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Pfaffians on Hilbert Space*

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We present a theory of relative Pfaffians on an infinite-dimensional Hilbert space.

1. INTRODUCTION

The generalization of the theory of determinants to operators on infinite-dimensional spaces is a cornerstone of analysis. This theory, developed by Fredholm and others, allows the definition of $\det(I+K)$ for K an element of the first Schatten class I_1 [1, 6, 7]. Analogous, regularized determinants $\det_n(I+K)$ exist for $K \in I_n$. In the applications, the operator $I+K$ is often the ratio of two unbounded operators, e.g., two Laplace or Dirac operators.

In this paper we investigate analoguous structures for the Pfaffian. In the finite-dimensional case, the Pfaffian is well known as a square root of the determinant of an antisymmetric matrix. It turns out that the notion of an infinite-dimensional Pfaffian emerges naturally in various applications, e.g., representation theory of loop groups [5], conformal field theory [8], and loop space index theorems related to $N=1$ supersymmetry [3].

We present a systematic treatment of the existence and continuity of Pfaffians in infinite dimensions. For $A, B \in I_2$ we define the relative Pfaffian $\mathrm{Pf}(A, B)$, which in finite dimensions and when A is invertible equals

$$\frac{\mathrm{Pf}(A^{-1}-B)}{\mathrm{Pf}(A^{-1})}. \tag{I.1}$$

The relative Paffian is an invariant, i.e., it is basis independent (the Pfaffian

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itself depends on an orientation class). In finite dimensions the Pfaffian satisfies

$$\operatorname{Pf}(A)^2 = \det(A) \tag{I.2}$$

while in infinite dimensions we have the corresponding relation,

$$\operatorname{Pf}(A, B)^2 = \det(I - AB). \tag{I.3}$$

In addition we define and study relative Pfaffian minors and regularized, relative Pfaffians. For example, for A, $B \in I_{2n}$, n odd, we define a regularized, relative Pfaffian $\operatorname{Pf}_n(A, B)$ with the property that

$$\operatorname{Pf}_n(A, B)^2 = \det{}_n(I - AB). \tag{I.4}$$

The relative Pfaffian $\operatorname{Pf}(A, B)$ may exist even though $\operatorname{Pf}(A^{-1})$ and $\operatorname{Pf}(A^{-1} - B)$ do not. For example, if A is a $2N$-dimensional, invertible matrix, then $(-1)^N \operatorname{Pf}(A^{-1}) = \operatorname{Pf}(A)^{-1}$. Thus both $\operatorname{Pf}(A)$ and $\operatorname{Pf}(A^{-1})$ cannot have limits as $N \to \infty$. On the other hand, a ratio of Pfaffians, and hence our relative Pfaffians, can have a well-behaved $N \to \infty$ limit. In other words, the theory of $\det(I + K)$ as a perturbation of the identity $\det(I) = 1$ only has an analog for the relative Pfaffian of two operators. In spite of the identities for the squares of the relative Pfaffian, the analytic problems for the existence and continuity of the Pfaffian cannot be immediately reduced to those for determinants: the Pfaffian does not obey a multiplication law like that for determinants. These issues are settled here by development of absolutely convergent expansions in finite-dimensional Pfaffians. As a consequence, if A and B depend analytically on a parameter, then $\operatorname{Pf}_n(A, B)$ is also analytic.

II. Finite-Dimensional Pfaffians

For the reader's convenience we review in this section the basic properties of finite-dimensional Pfaffians. We base our approach on expressing the Pfaffian as a Gaussian "Berezin integral" over a Grassmann algebra [2]. Parts of this section are also similar to Mathai and Quillen [4]. Our main goal is to derive the expansion (II.29) which we then formulate as Definition II.7 of the relative Pfaffian.

Let $\mathscr{V}$ denote a finite, even dimensional, real Euclidean space, and let $\{e_1, ..., e_{2N}\}$ denote an orthonormal basis for $\mathscr{V}$. Let $\mathscr{V}_c = \mathscr{V} \otimes \mathbb{C}$ be the complexification of $\mathscr{V}$, and let C be the complex conjugation on $\mathscr{V}_c$. By $\mathscr{L}(\mathscr{V}_c)$ we denote the set of endomorphisms of $\mathscr{V}_c$. An endomorphism $A \in \mathscr{L}(\mathscr{V}_c)$ is called skew-symmetric, if $A^{\mathrm{T}} = -A$, where the superscript T means transposition, or equivalently, $A^* = -CAC$, where $*$ means hermitian conjugation. Let $\mathscr{L}^a(\mathscr{V})$ be the set of skew-symmetric $A \in \mathscr{L}(\mathscr{V})$.

DEFINITION II.1. For $A \in \mathscr{L}^a(\mathscr{V})$, the Pfaffian is defined by

$$\mathrm{Pf}(A) = \frac{1}{2^N N!} \sum_{\pi \in S_{2N}} (-1)^\pi A_{\pi(1)\pi(2)} \cdots A_{\pi(2N-1)\pi(2N)}, \tag{II.1}$$

where S_{2N} is the symmetric group on $2N$ elements, and where A_{ij} are the matrix elements of A with respect to $\{e_1, ..., e_{2N}\}$.

A convenient representation of the Pfaffian is given by a Gaussian integral over a Grassmann algebra. Let $\mathscr{G}$ denote a real Grassmann algebra generated by $2N$ elements $\xi_1, ..., \xi_{2N}$. These elements span a vector space $\mathscr{M}$ and satisfy

$$\{\xi_i, \xi_j\} = \xi_i \xi_j + \xi_j \xi_i = 0, \qquad 1 \leqslant i, j \leqslant 2N. \tag{II.2}$$

Let $\mathscr{G}_c = \mathscr{G} \otimes \mathscr{C}$ be the complexification of $\mathscr{G}$. The Berezin integral [2] $\int \cdot \, d\xi$ is a linear funtional from $\mathscr{G}_c$ to $\mathbb{C}$. The monomials

$$\xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_{2N}^{\alpha_{2N}},$$

where $\alpha_j = 0$ or 1, define a basis for $\mathscr{G}_c$; on these elements we set

$$\int \xi_1^{\alpha_1} \cdots \xi_{2N}^{\alpha_{2N}} \, d\xi = \begin{cases} 1, & \text{if } \alpha_1 = \alpha_2 = \cdots = \alpha_{2N} = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{II.3}$$

The functional (II.3) can also be viewed as a multidimensional, iterated integral. Let $d\xi_j$, $j = 1, ..., 2N$, be a basis for $\mathscr{M}^*$ dual to $\{\xi_j\}$. Define one-dimensional integrals $\int \cdot \, d\xi_j$, $d\xi_j \in \mathscr{M}^*$, by

$$\int d\xi_j = 0, \qquad \int \xi_k \, d\xi_j = \delta_{jk}. \tag{II.4}$$

It is natural to regard ξ_j and $d\xi_j$ as elements of $\bigwedge (\mathscr{M}_c \oplus \mathscr{M}_c^*)$, so we have the algebraic relations

$$\{d\xi_i, d\xi_j\} = 0, \qquad \{\xi_i, d\xi_j\} = 0. \tag{II.5}$$

Then (II.3) can be regarded as an iterated integral with respect to

$$d\xi = d\xi_{2N} \, d\xi_{2N-1} \cdots d\xi_1. \tag{II.6}$$

The Grassmann algebra $\mathscr{G}_c$ can be naturally identified as the exterior algebra over $\mathscr{V}_c$,

$$\mathscr{G}_c = \bigwedge(\mathscr{V}_c).$$

Under this identification, a linear transformation $V \in \mathscr{L}(\mathscr{V}_c)$ has a natural action on $\mathscr{G}$. For $f \in \mathscr{G}_c$, let Vf denote this action. Linear transformations on the generators $\xi_1, ..., \xi_{2N}$ of $\mathscr{G}$ yield a transformation of the Berezin integral. This is the basic convariance property.

PROPOSITION II.2 [2]. *For $V \in \mathscr{L}(\mathscr{V}_c)$ and $f \in \mathscr{G}_c$,*

$$\int Vf\, d\xi = \det(V) \int f\, d\xi. \tag{II.7}$$

The Berezin integral transforms under a linear change of coordinates according to the inverse Jacobian. The proof of (II.7) follows immediately from

$$\int \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2N}}\, d\xi = \varepsilon_{i_1 \cdots i_{2N}}, \tag{II.8}$$

where $\varepsilon_{i_1 \cdots i_{2N}}$ is the sign of the permutation $i_1, ..., i_{2N}$ of $\{1, ..., 2N\}$.

Let $\langle \xi, A\xi \rangle$ denote the quadratic form

$$\langle \xi, A\xi \rangle = \sum_{i \leqslant i, j \leqslant 2N} A_{ij} \xi_i \xi_j. \tag{II.9}$$

Then the Pfaffian can be expressed as a Gaussian integral

$$\mathrm{Pf}(A) = \int \exp\left(\frac{1}{2} \langle \xi, A\xi \rangle\right) d\xi. \tag{II.10}$$

It follows from Proposition II.2 that for any $V \in \mathscr{L}(\mathscr{V}_c)$

$$\mathrm{Pf}(V^{\mathrm{T}} A V) = \det(V)\, \mathrm{Pf}(A). \tag{II.11}$$

This means that $\mathrm{Pf}(A)$ depends only on the orientation class of the basis $\{e_1, ..., e_{2N}\}$ of the underlying real Euclidean space $\mathscr{V}$.

The Pfaffian is also characterized (up to sign) as a homogeneous polynomial in the A_{ij} which is a square root of $\det(A)$. This fundamental property is summarized by

PROPOSITION II.3. *For $A \in \mathscr{L}^a(\mathscr{V}_c)$,*

$$\mathrm{Pf}(A)^2 = \det(A). \tag{II.12}$$

Before establishing (II.12), we introduce the notion of a complex Berezin integral. Let $\mathscr{G}'$ denote an isomorphic copy of $\mathscr{G}$ with generators ξ'_k, $k=1, ..., 2N$, and let $\bar{\mathscr{G}} = \mathscr{G}_c \otimes \mathscr{G}'_c$ be the graded tensor product of the $\mathbb{Z}_2$ graded algebras $\mathscr{G}_c$ and $\mathscr{G}'_c$. Then

$$\{\xi_k, \xi'_l\} = 0, \qquad k, l = 1, ..., 2N. \tag{II.13}$$

We define new generators for $\bar{\mathscr{G}}$,

$$\eta_k = \frac{1}{\sqrt{2}}(\xi_k + i\xi'_k), \quad \bar{\eta}_k = \frac{1}{\sqrt{2}}(\xi_k - i\xi'_k), \tag{II.14}$$

which satisfy

$$\{\eta_k, \eta_l\} = \{\eta_k, \bar{\eta}_l\} = \{\bar{\eta}_k, \bar{\eta}_l\} = 0. \tag{II.15}$$

Also let

$$d\eta_k = \frac{1}{\sqrt{2}}(d\xi_k + id\xi'_k), \quad d\bar{\eta}_k = \frac{1}{\sqrt{2}}(d\xi_k - id\xi'_k), \tag{II.16}$$

so the one-dimensional integrals

$$\int \bar{\eta}_k \, d\eta_k = \int \eta_k \, d\bar{\eta}_k = 1 \tag{II.17}$$

and

$$\int d\eta_k = \int d\bar{\eta}_k = \int \eta_k \, d\eta_k = \int \bar{\eta}_k \, d\bar{\eta}_k = 0 \tag{II.18}$$

yield as iterated integrals

$$\int (\eta_k \bar{\eta}_k) \, d\eta_k \, d\bar{\eta}_k = 1 \tag{II.19}$$

and

$$\int d\eta_k \, d\bar{\eta}_k = \int \eta_k \, d\eta_k \, d\bar{\eta}_k = \int \bar{\eta}_k \, d\eta_k \, d\bar{\eta}_k = 0. \tag{II.20}$$

Since $d\eta_k \, d\bar{\eta}_k$ commutes with $d\eta_l$ and $d\bar{\eta}_l$, a $2N$-complex dimensional Grassmann integral can be defined by

$$d\eta \, d\bar{\eta} = \prod_{k=1}^{2N} (d\eta_k \, d\bar{\eta}_k). \tag{II.21}$$

Then with

$$\langle \eta, A\bar{\eta}\rangle = \sum_{k,l=1}^{2N} A_{kl}\eta_k\bar{\eta}_l,$$

it follows that

$$\int \exp\langle \eta, A\bar{\eta}\rangle \, d\eta \, d\bar{\eta} = \det(A). \tag{II.22}$$

Proof of Proposition II.3. For $A \in \mathscr{L}^a(\mathscr{V}_c)$,

$$\langle \eta, A\bar{\eta}\rangle = \tfrac{1}{2}\langle \xi, A\xi\rangle + \tfrac{1}{2}\langle \xi', A\xi'\rangle,$$

so

$$\exp\langle \eta, A\bar{\eta}\rangle = \exp(\tfrac{1}{2}\langle \xi, A\xi\rangle)\exp(\tfrac{1}{2}\langle \xi', A\xi'\rangle),$$

and $\det(A) = \mathrm{Pf}(A)^2$, as claimed.

We now introduce the notion of the relative Pfaffian. As in the case of ratios of determinants, the relative Pfaffian plays an important role in the theory of perturbations of Gaussian measures. For $A \in \mathscr{L}(\mathscr{V}_c)$ and S a subset of $\{1, ..., 2N\}$, let A_S denote the restriction of A to the subspace of $\mathscr{V}_c$ spanned by $\{e_j\}_{j\in S}$ (we give $\{e_j\}_{j\in S}$ the orientation induced by $\{e_j\}_{j=1}^{2N}$). It is also convenient to define

$$\mathrm{Pf}(A_\varnothing) = 1, \qquad \varnothing = \text{empty set} \tag{II.23}$$

and

$$\mathrm{Pf}(A_S) = 0, \tag{II.24}$$

if $|S| = $ (Cardinality of S) is odd.

We now define the Pfaffian minor. Let σ be a subset of $\{1, ..., 2N\}$ and let σ^c denote its complement. Let $\varepsilon(\sigma, \sigma^c)$ denote the sign of the permutation of $\{1, ..., 2N\}$ given by $\{\sigma, \sigma^c\}$ with the elements arranged in increasing order.

DEFINITION II.4. Let $A \in \mathscr{L}^a(\mathscr{V}_c)$ be invertible. The Pfaffian minor corresponding to σ is defined by

$$\mathrm{Pf}^\sigma(A) = \varepsilon(\sigma, \sigma^c)\,\mathrm{Pf}(A_{\sigma^c}). \tag{II.25}$$

Let $\mathscr{T}_i$, $i = 1, ..., 2N$, denote an independent set of $2N$ Grassmann elements, satisfying

$$\{\xi_i, \mathscr{T}_k\} = \{\mathscr{T}_i, \mathscr{T}_k\} = 0. \tag{II.26}$$

PROPOSITION II.5. *Let $A \in \mathscr{L}^a(\mathscr{V}_c)$ be invertible. Then*

$$\int \exp\left(\frac{1}{2}\langle \xi, A^{-1}\xi\rangle + \langle \mathscr{T}, \xi\rangle\right) d\xi = \mathrm{Pf}(A^{-1}) \exp\left(\frac{1}{2}\langle \mathscr{T}, A\mathscr{T}\rangle\right) \tag{II.27}$$

and

$$\begin{aligned}\mathrm{Pf}^{\sigma}(A) &= (-1)^{|\sigma|/2}\, \mathrm{Pf}((A^{-1})_{\sigma})\, \mathrm{Pf}(A)\\ &= \int \left(\prod_{j\in\sigma} \xi_j\right) \exp\left(\frac{1}{2}\langle \xi, A\xi\rangle\right) d\xi,\end{aligned} \tag{II.28}$$

where $\prod$ denotes the product of ξ_j in order of increasing j.

Proof. Using the antisymmetry of A,

$$\begin{aligned}&-\tfrac{1}{2}\langle A^{-1}\xi - \mathscr{T}, A(A^{-1}\xi - \mathscr{T})\rangle\\ &\quad = \tfrac{1}{2}\langle \xi, A^{-1}\xi\rangle + \langle \mathscr{T}, \xi\rangle - \tfrac{1}{2}\langle \mathscr{T}, A\mathscr{T}\rangle.\end{aligned}$$

We exponentiate and integrate this identity over $d\xi$. Using Propositions II.2 and II.3 we obtain (II.27). The identity (II.28) now can be obtained by identifying the power series coefficients of (II.27) with respect to $\mathscr{T}_k$.

PROPOSITION II.6. *Let A, $B \in \mathscr{L}^a(\mathscr{V}_c)$ and let A be invertible. Then*

$$\frac{\mathrm{Pf}(A^{-1} - B)}{\mathrm{Pf}(A^{-1})} = \sum_{S} \mathrm{Pf}(A_S)\, \mathrm{Pf}(B_S), \tag{II.29}$$

where the summation extends over all subsets S of $\{1, ..., 2N\}$.

Proof. Using the representation (II.10), we write

$$\begin{aligned}\mathrm{Pf}(A^{-1} - B) &= \int \exp\left(\frac{1}{2}\langle \xi, A^{-1}\xi\rangle\right) \exp\left(-\frac{1}{2}\langle \xi, B\xi\rangle\right) d\xi\\ &= \sum_{k\geqslant 0} \frac{(-1)^k}{2^k k!} \sum_{1\leqslant j_i\leqslant 2N} B_{j_1 j_2} \cdots B_{j_{2k-1} j_{2k}}\\ &\quad \times \int \xi_{j_1} \cdots \xi_{j_{2k}} \exp\left(\frac{1}{2}\langle \xi, A^{-1}\xi\rangle\right) d\xi.\end{aligned}$$

Fix a set $S = \{j_1, ..., j_{2k}\}$ with $j_1 < j_2 < \cdots < j_{2k}$ and sum over the permutations of the elements of S. Using Proposition II.5,

$$\mathrm{Pf}(A^{-1} - B) = \sum_{k\geqslant 0} \sum_{S:|S|=2k} (-1)^k\, \mathrm{Pf}(B_S)\, \mathrm{Pf}^S(A^{-1}). \tag{II.30}$$

Thus

$$\mathrm{Pf}(A^{-1}-B)=\sum_S \mathrm{Pf}(A_S)\,\mathrm{Pf}(B_S)\,\mathrm{Pf}(A^{-1}), \tag{II.31}$$

and the proof is complete.

Having the nice expansion (II.30) which is symmetric in A and B, we can elevate this result to a definition.

DEFINITION II.7. For A, $B \in \mathscr{L}^a(\mathscr{V}_c)$, define the relative Pfaffian of A and B to be

$$\mathrm{Pf}(A, B)=\sum_S \mathrm{Pf}(A_S)\,\mathrm{Pf}(B_S), \tag{II.32}$$

where the summation extends over even subsets S of $\{1, ..., 2N\}$, with the convention (II.23).

Remark. It follows from Proposition II.6 and (II.11) that

$$\mathrm{Pf}(V^{-1}A(V^{-1})^{\mathrm{T}}, V^{\mathrm{T}}BV)=\mathrm{Pf}(A, B), \tag{II.33}$$

for all invertible $V \in \mathscr{L}(\mathscr{V}_c)$. In particular, $\mathrm{Pf}(A, B)$ is independent of the choice of basis in $\mathscr{V}$. It is an invariant.

PROPOSITION II.8. *For A, $B \in \mathscr{L}^a(\mathscr{V}_c)$*

$$\mathrm{Pf}(A, B)^2=\det(I-AB). \tag{II.34}$$

Proof. For A invertible, this follows from Propositions II.3 and II.6. If A is not invertible, perturb A and remove the perturbation.

III. THE INFINITE-DIMENSIONAL PFAFFIAN

In this section we extend the definition of the relative Pfaffian to the infinite-dimensional context. In Theorem III.3 we establish absolute convergence of the expansion (II.32) for $\mathrm{Pf}(A, B)$ under natural assumptions on A and B.

Let $\mathscr{H}$ be a real separable Hilbert space, and let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathscr{H}$. By $\mathscr{L}(\mathscr{H})$ we denote the set of bounded linear operators on $\mathscr{H}$, and by $I_p(\mathscr{H})$ we denote the pth Schatten class of operators with the usual norm $\|A\|_p=\{\mathrm{Tr}(|A|^p)\}^{1/p}$, where $|A|=(A^{\mathrm{T}}A)^{1/2}$, and where the superscript T means transposition.

Let $\mathscr{H}_c=\mathscr{H}\otimes\mathbb{C}$ be the complexification of $\mathscr{H}$, and let C denote the complex conjugation on $\mathscr{H}_c$. By $\mathscr{L}(\mathscr{H}_c)$ we denote the set of bounded linear operators on $\mathscr{H}_c$. Every operator $A \in \mathscr{L}(\mathscr{H}_c)$ can be uniquely represented in terms of its real and imaginary parts,

$$A = A_r + iA_i \tag{III.1}$$

with A_r, A_i real, i.e., $CA_rC = A_r$, etc. In fact,

$$\begin{aligned} A_r &= \frac{1}{2}(A + CAC), \\ A_i &= \frac{1}{2i}(A - CAC). \end{aligned} \tag{III.2}$$

By $\mathscr{L}^a(\mathscr{H}_c)$ we denote the set of skew elements of $\mathscr{H}_c$, namely $A \in \mathscr{L}(\mathscr{H}_c)$ satisfying $(A)^{\mathrm{T}} = -A$, or in other words, $A^* = -CAC$, where $*$ means Hermitian conjugation. Let $I_p(\mathscr{H}_c)$ be the pth Schatten class equipped with the norm $\|A\|_p = \{\mathrm{Tr}(A^*A)^{p/2}\}^{1/p}$. We also introduce the antisymmetric Schatten class $I_p^a(\mathscr{H}_c) = I_p(\mathscr{H}_c) \cap \mathscr{L}^a(\mathscr{H}_c)$. Finally, by $\mathscr{T}$ we denote the set of all finite subsets of $\mathbb{N}$ with an even number of elements. For $S \in \mathscr{T}$,

$$\mathrm{Pf}((CAC)_S) = \mathrm{Pf}((-A^*)_S). \tag{III.3}$$

PROPOSITION III.1. *For* $A \in I_2^a(\mathscr{H}_c)$,

$$\sum_{S \in \mathscr{T}} |\mathrm{Pf}(A_S)|^2 \leqslant \exp \frac{1}{2} \|A\|_2^2. \tag{III.4}$$

Proof. Let $S_0 \in \mathscr{T}$. Then using (III.3),

$$\begin{aligned} \sum_{S \subset S_0} |\mathrm{Pf}(A_S)|^2 &= \sum_{S \subset S_0} \mathrm{Pf}((-A^*)_S)\, \mathrm{Pf}(A_S) \\ &= \mathrm{Pf}(-A_{S_0}^*, A_{S_0}). \end{aligned} \tag{III.5}$$

Using (II.34) and the well-known estimate $|\det(I+K)| \leqslant \exp \|K\|_1$,

$$\begin{aligned} \mathrm{Pf}(-A_{S_0}^*, A_{S_0}) &= \{\det(I + A_{S_0}^* A_{S_0})\}^{1/2} \\ &\leqslant \exp \tfrac{1}{2} \|A_{S_0}\|_2^2. \end{aligned} \tag{III.6}$$

Since $A_{S_0} \to A$ in $I_2^a(\mathscr{H}_c)$, we can pass to the limit $S_0 \to \mathbb{N}$ in this inequality to obtain (III.4).

DEFINITION III.2. Let A, $B \in I_2^a(\mathscr{H}_c)$. Define the relative Pfaffian by

$$\mathrm{Pf}(A, B) = \sum_{S \in \mathscr{T}} \mathrm{Pf}(A_S)\, \mathrm{Pf}(B_S). \tag{III.7}$$

Remarks. It is clear that

$$\mathrm{Pf}(A, B) = \mathrm{Pf}(B, A). \tag{III.8}$$

Furthermore, Proposition III.1 ensures the absolute convergence of the series (III.7) for A, $B \in I_2^a$. By the Schwarz inequality we obtain

THEOREM III.3. *For A, $B \in I_2^a(\mathscr{H}_c)$,*

$$\begin{aligned} |\mathrm{Pf}(A, B)| &\leqslant \sum_{S \in \mathscr{F}} |\mathrm{Pf}(A_S)\, \mathrm{Pf}(B_S)| \\ &\leqslant \exp \frac{1}{4} (\|A\|_2^2 + \|B\|_2^2). \end{aligned} \tag{III.9}$$

COROLLARY III.4. *Let A, $B \in I_2^a(\mathscr{H}_c)$ and z_1, $z_2 \in \mathbb{C}$. Then*

$$\mathrm{Pf}(z_1 A, z_2 B)$$

is an entire function of z_1, z_2.

Next we establish Hölder continuity of the mapping A, $B \to \mathrm{Pf}(A, B)$.

THEOREM III.5. *For A, A', B, $B' \in I_2^a(\mathscr{H})$,*

$$\begin{aligned} |\mathrm{Pf}(A, B) - \mathrm{Pf}(A', B')| \leqslant{} & (\|A - A'\|_2 + \|B - B'\|_2) \\ & \times \exp \tfrac{1}{4}\{(\|A\|_2 + \|A'\|_2 + 1)^2 \\ & + (\|B\|_2 + \|B'\|_2 + 1)^2\}. \end{aligned} \tag{III.10}$$

Proof. The theorem follows immediately from Corollary II.4 and [6, Lemma 6.6], namely

LEMMA III.6. *Let X be a complex Banach space and $f\colon X \to \mathbb{C}$ a function such that*

(i) *$z \to f(T + zV)$ is entire for all T, $V \in X$.*

(ii) *There is a monotone nondecreasing function $G\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $T \in X$*

$$|f(T)| \leqslant G(\|T\|).$$

Then

$$|f(T) - f(V)| \leqslant \|T - V\|\, G(\|T\| + \|V\| + 1)$$

for all T, $V \in X$.

THEOREM III.7. *For* A, $B \in I_2^a(\mathscr{H}_c)$,

$$\mathrm{Pf}(A, B)^2 = \det(I - AB). \tag{III.11}$$

Proof. Using Proposition II.8 and Theorem III.4 we have

$$\begin{aligned}\mathrm{Pf}(A, B)^2 &= \lim_{S \to \mathbb{N}} \mathrm{Pf}(A_S, B_S)^2 \\ &= \lim_{S \to \mathbb{N}} \det(I - A_S B_S) = \det(I - AB),\end{aligned}$$

as $A_S B_S \to AB$ in $I_2(\mathscr{H}_c)$.

THEOREM III.8. *Let* A, $B \in I_2^a$ *and let* V *and* V^{-1} *be bounded. Then*

$$\mathrm{Pf}(A, B) = \mathrm{Pf}(V^{-1}A(V^{-1})^{\mathrm{T}}, V^{\mathrm{T}}BV). \tag{III.12}$$

Proof. This is a consequence of (II.33) and a restriction of the series (III.7) for (III.12) to finite-dimensional subspaces. Since $V^{-1}A(V^{-1})^{\mathrm{T}}$, $V^{\mathrm{T}}BV$, A, and B are all I_2, we can remove this approximation.

Remark. We infer from (III.12) that $\mathrm{Pf}(A, B)$ is a basis independent invariant.

We can use (III.12) to extend the definition of $\mathrm{Pf}(A, B)$. In typical applications $A \in I_{1+\varepsilon}^a$, B is an unbounded, skew-symmetric operator, and we wish to "transfer" regularity from A to B.

DEFINITION III.9. Let A, B be skew-symmetric, let V be invertible, and suppose

$$V^{-1}A(V^{-1})^{\mathrm{T}} \in I_2^a, \qquad V^{\mathrm{T}}BV \in I_2^a.$$

Then define $\mathrm{Pf}(A, B)$ by (III.12).

IV. THE RELATIVE PFAFFIAN MINOR

DEFINITION IV.1. Let A, $B \in I_2^a(\mathscr{H}_c)$ and let $\sigma \in \mathscr{T}$. The relative Pfaffian minor of (A, B) with respect to σ is

$$\mathrm{Pf}^{\sigma}(A, B) = \sum_{\substack{S \in \mathscr{T} \\ S \cap \sigma = \varnothing}} \mathrm{Pf}(A_{S \cup \sigma})\, \mathrm{Pf}(B_S). \tag{IV.1}$$

Clearly, $\mathrm{Pf}^{\sigma}(A, B)$ is a generalization of $\mathrm{Pf}(A, B)$,

$$\mathrm{Pf}^{\varnothing}(A, B) = \mathrm{Pf}(A, B). \tag{IV.2}$$

The following two results generalize the corresponding theorems about relative Pfaffians.

THEOREM IV.2. *For* A, $B \in I_2^a(\mathscr{H}_c)$, $\sigma \in \mathscr{T}$,

$$|\mathrm{Pf}^\sigma(A, B)| \leqslant \sum_{S \cap \sigma = \varnothing} |\mathrm{Pf}(A_{S \cup \sigma}) \, \mathrm{Pf}(B_S)| \leqslant \exp \frac{1}{4} (\|A\|_2^2 + \|B\|_2^2). \tag{IV.3}$$

Proof. We bound the middle term in (IV.3) by

$$\begin{aligned} &\Big\{ \sum_{S \cap \sigma = \varnothing} |\mathrm{Pf}(A_{S \cup \sigma})|^2 \Big\}^{1/2} \Big\{ \sum_{S \cap \sigma = \varnothing} |\mathrm{Pf}(B_S)|^2 \Big\}^{1/2} \\ &\qquad \leqslant \Big\{ \sum_{S \in \mathscr{T}} |\mathrm{Pf}(A_S)|^2 \Big\}^{1/2} \Big\{ \sum_{S \in \mathscr{T}} |\mathrm{Pf}(B_S)|^2 \Big\}^{1/2}. \end{aligned} \tag{IV.4}$$

Using (III.4) we bound (IV.4) by $\exp \frac{1}{4} (\|A\|_2^2 + \|B\|_2^2)$, which proves the bound (IV.3).

COROLLARY IV.3. *For* A, A', B, $B' \in I_2^a(\mathscr{H}_c)$ *and* $\sigma \in \mathscr{T}$,

$$\begin{aligned} |\mathrm{Pf}^\sigma(A, B) - \mathrm{Pf}^\sigma(A', B')| \leqslant {} & (\|A - A'\|_2 + \|B - B'\|_2) \\ & \times \exp \tfrac{1}{4} \{ (\|A\|_2 + \|A'\|_2 + 1)^2 \\ & + (\|B\|_2 + \|B'\|_2 + 1)^2 \}. \end{aligned} \tag{IV.5}$$

THEOREM IV.4. *Let* A, $B \in I_2^a(\mathscr{H}_c)$ *and suppose that* $(I - AB)^{-1} \in \mathscr{L}(\mathscr{H}_c)$. *Then*

$$\mathrm{Pf}^\sigma(A, B) = (-1)^{|\sigma|/2} \, \mathrm{Pf}(((I - AB)^{-1} A)_\sigma) \, \mathrm{Pf}(A, B). \tag{IV.6}$$

Proof. Formula (IV.6) holds if A and B are replaced by A_S and B_S, respectively, for $S \in \mathscr{T}$. Since $\|A_S B_S - AB\| = o(1)$, as $S \to \mathbb{N}$, there is an $S_0 \in \mathscr{T}$ such that $(I - A_S B_S)^{-1} \in \mathscr{L}(\mathscr{H}_c)$ for all $S \supset S_0$, and that

$$\|(I - AB)^{-1} - (I - A_S B_S)^{-1}\| = o(1), \tag{IV.7}$$

as $S \to \mathbb{N}$. This and Corollary IV.3 imply that

$$\begin{aligned} \mathrm{Pf}^\sigma(A, B) &= \lim_{S \to \mathbb{N}} \mathrm{Pf}^\sigma(A_S, B_S) \\ &= \lim_{S \to \mathbb{N}} \mathrm{Pf}(((I - A_S B_S)^{-1} A_S)_\sigma) \, \mathrm{Pf}(A_S, B_S) \\ &= \mathrm{Pf}(((I - AB)^{-1} A)_\sigma) \, \mathrm{Pf}(A, B), \end{aligned} \tag{IV.8}$$

where we have also used Theorem II.4. The claim follows.

COROLLARY IV.5. *With the same assumptions*

$$\mathrm{Pf}^{\sigma}(A, B)^2 = \det(((I-AB)^{-1}A)_{\sigma}) \det(I-AB). \tag{IV.9}$$

V. THE REGULARIZED, RELATIVE PFAFFIAN

In this section we generalize the notion of the relative Pfaffian to a regularized, relative Pfaffian $\mathrm{Pf}_n(A, B)$. For A, B elements of the Schatten class I_{2n}^a, the regularized, relative Pfaffian can be defined and $\mathrm{Pf}_1(A, B) = \mathrm{Pf}(A, B)$. The most important property of Pf_n is that

$$\mathrm{Pf}_n(A, B)^2 = \det{}_n(I-AB). \tag{V.1}$$

For A, $B \in I_{2n}^a$, it follows that $AB \in I_n$. The regularized Fredholm determinant $\det_n$ is defined on I_n, so (V.1) is a natural relation.

Let us begin by introducing the function

$$R_n(t) = (t-1)\exp\left(\sum_{k=1}^{n-1} t^k/k\right) + 1 = t^n T_n(t). \tag{V.2}$$

By interpolation, we obtain the representation

$$\begin{aligned} T_n(t) = t^{-n} R_n(t) &= t^{-n} \int_0^1 \frac{d}{ds} R(st)\, ds \\ &= \int_0^1 s^{n-1} \exp\left(\sum_{k=1}^{n-1} (st)^k/k\right) ds. \end{aligned} \tag{V.3}$$

Thus $T_n(t)$ is an entire function and

$$|T_n(t)| \leqslant \exp\left(\sum_{k=1}^{n-1} |t|^k/k\right).$$

For $A \in I_n$ it follows that $T_n(A)$ is bounded and $R_n(A) \in I_1$. Hence for $A \in I_n$, it follows that $\det(I - R_n(A))$ exists and [6]

$$\det{}_n(I-A) = \det(I - R_n(A)). \tag{V.4}$$

We also note that for A, $B \in I_{2n}^a$, n odd,

$$(AB)^{(n-1)/2} A \in I_2^a, \tag{V.5}$$

$$(BA)^{(n-1)/2} B T_n(AB) \in I_2^a. \tag{V.6}$$

The skew-symmetry of (V.6) follows from the identity

$$(BA)^r BT_n(AB) = T_n(BA)(BA)^r B, \tag{V.7}$$

where r is a positive integer.

DEFINITION V.1. For $A, B \in I_{2n}^a$, n odd,

$$\mathrm{Pf}_n(A, B) = \mathrm{Pf}((AB)^{(n-1)/2} A, (BA)^{(n-1)/2} BT_n(AB)). \tag{V.8}$$

PROPOSITION V.2. *For $A, B \in I_{2n}^a$, n odd, the relation* (V.1) *holds. Also*

$$|\mathrm{Pf}_n(A, B)| \leqslant \exp(\gamma_n \|AB\|_n^n), \tag{V.9}$$

for a constant γ_n independent of A, B.

Proof. We use (V.8) and the identity (III.11). Then

$$\mathrm{Pf}_n(A, B)^2 = \det(I - (AB)^n T_n(AB)) = \det{}_n(I - AB).$$

By means of [6, Theorem 6.4],

$$|\det{}_n(I - AB)| \leqslant \exp(2\gamma_n \|AB\|_n^n),$$

and the proposition follows.

We have the following continuity property of the regularized, relative Pfaffian.

PROPOSITION V.3. *Given n odd, there exists a constant γ_n such that for A, B, A', $B' \in I_{2n}^a$,*

$$\begin{aligned} |\mathrm{Pf}_n(A, B) - \mathrm{Pf}_n(A', B')| \leqslant (&\|A - A'\|_{2n} + \|B - B'\|_{2n}) \\ &\times \exp\{\gamma_n[(\|A\|_{2n} + \|A'\|_{2n} + 1)^n \\ &+ (\|B\|_{2n} + \|B'\|_{2n} + 1)^n]\}. \end{aligned} \tag{V.10}$$

Proof. This is a consequence of (V.8), Proposition III.5, and Hölder's inequality.

It is important that our definition of $\mathrm{Pf}_n(A, B)$ is natural, even though it appears complicated. When $AB \in I_1$,

$$\det{}_n(I - AB) = \det(I - AB) \exp\left(\sum_{j=1}^{n-1} \frac{1}{j} \mathrm{Tr}(AB)^j\right).$$

Our analogous result for Pf_n is:

COROLLARY V.4. *For A, $B \in I_2^a$,*

$$\mathrm{Pf}_n(A, B) = \mathrm{Pf}(A, B) \exp\left(\frac{1}{2} \sum_{j=1}^{n-1} \frac{1}{j} \mathrm{Tr}(AB)^j\right). \tag{V.11}$$

Proof. As a consequence of (V.1) and the representation of $\det_n$ above,

$$\mathrm{Pf}_n(A, B) = \varepsilon(A, B)\, \mathrm{Pf}(A, B) \exp\left(\frac{1}{2} \sum_{j=1}^{n-1} \frac{1}{j} \mathrm{Tr}(AB)^j\right),$$

where $\varepsilon(A, B) = \pm 1$. By continuity, as expressed in the proposition, $\varepsilon(A, B) = \varepsilon(A, 0) = 1$, and (V.11) holds.

DEFINITION V.5. Let n be odd, let A, $B \in I_{2n}^a(\mathscr{H}_c)$ be such that $(I - AB)^{-1} \in \mathscr{L}(\mathscr{H})$, and let $\sigma \in \mathscr{T}$. We define the regularized Pfaffian minor to be

$$\mathrm{Pf}_n^\sigma(A, B) = (-1)^{|\sigma|/2}\, \mathrm{Pf}(((I - AB)^{-1} A)_\sigma)\, \mathrm{Pf}_n(A, B). \tag{V.12}$$

THEOREM V.6. *With the same assumptions as in Definition* V.5,

$$|\mathrm{Pf}_n^\sigma(A, B)| \leqslant \beta_n^{|\sigma|}\, \|A\|_{2n}^{|\sigma|/2} \exp\{\gamma_n \|AB\|_n^n\} \tag{V.13}$$

with β_n and γ_n independent of A, B.

Remark. The estimate shows that the regularized, relative Pfaffian minor extends by continuity to all A, $B \in I_{2n}^a$. The singularity of the first factor in (V.12) cancels against a zero of the second factor yielding a well-defined object.

Proof. We have

$$\begin{aligned}
|\mathrm{Pf}_n^\sigma(A, B)|^2 &= \det(((I - AB)^{-1} A)_\sigma \det_n(I - AB) \\
&= \left(\bigwedge_{j \in \sigma} e_j, \bigwedge^{|\sigma|}(I - AB)^{-1} A \bigwedge_{j \in \sigma} e_j\right) \det_n(I - AB) \\
&\leqslant \|A\|_{2n}^{|\sigma|} \left\|\bigwedge^{|\sigma|}(I - AB)^{-1}\right\| \det_n(I - AB).
\end{aligned}$$

With the help of the inequality

$$\left\|\bigwedge^k(I - T)^{-1}\right\| \det_n(I - T) \leqslant \beta_n^{2k} \exp(2\gamma_n \|T\|_n^n),$$

this yields (V.13).

As above, this bound yields

COROLLARY V.7. *With the same assumptions,*

$$|\mathrm{Pf}_n^\sigma(A, B) - \mathrm{Pf}_n^\sigma(A', B')| \leqslant \beta_n^{|\sigma|}(\|A\|_{2n} + \|A'\|_{2n} + 1)^{|\sigma|/2} \times (\|A - A'\|_{2n} + \|B - B'\|_{2n}) \times \exp\{\gamma_n[(\|A\|_{2n} + \|A'\|_{2n} + 1)^n + (\|B\|_{2n} + \|B'\|_{2n} + 1)^n]\}. \tag{V.14}$$

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