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Quantum K-Theory.
II. Homotopy Invariance of the Chern Character*

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We prove that the Chern character of quantum algebras is invariant under a class of deformations of the Dirac operator. We also extend the definition of the Chern character to include certain unbounded operators.

I. INTRODUCTION

This is our second paper on entire cyclic cohomology, which generalizes non-commutative differential geometry to the case of infinite dimensions. In the first paper [5] we defined a quantum algebra, and we constructed an even cocycle τ for Connes' entire cohomology of the algebra, with $\mathbb{Z}_2$-grading. In Connes' terminology, the quantum algebra is a $\mathbb{Z}_2$-graded, θ-summable Fredholm module [2].

Here we carry on the study of τ in three directions. First, we extend the cocycle τ from the space of entire cochains $\mathscr{C}_+(\mathscr{A})$ which are even under the $\mathbb{Z}_2$ grading to the space $\mathscr{C}(\mathscr{A})$ of all entire cochains on $\mathscr{A}$. This extension has been investigated independently by Kastler [6].

Second we study a family τ^λ of cocycles arising from a family of Dirac operators parameterized by a real parameter λ. We give sufficient conditions on the Dirac operator $Q(\lambda)$ which generates τ^λ such that the elements of the family τ^λ are cohomologous.

Finally, we define a family of Banach space norms $\|\cdot\|_\eta$ on $\mathscr{A}$ such that the cocycles τ extend continuously to the closure $\mathscr{A}_\eta$ of $\mathscr{A}$ in these norms. This allows us to extend τ continuously to limiting elements of $\mathscr{A}$ in $\mathscr{A}_\eta$. These limiting elements are not necessarily bounded operators in $\mathscr{L}(\mathscr{H})$, but may be convenient for the computation of the cohomology class determined by τ. In a separate work we are investigating how the general

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estimates proposed here arise in some specific examples of geometry in infinite dimensions in the context of supersymmetric quantum field theory.

Recall that a quantum algebra $(\mathscr{A}, \mathscr{H}, \Gamma, Q)$ comprises a unital algebra $\mathscr{A}$ of bounded operators on a Hilbert space $\mathscr{H}$ with a $\mathbb{Z}_2$-grading Γ, which is self-adjoint and unitary. Furthermore, there is a self-adjoint operator Q on $\mathscr{H}$ with the two properties: (i) the heat kernel of Q^2 is trace class, namely,

$$\operatorname{Tr}(e^{-\beta Q^2}) < \infty, \qquad \text{for all} \quad \beta > 0, \tag{I.1}$$

and (ii) Q is odd, namely,

$$\Gamma Q + Q\Gamma = 0. \tag{I.2}$$

The operator Q defines a graded derivation d of $\mathscr{A}$,

$$da = i[Q, a]. \tag{I.3}$$

Here the graded commutator (I.3) is defined by

$$[Q, a] = (Qa_+ - a_+ Q) + (Qa_- + a_- Q), \tag{I.4}$$

with

$$a_\pm = \tfrac{1}{2}(a \pm \Gamma a \Gamma) \tag{I.5}$$

the homogeneous parts of a under the grading.

We assume that $d\colon \mathscr{A} \to \mathscr{A}$. In the commutative, nongraded case this assumption means that $\mathscr{A}$ is the algebra of smooth functions on a manifold. Instead [3], we would start with a $\mathbb{C}^*$-algebra $\mathscr{X}$ ("the algebra of continuous functions"), and assume that: (i) $\mathscr{A}$ is norm-dense in $\mathscr{X}$ and (ii) $d\mathscr{A} \subset \mathscr{X}$. Then $\mathscr{A}$ would be a non-commutative analog of continuously differentiable functions.

We define the supertrace form

$$F_n(a_0, \ldots, a_n) = \operatorname{Str}\left(\int_{\sigma_n} a_0 a_1(t_1)\, a_2(t_2) \cdots a_n(t_n)\, e^{-Q^2}\, dt\right), \tag{I.6}$$

where $a(t) = e^{-tH} a e^{tH}$ and where σ_n is the simplex $\sigma_n = \{t \in \mathbb{R}^n : 0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant 1\}$. Recall the definition from [5],

$$\tau_n(a_0, \ldots, a_n) = (-1)^{[n/2]}\, F_n(a_0, da_1^{\Gamma}, \ldots, da_j^{\Gamma^j}, \ldots, da_n^{\Gamma^n}). \tag{I.7}$$

For n even, this agrees with the Chern character $\tau^{\beta=1} = (\tau_0, \tau_2, \ldots,)$ defined in [5], while for n odd we show in Section II that $\tau = (\tau_1, \tau_3, \ldots,)$ is also a cocycle.

II. A Chern Character on $\mathscr{C}_-(\mathscr{A})$

Let $\mathscr{C}^n(\mathscr{A})$ denote the $(n+1)$-multilinear functionals on $\mathscr{A}$ which are continuous with respect to the norm

$$\|a\|_* = \|a\| + \|da\|. \tag{II.1}$$

This continuity induces a topology on $\mathscr{C}^n(\mathscr{A})$ given by a norm $\|f_n\|_*$, for $f_n \in \mathscr{C}^n(\mathscr{A})$. The entire cochains $f = (f_0, f_1, ..., f_n, ...)$ are elements of $\bigoplus_n \mathscr{C}^n(\mathscr{A})$ such that

$$\mathscr{C}(\mathscr{A}) = \{f : f_n \in \mathscr{C}^n(\mathscr{A}),\ n^{1/2}\|f_n\|_*^{1/n} \to 0\}. \tag{II.2}$$

The grading Γ lifts to $\mathscr{C}$ and induces the splitting

$$\mathscr{C} = \mathscr{C}_+ \oplus \mathscr{C}_-, \qquad \mathscr{C}^n = \mathscr{C}^n_+ \oplus \mathscr{C}^n_-. \tag{II.3}$$

In [5] we restricted attention to $\mathscr{C}_+(\mathscr{A})$. Here we define b, B on $\mathscr{C}_-(\mathscr{A})$ which agrees on homogeneous elements with the formula of Connes [1, p. 276]. Let $f \in \mathscr{C}(\mathscr{A})$ be homogeneous, namely, $f \in \mathscr{C}_+(\mathscr{A})$ or $f \in \mathscr{C}_-(\mathscr{A})$. Then define

$$a^{\sim} = \begin{cases} a^\Gamma & f \in \mathscr{C}_+ \\ a & f \in \mathscr{C}_-. \end{cases} \tag{II.4}$$

We take

$$\begin{aligned}(bf_n)(a_0, ..., a_{n+1}) = \sum_{j=0}^{n} &(-1)^j f_n(a_0, ..., a_j a_{j+1}, ..., a_n) \\ &+ (-1)^{(n+1)} f_n(a^{\sim}_{n+1} a_0, a_1, ..., a_n)\end{aligned} \tag{II.5}$$

and

$$\begin{aligned}(Bf_n)(a_0, ..., a_{n-1}) = \sum_{j=0}^{n} (-1)^{(n-1)j}(&f_n(\mathbf{1}, a^{\sim}_{n-j}, ..., a^{\sim}_{n-1}, ..., a_0, ..., a_{n-j-1}) \\ &+ (-1)^{(n+1)} f_n(a^{\sim}_{n-1}, ..., a^{\sim}_{n-1}, a_0, ..., a_{n-j-1}, \mathbf{1})).\end{aligned} \tag{II.6}$$

With this definition,

$$b, B : \mathscr{C}_\pm \to \mathscr{C}_\pm$$

and

$$B^2 = b^2 = 0, \qquad Bb + bB = 0. \tag{II.7}$$

This can be shown using similar arguments as in [5]. Thus, b, B, and $\partial = b + B$ extend by linearity to $\mathscr{C}(\mathscr{A})$ and satisfy (II.10) and $\partial^2 = 0$.

With this definition, we claim (and Kastler [6] independently shows) that the arguments of [5] yield

THEOREM II.1. *The entire cochain* τ *defined by* (I.7) *is a* ∂*-cocycle and* $\tau_{2k} \in \mathscr{C}_+(\mathscr{A})$, $\tau_{2k+1} \in \mathscr{C}_-(\mathscr{A})$.

III. HOMOTOPY INVARIANCE OF THE CHERN CHARACTER

In this section we consider deformations of the Chern character τ. Let $\lambda \in I$, a closed interval; it is no loss of generality to assume that $I = [0, 1]$. Consider a family of odd, self-adjoint operators $Q(\lambda)$, whose heat kernels are trace class, with $Q \equiv Q(\lambda = 0)$. Define

$$d_\lambda a = i[Q(\lambda), a], \qquad da \equiv d_0 a = i[Q, a], \tag{III.1}$$

and

$$F_n^\lambda(a_0, ..., a_n) = \mathrm{Str} \int_{\sigma_n} a_0 e^{-t_1 Q(\lambda)^2} a_1 e^{-(t_2 - t_1) Q(\lambda)^2} \cdots a_n e^{-(1 - t_n) Q(\lambda)^2} \, dt. \tag{III.2}$$

As in (I.7), we set

$$\tau_n^\lambda(a_0, ..., a_n) = (-1)^{[n/2]} F_n^\lambda(a_0, d_\lambda a_1^\Gamma, ..., d_\lambda a_j^{\Gamma^j}, ..., d_\lambda a_n^{\Gamma^n}). \tag{III.3}$$

Our main result in this section is to give sufficient conditions on $Q(\lambda)$, such that

$$\tau^\lambda - \tau = \int_0^\lambda \left(\frac{d\tau^{\lambda'}}{d\lambda'} \right) d\lambda'$$

and

$$\frac{d\tau^\lambda}{d\lambda} = \partial G^\lambda \tag{III.4}$$

with $G^\lambda \in \mathscr{C}(\mathscr{A})$. Hence

$$\tau^\lambda - \tau = \partial H^\lambda,$$

where

$$H^\lambda = \int_0^\lambda G^{\lambda'} \, d\lambda',$$

which implies that τ^λ and τ are cohomologous.

Let us now formulate our conditions on $Q(\lambda)$. The analytic details are somewhat involved, but clearly some precise continuity hypotheses are required. We assume that $Q(\lambda)$ is a family of self-adjoint operators, with $H(\lambda)=Q(\lambda)^2$. We assume that $\lambda \to (Q(\lambda) \pm i)^{-1}$ is norm continuous in $\mathscr{L}(\mathscr{H})$ for $\lambda \in I$. Define the self-adjoint contraction

$$f(\lambda)=e^{-H(\lambda)}.$$

From norm-continuity of $(Q(\lambda) \pm i)^{-1}$ we infer norm-continuity of $f(\lambda)$ in λ. By definition, the range $\mathscr{R}(f(\lambda)^s)$ is dense in $\mathscr{H}$, for $s \geqslant 0$. Let us define the basic smooth domains

$$\mathscr{D}(\lambda)=\bigcup_{s>0} \mathscr{R}(f(\lambda)^s)$$

and

$$\mathscr{D}=\bigcup_{\lambda \in I} \mathscr{D}(\lambda). \tag{III.5}$$

By the spectral theorem, we can define the bilinear forms

$$\Delta_H(\lambda, \mu)=\frac{H(\lambda)-H(\mu)}{\lambda-\mu} \quad \text{and} \quad \Delta_Q(\lambda, \mu)=\frac{Q(\lambda)-Q(\mu)}{\lambda-\mu} \tag{III.6}$$

for $\lambda, \mu \in I$ with the domain $\mathscr{D}(\lambda) \times \mathscr{D}(\mu)$. We assume the following estimates for these difference quotients of the putative derivatives $H'(\lambda)$ and $Q'(\lambda)$:

(i) The form $\Delta_Q(\lambda, \mu)$ extends to a symmetric operator on the domain $\mathscr{D}$. For some $\eta>0$ there exists $c<\infty$ such that for, $s, s'>0$,

$$\|\Delta_Q(\lambda, \mu) f(\lambda)^s\| \leqslant c s^{-1/2+\eta}, \tag{III.7}$$

for all $\lambda, \mu \in I$.

(ii) There exists a symmetric operator $Q'(\lambda)$ with domain $\mathscr{D}$ and a form $H'(\lambda)$ with domain $\mathscr{D} \times \mathscr{D}$ such that for some $\eta>0$, and all $s, s'>0$,

$$\|(\Delta_Q(\lambda, \mu)-Q'(\lambda)) f(\lambda)^s\| \leqslant o(1)\, s^{-1/2+\eta}, \tag{III.8}$$

$$\| f(\mu)^s (\Delta_H(\lambda, \mu)-H'(\lambda)) f(\lambda)) f(\lambda)^{s'}\| \leqslant o(1)(ss')^{-1+\eta}, \tag{III.9}$$

as $\mu \to \lambda$.

An alternative form of these assumptions can be posed in terms of estimates on perturbations of the Hamiltonian. We state these stronger estimates as (i′) and (ii′), from which (i) and (ii) can be derived.

(i′) The form $\Delta_Q(\lambda, \mu)$ extends to a symmetric operator on the domain $\mathscr{D}$. For some $\eta > 0$ and $c < \infty$ the form Δ_Q^2 satisfies

$$\Delta_Q(\lambda, \mu)^2 \leqslant c(H(\lambda) + I)^{1-\eta} \tag{III.7′}$$

for all $\lambda, \mu \in I$.

(ii′) There exists a symmetric operator $Q'(\lambda)$ with domain $\mathscr{S}$ and a form $H'(\lambda)$ with domain $\mathscr{S} \times \mathscr{S}$ such that for some $\eta > 0$,

$$\|(\Delta_Q(\lambda, \mu) - Q'(\lambda))(H(\lambda) + I)^{-1/2+\eta}\| \leqslant o(1), \tag{III.8′}$$

and

$$\|(H(\mu) + I)^{-1/2+\eta}(\Delta_H(\lambda, \mu) - H'(\lambda))(H(\lambda) + I)^{-1/2+\eta}\| \leqslant o(1), \tag{III.9′}$$

as $\mu \to \lambda$.

Proposition III.1. *Under the assumptions* (i)–(ii) *above*:

(1) *For all $\lambda \in I$, τ_n^λ satisfies the growth condition, with respect to the norm* (II.1), *namely,*

$$n^{1/2} \|\tau_n^\lambda\|_* \to 0;$$

i.e.,

$$\{\tau_0^\lambda, \tau_2^\lambda, ...\} \in \mathscr{C}_+(\mathscr{A}), \qquad \{\tau_1^\lambda, \tau_3^\lambda, ...\} \in \mathscr{C}_-(\mathscr{A}).$$

(2) *Also,*

$$b\tau_n^\lambda - B\tau_{n+2}^\lambda = 0.$$

Proof. (1) To estimate the growth of τ_n^λ, we notice that (III.7) implies

$$\|\Delta_\lambda a e^{-sQ(\lambda)^2}\| \leqslant c' s^{-1/2+\eta} \|a\| \leqslant c' s^{-1/2+\eta} \|a\|_*,$$

where

$$\Delta_\lambda a = i[Q(\lambda) - Q, a].$$

Since we only consider $s = t_{k+1} - t_k \in [0, 1]$, we also have

$$\|dae^{-sQ(\lambda)^2}\| \leqslant c'' s^{-1/2+\eta} \|a\|_*.$$

Since

$$d_\lambda a = da + \Delta_\lambda a,$$

this yields

$$\| d_\lambda a e^{-sQ(\lambda)^2} \| \leqslant c s^{-1/2+\eta} \| a \|_* . \tag{III.10}$$

In particular,

$$\| d_\lambda a_j^{\Gamma^j} e^{-(t_{j+1}-t_j)Q(\lambda)^2/2} \| \leqslant c(t_{j+1}-t_j)^{-1/2+\eta} \| a_j \|_*, \qquad j=1, ..., n$$

and

$$\| a_0 e^{-t_1 Q(\lambda)^2/2} \| \leqslant c t_1^{-1/2+\eta} \| a_0 \|_* .$$

This implies the estimate on τ_n^λ,

$$| \tau_n^\lambda(a_0, ..., a_n)| \leqslant c^{n+1} \left(\prod_{j=0}^{n} \| a_j \|_* \right) \mathrm{Tr}(e^{-Q(\lambda)^2/2})\, I(\eta + \tfrac{1}{2}),$$

where

$$I(\xi) = \int_{\sigma_n} \prod_{j=0}^{n} (t_{j+1} - t_j)^{-1+\xi}\, dt = \prod_{j=1}^{n} B(\xi, j\xi). \tag{III.11}$$

Here $B(m, n)$ is the beta function defined by

$$B(m, n) = \Gamma(m)\, \Gamma(n)/\Gamma(m+n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, dt.$$

Simple estimates show that

$$I(\xi) = \frac{\Gamma(\xi)^{n+1}}{\Gamma((n+1)\,\xi)} \leqslant b' b^n (n!)^{-\xi}. \tag{III.12}$$

Thus

$$\| \tau_n^\lambda \|_* \leqslant b'(bC)^n \left(\frac{1}{n!} \right)^{1/2+\eta} \mathrm{Tr}(e^{-Q(\lambda)^2/2}).$$

As $\eta > 0$, we infer the growth condition.

(2) The proof is analoguous to the one in [5].

We have thus constructed a family of cocyles. To show they are cohomologous we prove that the derivative $d\tau^\lambda/d\lambda$ equals a coboundary ∂G^λ, where $G^\lambda \in \mathscr{C}(\mathscr{A})$. The two main steps are: (a) to show that the derivative with respect to λ exists, (b) to check the algebraic identity (III.4).

LEMMA III.2. *Under the assumptions* (III.7)–(III.9), $f(\lambda)$ *is norm-differentiable and the Duhamel formula holds*:

$$\frac{df(\lambda)}{d\lambda}=\int_0^1 f(\lambda)^s\, H'(\lambda)\, f(\lambda)^{1-s}\, ds. \tag{III.13}$$

Proof. The function $h(s)=f(\lambda)^s f(\mu)^{1-s}$ maps the unit interval to $\mathscr{L}(\mathscr{H})$ and is weakly differentiable on the open interval (0, 1). Since $h(s)$ is continuous at $s=0$, 1, we infer from the fundamental theorem of calculus that

$$\frac{f(\lambda)-f(\mu)}{\lambda-\mu}=\int_0^1 f(\lambda)^s\, \Delta_H(\lambda, \mu)\, f(\mu)^{1-s}\, ds. \tag{III.14}$$

Note that our assumptions imply that

$$\| f(\mu)^{s'}\, \Delta_H(\lambda, \mu)\, f(\lambda)^s \| \leqslant c(s's)^{-1+\eta}.$$

Together with (III.9), we see that the right side of (III.14) is uniformly bounded and uniformly convergent as $\mu \to \lambda$. Thus the norm derivative of $f(\lambda)$ exists as claimed.

LEMMA III.3. *Under the above assumptions, with* $s>0$, *the operator* $\exp(-sH(\lambda))$ *is norm-differentiable in* λ. *Also*

$$\frac{d}{d\lambda} e^{-sH(\lambda)}=\int_0^s e^{-(s-\alpha)H(\lambda)}(Q'(\lambda)\, Q(\lambda)+Q(\lambda)\, Q'(\lambda))\, e^{-\alpha H(\lambda)}\, d\alpha. \tag{III.15}$$

Proof. The differentiability of $\exp(-sH(\lambda))$ follows from Lemma III.2 (with sH replacing H) and its derivative equals $\int_0^s f(\lambda)^{s-\alpha}\, H'(\lambda)\, f(\lambda)^{\alpha}\, d\alpha$. The difference quotient $\Delta_H(\lambda, \mu)$ in (III.14) can be written as

$$\Delta_H(\lambda, \mu)=Q(\lambda)\, \Delta_Q(\lambda, \mu)+\Delta_Q(\lambda, \mu)\, Q(\mu). \tag{III.16}$$

The estimates (III.7–(III.8) ensure uniform convergence of (III.14) to (III.15), as $\mu \to \lambda$. Here we also use the fact that norm continuity of $(Q(\lambda)\pm i)^{-1}$ in λ ensures the norm continuity of the bounded function $Q(\lambda)\exp(-sQ(\lambda)^2)$.

We are now ready to establish the existence of $d\tau^{\lambda}/d\lambda$. Let us introduce the notation

$$d'_{\lambda} a=i[Q'(\lambda), a] \tag{III.17}$$

and

$$H'=Q'Q+QQ'=-id_{\lambda} Q'. \tag{III.18}$$

PROPOSITION III.4. *Under the assumptions* (i)–(ii) *abive, τ^λ is differentiable in λ, $d\tau^\lambda/d\lambda \in \mathscr{C}(\mathscr{A})$, and*

$$\frac{d\tau_n^\lambda}{d\lambda}(a_0, ..., a_n)$$

$$= (-1)^{[n/2]} \sum_{j=1}^{n} F_n^\lambda(a_0, d_\lambda a_1^\Gamma, ..., d'_\lambda a_j^{\Gamma^j}, ..., d_\lambda a_n^{\Gamma^n})$$

$$+ (-1)^{[n/2]} \sum_{j=0}^{n} F_{n+1}^\lambda(a_0, d_\lambda a_1^\Gamma, ..., d_\lambda a_j^{\Gamma^j}, id_\lambda Q', ..., d_\lambda a_n^{\Gamma^n}). \qquad \text{(III.19)}$$

Proof. We claim that the functional F_{n+1}^λ extends from $\mathscr{C}^{n+1}(\mathscr{A})$ to a functional on $\mathscr{A}^j \times H' \times \mathscr{A}^{n+1-j}$ and defines an element of $\mathscr{C}^n(\mathscr{A})$. In fact, as a consequence of our fundamental assumptions (i) and (ii) we have

$$\| e^{-sQ(\lambda)^2/2}\, d_\lambda Q'(\lambda)\, e^{-s'Q(\lambda)^2/2} \| \leqslant c(s's)^{-1+\eta}. \qquad \text{(III.20)}$$

Now we proceed as in the proof of Proposition IV.2 of [5], using half of the $e^{-Q(\lambda)^2(t_{k+1}-t_k)}$ factors to control $d_\lambda Q'(\lambda)$, namely,

$$\begin{aligned}
&| F_{n+1}^\lambda(a_0, a_1, ..., a_j, id_\lambda Q'(\lambda), a_{j+1}, ..., a_n)| \\
&\quad \leqslant \left(\prod_{j=0}^{n} \|a_j\|\right) \operatorname{Tr}(e^{-Q(\lambda)^2/2}) \\
&\quad\quad \times \int_{\sigma_{n+1}} \| e^{-(t_{j+1}-t_j)Q(\lambda)^2/2} d_\lambda Q'(\lambda)\, e^{-(t_{j+2}-t_{j+1})Q(\lambda)^2/2} \|\, dt \\
&\quad \leqslant \left(\prod_{j=0}^{n} \|a_j\|\right) \operatorname{Tr}(e^{-Q(\lambda)^2/2})\, c\, \frac{\Gamma(\eta)^2}{\Gamma(n+2\eta)} \\
&\quad \leqslant \frac{C}{n!} \left(\prod_{j=0}^{n} \|a_j\|\right) \operatorname{Tr}(e^{-Q(\lambda)^2/2}).
\end{aligned} \qquad \text{(III.21)}$$

The desired continuity follows.

Now we show that $d\tau^\lambda/d\lambda \in \mathscr{C}(\mathscr{A})$. First note that (III.19) holds as an identity; this is a consequence of differentiating (III.3). Lemmas III.2 and III.3 justify the calculation. From (III.7)–(III.8) we infer that

$$\| Q'(\lambda)\, e^{-sQ(\lambda)^2} \| \leqslant cs^{-1/2+\eta},$$

and thus

$$\| d'_\lambda a \| \leqslant cs^{-1/2+\eta} \| a \|_*. \qquad \text{(III.22)}$$

Using half of the heat kernel factors $e^{-Q(\lambda)^2(t_{k+1}-t_k)}$ to bound $d_\lambda a_j$ and $d'_\lambda a_j$, we obtain the estimate

$$\begin{aligned}
&|F_n^\lambda(a_0, d_\lambda a_1^{\Gamma}, ..., d'_\lambda a_j^{\Gamma^j}, ..., d_\lambda a_n^{\Gamma^n})| \\
&\quad\leqslant c^n\left(\prod_{j=0}^{n} \|a_j\|_*\right)\mathrm{Tr}(e^{-Q(\lambda)^2/2})\, I\left(\frac{1}{2}+\eta\right) \\
&\quad\leqslant b'(bc)^n\left(\prod_{j=0}^{n} \|a_j\|_*\right)\mathrm{Tr}(e^{-Q(\lambda)^2/2})\left(\frac{1}{n!}\right)^{1/2+\eta}
\end{aligned}$$

Thus the growth condition holds for the first sum in (III.19).

Now let us estimate the terms in the second sum. The term $id_\lambda Q'(\lambda)$ needs special care. Note $id_\lambda Q'(\lambda) = -(Q'Q+QQ')$, and

$$\|e^{-s'Q(\lambda)^2/4}Q'Qe^{-sQ(\lambda)^2/4}\| \leqslant cs^{-1+\eta} = cs^{-1/2+\eta}\cdot s^{-1/2}$$

and

$$\|e^{-s'Q(\lambda)^2/4}QQ'e^{-sQ(\lambda)^2/4}\| \leqslant cs'^{-1+\eta}s^{-1/2+\eta}.$$

This and the inequality

$$\|dae^{-sQ(\lambda)^2/4}\| \leqslant cs^{-1/2+\eta}\,\|a\|_*$$

imply that terms in the second sum in (III.19) can be bounded by

$$\begin{aligned}
&c^n\left(\prod_{j=0}^{n} \|a_j\|_*\right)\mathrm{Tr}(e^{-Q(\lambda)^2/2}) \\
&\quad\cdot\int_{\sigma_n}\prod_{j=0}^{n}(t_{j+1}-t_j)^{-1/2+\eta}\cdot(t_{k+1}-t_k)^{-1/2}\,dt, \qquad \text{(III.23)}
\end{aligned}$$

where the extra factor $(t_{k+1}-t_k)^{-1/2}$ comes from the above estimates on $(Q'Q+QQ')$. The integral in (III.23) is

$$\frac{\Gamma(\eta+1/2)^n\,\Gamma(\eta)}{\Gamma((n+1)(\eta+1/2)-1/2)} \leqslant b'b^n\left(\frac{1}{n!}\right)^{1/2+\eta}.$$

This implies the growth condition. The proof is complete.

Now we construct the coboundary. Define

$$\begin{aligned}
G_{n-1}^\lambda(a_0, ..., a_{n-1}) = &-(-1)^{[n/2]}\sum_{j=0}^{n-1}(-1)^j \\
&\times F_n^\lambda(a_0, d_\lambda a_1^{\Gamma}, ..., d_\lambda a_j^{\Gamma^j}, iQ', d_\lambda a_{j+1}^{\Gamma^{j+2}}, ..., d_\lambda a_{n-1}^{\Gamma^n}).
\end{aligned} \qquad \text{(III.24)}$$

PROPOSITION III.5. (1) *The functional G^λ is an element of $\mathscr{C}(\mathscr{A})$.*

(2) *It satisfies*

$$(\partial G^\lambda)_n (a_0, ..., a_n)$$
$$= (-1)^{[n/2]} \sum_{j=0}^{n} F_n^\lambda(a_0, d_\lambda a_1^\Gamma, ..., d'_\lambda a_j^{\Gamma^j}, ..., d_\lambda a_n^{\Gamma^n})$$
$$+ (-1)^{[n/2]} \sum_{j=0}^{n} F_{n+1}^\lambda(a_0, d_\lambda a_1^\Gamma, ..., d_\lambda a_j^{\Gamma^j}, id_\lambda Q', d_\lambda a_{j+1}^{\Gamma^{j+1}}, ..., d_\lambda a_n^{\Gamma^n}). \tag{III.25}$$

Proof. Part (1) is proved in the same way as Proposition III.4. Part (2) is a consequence of the identities

$$(bG_{n-1}^\lambda)(a_0, ..., a_n)$$
$$= (-1)^{[n/2]} \sum_{j=1}^{n} F_n^\lambda(a_0, d_\lambda a_1^\Gamma, ..., d_\lambda a_{j-1}^{\Gamma^{j-1}}, d'_\lambda a_j^{\Gamma^j}, d_\lambda a_{j+1}^{\Gamma^{j+1}}, ..., d_\lambda a_n^{\Gamma^n})$$
$$+ (-1)^{[n/2]} \sum_{j=0}^{n} F_{n+1}^\lambda(a_0, d_\lambda a_1^\Gamma, ..., d_\lambda a_j^{\Gamma^j}, id_\lambda Q', d_\lambda a_{j+1}^{\Gamma^{j+1}}, ..., d_\lambda a_n^{\Gamma^n})$$
$$+ (-1)^{[n/2]} \sum_{j=0}^{n} (-1)^j F_{n+1}^\lambda(d_\lambda a_0^\Gamma, ..., d_\lambda a_j^{\Gamma^{j-1}}, iQ', d_\lambda a_{j+1}^{\Gamma^{j+1}}, ..., d_\lambda a_n^{\Gamma^n}), \tag{III.26}$$

and

$$(BG_{n+1}^\lambda)(a_0, ..., a_n)$$
$$= -(-1)^{[n/2]} \sum_{j=0}^{n} (-1)^j$$
$$\times F_{n+1}^\lambda(d_\lambda a_0^\Gamma, ..., d_\lambda a_j^{\Gamma^{j+1}}, iQ', d_\lambda a_{j+1}^{\Gamma^{j+1}}, ..., d_\lambda a_n^{\Gamma^n}). \tag{III.27}$$

These identities are established by using Propositions (IV.4) and (IV.6) of [5], and we omit the details.

We summarize the above results as

THEOREM III.6. *Under assumptions* (i) *and* (ii) *above, the function $\tau^\lambda : I \to \mathscr{C}(\mathscr{A})$ is continuously differentiable in λ and*

$$\frac{d\tau^\lambda}{d\lambda} = \partial G^\lambda, \tag{III.28}$$

where G^λ is defined by (III.24). *As a consequence, the family τ^λ of cocycles is cohomologous.*

Remarks. 1. Our method actually requires that the family τ^λ of deformations is differentiable. Presumably one can generalize the result by other methods to cases where the family $\lambda \to \tau^\lambda$ describes topological (continuous) deformations, which are not differentiable. Certain special cases of this result are well understood. The index map

$$\mathrm{Ind}(Q(\lambda)_+) = \tau_0^\lambda(1)$$

is constant if the resolvent map $(Q(\lambda) \pm i)^{-1}$ is norm continuous in λ. In that case, for any $G \in \mathscr{C}(\mathscr{A})$,

$$\frac{d}{d\lambda}\tau_0^\lambda(1) = 0 = G_1(\mathbf{1}, \mathbf{1}) - G_1(\mathbf{1}, \mathbf{1}) = (BG)_0\,(\mathbf{1}) = (\partial G)_0(\mathbf{1}).$$

2. The family τ^β of cocycles defined in [5] by

$$\tau_{2n}^\beta(a_0, ..., a_{2n})$$
$$= \beta^{-n}\,\mathrm{Str}\left(\int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_{2n} \leqslant \beta} (a_0 F\, da_1(t_1)\, \Gamma\, da_2(t_2) \cdots \Gamma\, da_{2n}(t_{2n})\, e^{-\beta Q^2}\, dt\right)$$

is cohomologous for $\beta \in (0, \infty)$. By scaling, this reduces to a special case of the study of (III.1) with $Q(\beta) = \beta^{1/2}Q$. Note that this deformation does not satisfy assumptions (i) and (ii).

3. The case $Q(\lambda) = Q + B(\lambda)$ where $\lambda \in I$, and $B'(\lambda)$ is a bounded transformation on $\mathscr{H}$ has been studied in [4].

IV. Extension of the Character τ to Limiting Elements

In [5] we assumed that the algebra $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is invariant under d. We then constructed the Chern character τ, so that τ_n satisfies the continuity estimate

$$|\tau_n(a_0, ..., a_n)| \leqslant (n!)^{-1}\left(\prod_{j=0}^{n} \|a_j\|_*\right)\mathrm{Tr}(e^{-H}). \tag{IV.1}$$

Here we extend the definition of τ. We define a Banach space $\mathscr{A}_\eta$ which is the completion of $\mathscr{A}$ in a norm $\|\cdot\|_\eta$ such that

$$\|a\|_\eta \leqslant \|a\|_*, \qquad a \in \mathscr{A}. \tag{IV.2}$$

We show that with some constants b, c,

$$|\tau_n(a_0, ..., a_n)| \leqslant bc^n(n!)^{-1+2\eta} \left(\prod_{j=0}^{n} \|a_j\|_\eta \right) \mathrm{Tr}(e^{-H/3}). \tag{IV.3}$$

It follows that τ_n extends as a continuous, multilinear functional on $\mathscr{A}_\eta^{n+1}$, and that $\tau = \{\tau_n\}$ satisfies the entire growth condition with respect to the norm $\|\cdot\|_\eta$. Elements of $\mathscr{A}_\eta$ need not be bounded operators on $\mathscr{H}$. This extension will be helpful in related works pertaining to working out examples of this construction.

In order to define our new norm, we introduce a real parameter $\eta \in [0, \frac{1}{4})$. Let

$$\|a\|_\eta = \sup_{s,\, t \in (0,\, 1)} (st)^\eta \, \|e^{-sH} a e^{-tH}\|_*. \tag{IV.4}$$

Thus our new norm measures the rate of divergence of the heat kernel regularization of a and da as s, $t \to 0$. Clearly,

$$\begin{aligned} \|a\|_\eta &= \sup_{s,\, t \in (0,\, 1)} (st)^\eta \, (\|e^{-sH} a e^{-tH}\| + \|e^{-sH} da e^{-tH}\|) \\ &\leqslant \|a\| + \|da\| = \|a\|_*, \end{aligned} \tag{IV.5}$$

so (IV.2) holds. Also, the special choice $\eta = 0$ yields $\|a\|_\eta = \|a\|_*$, so we are interested in the case $\eta \in (0, \frac{1}{4})$, for which

$$\|e^{-sH} a e^{-tH}\|_* \leqslant (st)^{-\eta} \, \|a\|_\eta, \tag{IV.6}$$

for all s, $t \in (0, 1)$.

We remark that a sufficient condition that a sequence $a_n \in \mathscr{A}$ of operators converge in $\mathscr{A}_\eta$ is the $\|\cdot\|_*$-norm convergence of the sequence

$$\hat{a}_n \equiv (H+I)^{-\eta} \, a_n (H+I)^{-\eta}. \tag{IV.7}$$

It follows that for s, $t \in (0, 1)$,

$$\|e^{-sH}(a_n - a_m) \, e^{-tH}\|_* \leqslant \mathrm{const}(st)^{-\eta} \, \|(H+I)^{-\eta} \, (a_n - a_m)(H+I)^{-\eta}\|_*.$$

Thus convergence of the $\hat{a}_n$ ensures that $\{a_n\}$ is a Cauchy sequence in $\mathscr{A}_\eta$.

Our main result is:

THEOREM IV.1. *Given $\eta \in (0, \frac{1}{4})$, the corresponding Chern character τ on $\mathscr{C}(\mathscr{A})$ given by* (1.7) *satisfies* (IV.3), *where b and c are independent of η.*

Remark. It follows that τ is an entire cocycle in the topology determined by $\|\cdot\|_\eta$. In particular, τ_n extends continuously to $\mathscr{A}_\eta^{n+1}$, which can be used in the computation of invariants. For the case $\eta = \frac{1}{4}$ we obtain a cocycle with a finite radius of convergence.

Proof. We estimate $\tau_n(a_0, ..., a_n)$ using the following method: each factor $\exp(-(t_{j+1} - t_j) H)$ is divided into three pieces of the form $\exp(-(t_{j+1} - t_j) H/3)$. One factor is used to estimate the operator a_{j-1} on its left. The second factor remains under the trace. The third factor is used to estimate the operator a_j on its right. (For $j = n$, take $a_{j+1} = a_0$, $t_{j+1} = 1$, while for $j = 0$ take $t_j = 0$.) We apply Hölder's inequality to $\mathrm{Tr}(\Gamma^{n+1} a_0 da_1(t_1) \cdots da_n(t_n) e^{-H})$, estimating the operators with I_p norms. Use the I_∞ norm on Γ and on the factors $E^{-(t_{j+1} - t_j)H/3} da_{j+1} e^{-(t_{j+2} - t_{j+1})H/3}$, for $j = 0, 1, ..., n-1$, use the I_∞ norm also on $e^{-(1-t_n)H/3} a_0 e^{-t_1 H/3}$, and use the I_{p_j} norms on the "middle" factor $\exp(-(t_{j+1} - t_j) H/3)$ with $p_j = (t_{j+1} - t_j)^{-1}$ and $j = 0, ..., n$. Since $\sum_0^n p_j^{-1} = 1$, we have for $0 < t_1 < \cdots < t_n < 1$,

$$\begin{aligned} &|\mathrm{Tr}(\Gamma^{n+1} a_0 da_1(t_1) \cdots da_n(t_n) e^{-H})| \\ &\quad \leqslant \mathrm{Tr}(e^{-H/3}) \left(\prod_{j=0}^{n} \|a_j\|_\eta \right) \int_{\sigma_n} \prod_{j=0}^{n} (t_{j+1} - t_j)^{-2\eta} \, dt. \end{aligned} \tag{IV.8}$$

Using the bound (III.12) completes the proof.

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