^k.

LEMMA V.3. *Let $A, B \in \wedge^2_{\mathrm{Hilb}}(L^2(X, dx))$. Then*

$$\langle \wedge^n A, \wedge^n B \rangle = \frac{(n!)^2}{(2n)!} \int_{X^{2n}} \mathrm{Pf}(A(x_j, x_k)) \, \mathrm{Pf}(B(x_j, x_k)) \bigotimes_{j=1}^{2n} dx_j. \tag{V.6}$$

Proof. Let $f_j(x)$, $j = 1, 2, \ldots$, be an orthonormal basis for $L^2(X, dx)$. Then $f_j \wedge f_k(x, y)$, $j < k$, is an orthonormal basis for $\wedge^2(L^2(X, dx))$ and

$$A(x, y) = \tfrac{1}{2} \sum_{j,k} A_{jk}(f_j \wedge f_k)(x, y), \tag{V.7}$$

with $A_{jk} = \int_{X^2} A(x, y)(f_j \wedge f_k)(x, y) \, dx \otimes dy$. Inserting this into the integral on the right-hand side of (V.6), we obtain

$$\int_{X^{2n}} \mathrm{Pf}(A(x_j, x_k)) \, \mathrm{Pf}(B(x_j, x_k)) \bigotimes_{j=1}^{2n} dx_j$$
$$= \sum_{k_1, \ldots, k_{2n}} \sum_{l_1, \ldots, l_{2n}} \frac{1}{(2^n n!)^2}$$
$$\times \sum_{\pi_1, \rho \in S_{2n}} (-1)^\pi (-1)^\rho A_{k_{\pi(1)} k_{\pi(2)}} \cdots A_{k_{\pi(2n-1)} k_{\pi(2n)}}$$
$$\cdots B_{l_{\rho(1)} l_{\rho(2)}} \cdots B_{l_{\rho(2n-1)} l_{\rho(2n)}}$$
$$\times \int_{X^{2n}} f_{\pi(1)}(x_{\pi(1)}) \cdots f_{\pi(2n)}(x_{\pi(2n)})$$
$$\times f_{\rho(1)}(x_{\rho(1)}) \cdots f_{\rho(2n)}(x_{\rho(2n)}) \bigotimes_{j=1}^{2n} dx_j. \tag{V.8}$$

The integral on the right-hand side of (V.8) vanishes unless $k_{\pi(j)} = l_{\rho(j)}$, for $j = 1, 2, ..., 2n$. Therefore, (V.8) can be written as

$$(2n)! \sum_{\substack{S \subset \mathbb{N} \\ |S| = 2n}} \mathrm{Pf}(A_S)\, \mathrm{Pf}(B_S).$$

Comparing this with (V.4) yields (V.6). ∎

An immediate consequence of this lemma is:

THEOREM V.4. *For $A \in \bigwedge^2_{\mathrm{Hilb}}(L^2(X, dx))$ we let*

$$A_n(x_1, ..., x_n) := \begin{cases} \mathrm{Pf}(A(x_j, x_k)), & \text{if } n = 2m, \\ 0, & \text{if } n = 2m+1. \end{cases} \tag{V.9}$$

Then:

(i) *The Pfaffian* $\mathrm{Pf}(A, B)$, $A, B \in \bigwedge^2_{\mathrm{Hilb}}(\mathscr{H})$, *has the expansion*

$$\mathrm{Pf}(A, B) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} A_n(x_1, ..., x_n)\, B_n(x_1, ..., x_n) \bigotimes_{j=1}^{n} dx_j. \tag{V.10}$$

(ii) *Let* $\mathrm{Pf}^{(m)}(A, B)(y_1, ..., y_{2m})$ *denote the integral kernel of the* mth *Pfaffian minor of* (A, B). *Then*

$$\mathrm{Pf}^{(m)}(A, B)(y_1, ..., y_{2m}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} A_{n+2m}(y_1, ..., y_{2m}, x_1, ..., x_n)\, B_n(x_1, ..., x_n) \bigotimes_{j=1}^{n} dx_j. \tag{V.11}$$

ACKNOWLEDGMENT

We thank Arthur Jaffe for very helpful discussions and comments on the manuscript.

REFERENCES

[G] A. GROTHENDIECK, La Théorie de Fredholm, *Bull. Soc. Math. France* **84** (1956), 319–384.

[JKL] A. JAFFE, S. KLIMEK, AND A. LESNIEWSKI, in preparation.

[JLW] A. JAFFE, A. LESNIEWSKI, AND J. WEITSMAN, Pfaffians on Hilbert space, *J. Funct. Anal.* **83** (1989), 348–363.

[PS] A. PRESSLEY AND G. SEGAL, "Loop Groups," Oxford Univ. Press, London/New York, 1986.

[RS] M. REED AND B. SIMON, "Methods of Modern Mathematical Physics, IV." Academic Press, London/New York, 1978.

[S] H. H. SCHAEFER, "Topological Vector Spaces," Springer, Berlin/New York, 1971.