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Quantum Riemann Surfaces: II. The Discrete Series*

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Abstract. We continue our study of noncommutative deformations of two-dimensional hyperbolic manifolds which we initiated in Part I. We construct a sequence of $\mathbb{C}^*$-algebras which are quantizations of a compact Riemann surface of genus g corresponding to special values of the Planck constant. These algebras are direct integrals of finite-dimensional $\mathbb{C}^*$-algebras.

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1. Introduction

In this paper we continue our study of quantum deformations of Riemann surfaces [4, 5]. Quantum Riemann surfaces can be thought of as nontrivial, noncommutative manifolds of lowest dimension. In [4], we constructed a one-parameter quantum deformation of the universal covering U of hyperbolic Riemann surfaces. To be more specific, we constructed a family $C_\mu(\bar{U})$ of $\mathbb{C}^*$-algebras, where $0 < \mu < 1$ is a deformation parameter. These $\mathbb{C}^*$-algebras can be thought of as the $\mathbb{C}^*$-algebras of 'continuous functions having limits at infinity' on the quantum unit disc. We have shown that $C_\mu(\bar{U})$ is isomorphic with the $\mathbb{C}^*$-algebra $\mathscr{T}'(\bar{U})$ generated by a class of Toeplitz operators. It can be shown that these Toeplitz operators arise naturally in the process of geometric quantization of U [2, 9, 10].

Recall that a quantum deformation (see [4] for relevant references) of a manifold M is a fibration $(\mathscr{A}, \pi, S)$ over a parameter space S with a fixed base point O. Each fiber $\mathscr{A}_s = \pi^{-1}(s)$ is a $\mathbb{C}^*$-algebra with the property that $\mathscr{A}_O = C(M)$, a $\mathbb{C}^*$-algebra of continuous functions on M. A deformation map D is a connection in $(\mathscr{A}, \pi, S)$. If $\gamma : [0, 1] \to S$ is a curve in S starting at O, then by $D_{\gamma(t)}(f)$ we denote its lift to $\mathscr{A}$ starting at $f \in C(M)$. The crucial requirement is that for f, g

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smooth the limit

$$\lim_{t\to 0}\frac{1}{t}(D_{\gamma(t)}(f)D_{\gamma(t)}(g)-D_{\gamma(t)}(fg)) \tag{1.1}$$

exists. As a consequence,

$$\{f,g\}_\gamma := \lim_{t\to 0}\frac{1}{t}[D_{\gamma(t)}(f), D_{\gamma(t)}(g)] \tag{1.2}$$

is a Poisson bracket on M. One often refers to $\{\cdot,\cdot\}_\gamma$ as a direction of deformation. In the situation of [4], $S=[0,1)$, $O=0$ and the deformation map is given by the Toeplitz operator $D_t(f)=T^r(f)$, where $r=1+t^{-1}$.

Let now $S=U/\Gamma$ be a compact Riemann surface arising as the quotient of U by a Fuchsian group Γ. Consider the family of $\mathbb{C}^*$-algebras $\{\mathscr{T}_\Gamma^r(U)\}$ of Toeplitz operators with symbols invariant under Γ. This family is a quantum deformation of the algebra of continuous functions on S. In this Letter, we study the structure of $\mathscr{T}_\Gamma^r(U)$ for the discrete sequence of values of the deformation parameter, $r=(2g-2)^{-1}n$, where g is the genus of S, and where $n=1,2,\ldots$. The algebra $\mathscr{T}_\Gamma^r(U)$ is a direct integral of finite-dimensional $\mathbb{C}^*$-algebras. These finite-dimensional $\mathbb{C}^*$-algebras are the full matrix algebras on a family of Hilbert spaces of automorphic forms. This property is a unique feature of the discrete series of values of the deformation parameter. In fact, the values $r=(2g-2)^{-1}n$ are precisely the values of deformation parameter obtained for geometric quantization.

The paper is organized as follows. In Section 2 we summarize our results and fix notation. In Section 3 we review the technique of geometric quantization as applied to compact Riemann surfaces. Sections 4 and 5 contain the proofs of our main results, Theorems A and B of Section 2. The proofs involve rather detailed estimates on the r-dependence of automorphic forms of weight r.

2. Main Results

Let S be a compact Riemann surface of genus $g\geqslant 2$ and let Γ_0 be a Fuchsian group uniformizing it. In other words, Γ_0 is a discrete subgroup of $\mathrm{PSU}(1,1)$ acting fix point free on the unit disc U, and $S\cong U/\Gamma_0$. By $\Gamma\subset\mathrm{SU}(1,1)$, we denote a subgroup of $\mathrm{SU}(1,1)$ which covers Γ_0. Let $\pi:\Gamma\to\Gamma_0$ denote the covering homomorphism. For

$$\gamma=\begin{pmatrix} a & b\\ \bar b & \bar a\end{pmatrix}\in\Gamma$$

we denote

$$\gamma(\zeta)=\frac{a\zeta+b}{\bar b\zeta+\bar a},\quad \zeta\in U. \tag{2.1}$$

Then the derivative of (2.1) is $\gamma'(\zeta) = (\bar{b}\zeta + \bar{a})^{-2}$. We set

$$\gamma'(\zeta)^{1/2} := (\bar{b}\zeta + \bar{a})^{-1}. \tag{2.2}$$

Let $r > 0$. Recall (see, e.g. [6]) that $v : \Gamma \to \mathbb{C}$ is a multiplier for the group Γ and the weight r, if

$$|v(\gamma)| = 1, \tag{2.3}$$

$$\gamma_1'(\gamma_2(\zeta))^{-r/2}\gamma_2'(\zeta)^{-r/2}v(\gamma_1)v(\gamma_2) = (\gamma_1\gamma_2)'(\zeta)^{-r/2}v(\gamma_1\gamma_2). \tag{2.4}$$

Here $\gamma'(\zeta)^{-r/2}$ is defined as $\exp\{-r \log \gamma'(\zeta)^{1/2}\}$ and $\log \alpha := \log|\alpha| + i \arg \alpha$ where $-\pi < \arg \alpha \leqslant \pi$. It is well known [7] that a multiplier exists if and only if $r = n(2g-2)^{-1}$, $n = 1, 2, \ldots$. If v_1 and v_2 are multipliers then, as a consequence of (2.4), their ratio is a character of Γ. As a consequence, it is natural to identify the set $M(\Gamma, r)$ of multipliers for Γ and r with the set $\hat{\Gamma}$ of characters of Γ. Since Γ has the presentation $A_1B_1A_1^{-1}B_1^{-1} \cdots A_gB_gA_g^{-1}B_g^{-1} = \pm I$, $\hat{\Gamma}$ is a $2g$-dimensional torus which we normalize so that it can be identified with $\mathrm{Jac}(S)$, the Jacobian of S. In other words, as sets, $M(\Gamma, r) \cong \mathrm{Jac}(S)$.

Recall that a holomorphic function $\phi : U \to \mathbb{C}$ is called an automorphic form of weight $r > 0$ and multiplier v if

$$\phi(\gamma(\zeta)) = v(\gamma)\gamma'(\zeta)^{-r/2}\phi(\zeta), \tag{2.5}$$

for each $\gamma \in \Gamma$. Let $\mathscr{H}^r(\Gamma, v)$ denote the complex vector space of automorphic forms of weight r and multiplier v. There is a natural definition of an inner product on $\mathscr{H}^r(\Gamma, v)$ due to Petersson [8]. For $\phi, \psi \in \mathscr{H}^r(\Gamma, v)$ we set

$$(\phi, \psi) := \int_D \overline{\phi(\zeta)}\psi(\zeta)\, \mathrm{d}\mu^r(\zeta), \tag{2.6}$$

where D is a fundamental domain of Γ, and where

$$\mathrm{d}\mu^r(\zeta) := \frac{r-1}{\pi}(1 - |\zeta|^2)^{r-2}\, \mathrm{d}^2\zeta. \tag{2.7}$$

As a consequence of (2.5), (ϕ, ψ) is independent of the choice of D. For the future convenience we will assume that $0 \in D$. The space $\mathscr{H}^r(\Gamma, v)$ thus becomes a (finite-dimensional) Hilbert space.

For $\phi : U \to \mathbb{C}$ holomorphic and bounded we set

$$\Theta_v^r\phi(\zeta) := \sum_{\gamma \in \Gamma} v(\gamma)^{-1}\gamma'(\zeta)^{r/2}\phi(\gamma(\zeta)). \tag{2.8}$$

Classical theorems going back to Poincaré (see, e.g., [6]) state that the series (2.8) converges almost uniformly in U and that $\Theta_v^r\phi$ is an automorphic form (whenever there is no danger of confusion, we will write $\Theta\phi$ rather than $\Theta_v^r\phi$). The series (2.8) is called the Poincaré theta series of ϕ.

Of particular importance will be the following theta series:

$$K^r_{\Gamma,v}(\zeta,\eta) := \sum_{\gamma\in\Gamma} v(\gamma)^{-1}\gamma'(\zeta)^{r/2}K^r(\gamma(\zeta),\eta), \quad \eta\in U, \tag{2.9}$$

where

$$K^r(\zeta,\eta) := (1-\zeta\bar{\eta})^{-r}. \tag{2.10}$$

Recall that $K^r(\zeta,\eta)$ is the reproducing kernel for the Hilbert space $\mathscr{H}^r$ of functions holomorphic in U and square integrable with respect to $\mathrm{d}\mu^r$. This implies that $K^r_{\Gamma,v}(\zeta,\eta)$ is the reproducing kernel for $\mathscr{H}^r(\Gamma,v)$. Furthermore,

$$K^r_{\Gamma,v}(\zeta,\eta)^* = K^r_{\Gamma,v}(\eta,\zeta). \tag{2.11}$$

Let $L^2_{\Gamma,v}(D,\mathrm{d}\mu^r)$ denote the Hilbert space of functions on U satisfying (2.5) and square integrable on D with respect to $\mathrm{d}\mu^r$. Then, $K^r_{\Gamma,v}(\zeta,\eta)$ is the integral kernel of the orthogonal projection $P\colon L^2_{\Gamma,v}(D,\mathrm{d}\mu^r)\to\mathscr{H}^r(\Gamma,v)$,

$$(P\phi)(\zeta) = \int_D K^r_{\Gamma,v}(\zeta,\eta)\phi(\eta)\,\mathrm{d}\mu^r(\eta). \tag{2.12}$$

Let $C_\Gamma(U)$ denote the Banach space of bounded continuous functions on U which are invariant under Γ. For $f\in C_\Gamma(U)$ we define $T^r_{\Gamma,v}(f) := PM(f)$, where $M(f)$ denotes multiplication by f, or explicitly

$$(T^r_{\Gamma,v}(f)\phi)(\zeta) = \int_D K^r_{\Gamma,v}(\zeta,\eta)f(\eta)\phi(\eta)\,\mathrm{d}\mu^r(\eta). \tag{2.13}$$

Then $T^r_{\Gamma,v}(f)$ is a linear operator on $\mathscr{H}^r(\Gamma,v)$ called the Toeplitz operator with symbol f. As $\dim\mathscr{H}^r(\Gamma,v)<\infty$, the $\mathbb{C}^*$-algebra generated by all Toeplitz operators is the full operator algebra $\mathrm{End}(\mathscr{H}^r(\Gamma,v))$.

In Section 3 we will explain how the spaces $\mathscr{H}^r(\Gamma,v)$ arise as a result of geometric quantization of S. Our main concern in this Letter is to study the properties of the mapping $C(S)\ni f\to T^r_{\Gamma,v}(f)$ (we have identified a continuous function f with its lift to a Γ-invariant function on U). Our first main result is

THEOREM A. *Let $f\in C_\Gamma(U)$. Then*

$$\|T^r_{\Gamma,v}(f)\| \leqslant \|f\|_\infty \leqslant \|T^r_{\Gamma,v}(f)\| + o(1), \quad \textit{as } r\to\infty, \tag{2.14}$$

where $\|f\|_\infty$ is the sup-norm of f, and where $\|T^r_{\Gamma,v}(f)\|$ is the operator norm on $\mathscr{H}^r(\Gamma,v)$. In particular,

$$\lim_{r\to\infty}\|T^r_{\Gamma,v}(f)\| = \|f\|_\infty. \tag{2.15}$$

Let $C^\infty_\Gamma(U)$ denote the Fréchet space of smooth Γ-invariant functions on U such that

$$\|f\|_{p,\infty} := \sum_{j+k\leqslant p}\sup_{\zeta\in D}|\partial^j\bar{\partial}^k f(\zeta)| < \infty \quad (\text{in particular, } \|f\|_\infty = \|f\|_{0,\infty}).$$

Our second main result is

THEOREM B. *Let $f, g \in C^\infty_\Gamma(U)$. Then, for r sufficiently large,*

$$\begin{aligned}&\| r(T^r_{\Gamma,v}(f)T^r_{\Gamma,v}(g) - T^r_{\Gamma,v}(fg)) + T^r_{\Gamma,v}((1-|\zeta|^2)^2\,\partial f\,\bar\partial g)\| \\ &\quad \leqslant Cr^{-1/2}\|f\|_{4,\infty}\|g\|_{4,\infty},\end{aligned} \tag{2.16}$$

where C is a constant.

The above theorem states that the algebra (of operators on the finite-dimensional Hilbert space $\mathscr{H}^r(\Gamma, v)$) generated by all $T^r_{\Gamma,v}(f)$ with smooth f is a deformation of the algebra of smooth functions on S in the sense explained in the Introduction. The deformed product is given by $f \cdot_r g = fh - (1/r)(1-|\zeta|^2)^2\,\partial f\,\bar\partial g + \cdots$. Furthermore, the corollary below states that this algebra is a deformation of $C^\infty(S)$ regarded as the Poisson algebra with the Poisson structure induced by the Poincaré symplectic form,

$$\{f, g\} := i(1-|\zeta|^2)^2[\partial f(\zeta)\,\bar\partial g(\zeta) - \bar\partial f(\zeta)\,\partial g(\zeta)]. \tag{2.16}$$

COROLLARY TO THEOREM B. *Let $f, g \in C^\infty_\Gamma(U)$. Then*

$$\left\| \frac{r}{i}[T^r_{\Gamma,v}(f), T^r_{\Gamma,v}(g)] - T^r_{\Gamma,v}(\{f, g\}) \right\| \leqslant Cr^{-1/2}\|f\|_{4,\infty}\|g\|_{4,\infty}, \tag{2.17}$$

for r sufficiently large.

We will prove Theorems A and B in Sections 4 and 5, respectively.

3. Geometric Quantization

In this section, we show that the Hilbert spaces $\mathscr{H}^r(\Gamma, v)$ introduced in Section 2 arise naturally in the process of geometric quantization (see, e.g., [2, 9, 10]) of S.

We choose the Poincaré Hermitian metric on the universal covering U of S, $\mathrm{d}s^2 = (1-|\zeta|^2)^2\,\mathrm{d}\zeta \otimes \mathrm{d}\bar\zeta$. Its curvature,

$$\omega := -2(1-|\zeta|^2)^2\,\mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta, \tag{3.1}$$

coincides with the Poincaré symplectic form on U. Let $L \to S$ be a holomorphic line bundle over S whose Chern number is $n \in \mathbb{Z}$. The space of sections of L can naturally be identified with the space of smooth functions $s\colon U \to \mathbb{C}$ such that for $\gamma \in \Gamma$,

$$\gamma^* s(\zeta) = v(\gamma)\gamma'(\zeta)^{\lambda/2}s(\zeta), \tag{3.2}$$

where v is a multiplier of Γ and where $\lambda = (2g-2)^{-1}n$.

Let $h \in \mathbb{R}$. The first step in geometric quantization is to construct a holomorphic line bundle $L \to S$ such that $c_1(L) = (i/2\pi)h^{-1}\omega \in H^2(S, \mathbb{Z})$. Since $\int_D \omega = 2\pi i(2g-2)$, it follows that such an L

exists if and only if

$$h = \lambda^{-1} = (2g-2)/n. \tag{3.3}$$

This is the quantization condition of geometric quantization. Hence, for each h given by (3.3), we obtain a family $\{L_{h,v}\}$ of holomorphic line bundles parametrized by the set of multipliers of Γ.

The second step is to construct a complex connection ∇ in $L_{h,v}$ such that $\mathrm{curv}(\nabla) = h^{-1}\omega$. Writing $\nabla = \mathrm{d} + \alpha(\zeta)\,\mathrm{d}\zeta$ with α such that

$$\gamma^*\alpha(\zeta) = \gamma'(\zeta)^{-1/2}\alpha(\zeta) + h^{-1}\gamma'(\zeta)^{-1}\gamma''(\zeta),$$

and using $\mathrm{curv}(\nabla) = -\partial/\partial\bar{\zeta}\alpha(\zeta)\,\mathrm{d}\zeta \wedge \mathrm{d}\zeta$, we find that

$$\alpha(\zeta) = h^{-1}\frac{\bar{\zeta}}{1-|\zeta|^2}. \tag{3.4}$$

Next, we find a ∇-invariant hermitian structure $\langle\cdot,\cdot\rangle$ on $L_{h,v}$. In other words, we require that $\langle\cdot,\cdot\rangle$ satisfies

$$X\langle s,t\rangle(\zeta) = \langle\nabla_X s, t\rangle(\zeta) + \langle s, \nabla_X t\rangle(\zeta), \tag{3.5}$$

for each real vector field X. Writing $\langle s,t\rangle(\zeta) = \chi(\zeta)\overline{s(\zeta)}t(\zeta)$ with $\chi(\zeta) > 0$ we find that (3.6) is equivalent to

$$\frac{\partial}{\partial\zeta}\chi(\zeta) = \alpha(\zeta)\chi(\zeta), \qquad \frac{\partial}{\partial\bar{\zeta}}\chi(\zeta) = \overline{\alpha(\zeta)}\chi(\zeta).$$

Solving this system of equations yields $\chi(\zeta) = C(1-|\zeta|^2)^{1/h}$, where $C > 0$. As a consequence, $\langle s,t\rangle(\zeta) = C(1-|\zeta|^2)^{1/h}\overline{s(\zeta)}t(\zeta)$.

The next step of geometric quantization is to construct the prequantum Hilbert space. This is the completion of $\Gamma(L_{h,v})$, the space of sections of $L_{h,v}$, in the norm induced by the following inner product:

$$(s,t) := \int_D \langle s,t\rangle(\zeta)\,\mathrm{d}\mu_0(\zeta), \tag{3.6}$$

where $\mathrm{d}\mu_0(\zeta) = 2(1-|\zeta|^2)^{-2}\,\mathrm{d}^2\zeta$ is the Poincaré measure.

The final step of geometric quantization is to choose a polarization on S. Clearly, S has a Kähler polarization, namely the vector field $\partial/\partial\bar{\zeta}$. The corresponding subspace of the prequantum Hilbert space consists of holomorphic sections of $L_{h,v}$ and coincides with $H^{1/h}(\Gamma, v)$ if $0 < h < 1$, and if we choose the constant C in $\chi(\zeta)$ to be $(2\pi)^{-1}(h^{-1}-1)$.

4. Proof of Theorem A

The proof of Theorem A is based on two technical lemmas which we now formulate. For $\eta \in U$ we set

$$\gamma_\eta := (1-|\eta|^2)^{-1/2}\begin{pmatrix} 1 & \eta \\ \bar{\eta} & 1 \end{pmatrix} \in \mathrm{SU}(1,1). \tag{4.1}$$

The lemmas are concerned with the properties of the series

$$\sum_{\gamma\in\Gamma} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)|^{r/2}, \quad \zeta \in U,$$

as η varies over a compact subset $K \subset U$ and $r \to \infty$. They improve slightly on the classical results, see e.g. [3, 6].

LEMMA 4.1. *Let $K \subset U$ be compact. There exist constants $A > 0$, $C > 1$, $\delta > 0$ and $r_0 > 2$ such that*

$$\sum_{\Gamma \ni \gamma \neq \pm I} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)|^{r/2} \leqslant AC^{-r}, \tag{4.2}$$

for $\eta \in K$, $|\zeta| \leqslant \delta$ and $r \geqslant r_0$.

LEMMA 4.2. *There exist constants $B > 0$ and $r_0 > 2$ such that*

$$\sum_{\gamma\in\Gamma} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)|^{r/2} \leqslant Br(1-|\zeta|^2)^{-r/2}, \tag{4.3}$$

for $\eta \in U$, $\zeta \in U$ and $r \geqslant r_0$.

We will prove these lemmas after we have completed the main line of the argument.

The idea of the proof of the theorem is to construct a family of unit vectors $\phi_\eta = \phi_\eta^{(r,v)} \in \mathscr{H}^r(\Gamma, v)$ such that

$$\sup_{\zeta\in D} |f(\eta) - (\phi_\eta^{(r,v)}, T^r_{\Gamma,v}(f)\phi_\eta^{(r,v)})| = 0, \quad \text{as } r \to \infty. \tag{4.4}$$

Having done so, we obtain

$$\begin{aligned}\sup_{\zeta\in D} |f(\eta)| &\leqslant |(\phi_\eta, T^r_{\Gamma,v}(f)\phi_\eta)| + \sup_{\zeta\in D} |f(\eta) - (\phi_\eta, T^r_{\Gamma,v}(f)\phi_\eta)|\\ &\leqslant \|T^r_{\Gamma,v}(f)\| + o(1),\end{aligned}$$

and the claim follows. The vectors $\phi_\eta^{(r,v)}$ are defined as follows:

$$\phi_\eta^{(r,v)}(\eta) := K^r_{\Gamma,v}(\eta,\eta)^{-1/2} K^r_{\Gamma,v}(\zeta,\eta). \tag{4.5}$$

To prove (4.5) we write

$$\begin{aligned}&f(\eta) - (\phi_\eta, T^r_{\Gamma,v}(f)\phi_\eta)\\ &\quad= \int_D |\phi_\eta(\zeta)|^2 (f(\eta) - f(\zeta)) \, \mathrm{d}\mu^r(\zeta)\\ &\quad= K^r_{\Gamma,v}(\eta,\eta)^{-1} \int_U K^r_{\Gamma,v}(\zeta,\eta)(f(\eta) - f(\zeta)) K^r_{\Gamma,v}(\zeta,\eta) \, \mathrm{d}\mu^r(\zeta),\end{aligned}$$

where, in the second equality we have used (2.9). Substituting $\zeta \to \gamma_\eta(\zeta)$ we write this as

$$\begin{aligned}
&K^r_{\Gamma,v}(\eta,\eta)^{-1}\int_U \overline{K^r_{\Gamma,v}(\gamma_\eta(\zeta),\eta)}(f(\gamma_\eta(\zeta))\\
&\quad -f(\gamma_\eta(0)))K^r(\gamma_\eta(\zeta),\gamma_\eta(0))|\gamma'_\eta(\zeta)|^r\,\mathrm{d}\mu^r(\zeta)\\
&=(\gamma'_\eta(0)^{r/2}K^r_{\Gamma,v}(\eta,\eta))^{-1}\int_U \gamma'_\eta(\zeta)^{r/2}K^r_{\Gamma,v}(\gamma_\eta(\zeta),\eta)(f(\gamma_\eta(\zeta))\\
&\quad -f(\gamma_\eta(0)))\,\mathrm{d}\mu^r(\zeta).
\end{aligned}\tag{4.6}$$

But

$$\begin{aligned}
&\gamma'_\eta(\zeta)^{r/2}K^r_{\Gamma,v}(\gamma_\eta(\zeta),\eta)\\
&=\sum_{\gamma\in\Gamma}\frac{1}{v(\gamma)}\gamma'_\eta(\zeta)^{r/2}\gamma'(\gamma_\eta(\zeta))^{r/2}K^r(\gamma\gamma_\eta(\zeta),\eta)\\
&=\sum_{\gamma\in\Gamma}\frac{1}{v(\gamma)}(\gamma\gamma_\eta)'(\zeta)^{r/2}K^r(\gamma_\eta(\gamma_\eta^{-1}\gamma\gamma_\eta(\zeta)),\gamma_\eta(0))\\
&=\gamma'_\eta(0)^{-r/2}\sum_{\gamma\in\Gamma}\frac{1}{v(\gamma)}(\gamma\gamma_\eta)'(\zeta)^{r/2}\gamma'_\eta(\gamma_\eta^{-1}\gamma\gamma_\eta(\zeta))^{-r/2}\\
&=\gamma'_\eta(0)^{-r/2}\sum_{\gamma\in\Gamma}\frac{1}{v(\gamma)}(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)^{r/2},
\end{aligned}$$

and so (4.6) can be written as

$$F^{(r,v)}_\eta(0)^{-1}\int_U \overline{F^{(r,v)}_\eta(\zeta)}(f(\gamma_\eta(\zeta))-f(\gamma_\eta(0)))\,\mathrm{d}\mu^r(\zeta),\tag{4.7}$$

where

$$F^{(r,v)}_\eta(\zeta):=\sum_{\gamma\in\Gamma}\frac{1}{v(\gamma)}(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)^{r/2}.\tag{4.8}$$

To bound (4.7) we first observe that, as a consequence of Lemma 4.1, $F^{(r,v)}_\eta(0)\leqslant$ Const, uniformly in $\eta\in D$, and $r\geqslant r_0$. Then we write the integral in (4.7) as a sum of two integrals: over $|\zeta|\leqslant\delta$ and over $\delta<|\zeta|<1$, where δ will be chosen later. Since D is compact,

$$|\gamma_\eta(\zeta)-\gamma_\eta(\zeta')|\leqslant\frac{1+\mathrm{d}}{1-\mathrm{d}}|\zeta-\zeta'|,\tag{4.9}$$

where $\mathrm{d}=\max_{\eta\in D}|\eta|<1$. Therefore,

$$\begin{aligned}
&\left|\int_{|\zeta|<\delta'}\overline{F^{(r,v)}_\eta(\zeta)}(f(\gamma_\eta(\zeta))-f(\gamma_\eta(0)))\,\mathrm{d}\mu^r(\zeta)\right|\\
&\quad\leqslant\left(2+\sup_{|\zeta|<\delta'}\sup_{\eta\in D}\sum_{\Gamma\ni\gamma\neq\pm I}|(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)^{r/2}|\right)\sup_{|\zeta-\eta|\leqslant\delta'}|f(\zeta)-f(\eta)|,
\end{aligned}$$

where $\delta' := 2\delta(1+\mathrm{d})(1-\mathrm{d})^{-1}$. Let $\varepsilon > 0$ be given. We choose δ as in Lemma 4.1. Since f is Γ invariant with D compact, it is uniformly continuous on U. Making δ smaller, if necessary, we bound (4.10) by $\varepsilon/2$.

On the other hand, using Lemma 4.2,

$$\begin{aligned}
&\left|\int_{\delta < |\zeta| < 1} \overline{F_\eta^{(r,\varepsilon)}(\zeta)}(f(\gamma_\eta(\zeta)) - f(\gamma_\eta(0)))\, \mathrm{d}\mu^r(\zeta)\right| \\
&\quad \leqslant 2B\|f\|_\infty \frac{r(r-1)}{\pi} \int_{\delta < |\zeta| < 1} (1-|\zeta|^2)^{r/2-2}\, \mathrm{d}^2\zeta \\
&\quad \leqslant \text{const. } r(1-\delta^2)^{r/2-1} \to 0, \quad \text{as } r \to \infty.
\end{aligned}$$

As a consequence, for r large enough, the integral over $\delta < |\zeta| < 1$ is less than $\varepsilon/2$ and the theorem is proven.

Proof of Lemma 4.1. We have

$$\begin{aligned}
1 &= \int_U \mathrm{d}\mu^\rho(\zeta) = \sum_{\gamma \in \Gamma} \int_{\gamma(D)} \mathrm{d}\mu^\rho(\zeta) \\
&= \sum_{\gamma \in \Gamma} \int_D |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)|^\rho\, \mathrm{d}\mu^\rho(\zeta) \geqslant \mu^\rho(D)(1-\mathrm{d})^{-\rho} \sum_{\gamma \in \Gamma} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(0)|^\rho, \qquad (4.10)
\end{aligned}$$

for $\rho > 1$, where we have denoted $\mathrm{d} := \max_{\zeta \in D}|\zeta|$. Denoting $r_0 = 2\rho$, we thus find that the series

$$\sum_{\gamma \in \Gamma} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(0)|^{r_0/2}$$

is convergent. Therefore, there is $\kappa(\eta) > 0$ such that $|(\gamma_\eta^{-1}\gamma\gamma_\eta)'(0)| \leqslant (1+\kappa(\eta))^{-1}$, for $\eta \neq \pm e$. Since η ranges over a compact set K, we can choose $\kappa(\eta)$ so that $\kappa := \min_{\eta \in K} \kappa(\eta) > 0$. In other words,

$$|(\gamma_\eta^{-1}\gamma\gamma_\eta)'(0)| \leqslant (1+\kappa)^{-1}, \quad \gamma \neq e,$$

uniformly in $\eta \in D$. Hence, for $r > r_0$

$$\begin{aligned}
&\sum_{\gamma \neq \pm I} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)|^{r/2} \\
&\quad \leqslant (1-|\zeta|)^{-r/2} \sum_{\gamma \neq \pm I} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(0)|^{r/2} \\
&\quad \leqslant (1-|\zeta|)^{-r/2}(1+\kappa)^{-(r-r_0)/2} \sum_{\gamma \neq \pm I} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(0)|^{r/2}.
\end{aligned}$$

We now choose δ so that $(1-\delta)(1+\kappa) > 1$ and (4.2) follows with

$$A := (1+\kappa)^{r_0/2}(1-\mathrm{d})^{r_0/2}\mu_{r_0/2}(D)^{-1} \quad \text{and} \quad C := (1-\delta)(1+\kappa). \qquad \square$$

Proof of Lemma 4.2. We first show that there are $B > 0$ and $r_0 > 2$ such that

$$\int_D |K^r(\zeta, \eta)|^2 \, \mathrm{d}\mu^r(\zeta) \geqslant (Br)^{-1}, \tag{4.11}$$

uniformly in $\eta \in U$ and $r \geqslant r_0$. Indeed,

$$|K^r(\xi, \eta)| \geqslant (1 + |\zeta|)^{-r}, \tag{4.12}$$

and so

$$\begin{aligned}
&\int_D |K^r(\zeta, \eta)|^2 \, \mathrm{d}\mu^r(\zeta) \\
&\quad \geqslant \frac{r-1}{\pi} \int_D \frac{(1-|\zeta|^2)^r}{(1+|\zeta|)^{2r}} (1-|\zeta|^2)^{-2} \, \mathrm{d}^2\zeta \\
&\quad = \frac{r-1}{2\pi} \int_D \exp\{-r\delta(0, \zeta)\} \, \mathrm{d}\mu_0(\zeta),
\end{aligned}$$

where $\mathrm{d}\mu_0(\zeta) = 2(1-|\zeta|^2)^{-2} \, \mathrm{d}^2\zeta$ is the Poincaré measure on U, and where

$$\delta(\zeta, \eta) := \log \frac{|1-\zeta\bar{\eta}| + |\zeta - \eta|}{|1-\zeta\bar{\eta}| - |\zeta - \eta|}$$

is the geodesic distance on U (see, e.g., [1]). Choose r_0 so that D contains the disc $B_{r_0} := \{\zeta : \delta(0, \zeta) \leqslant r_0^{-1}\}$. Then, for all $r \geqslant r_0$,

$$\begin{aligned}
&\int_D \exp\{-r\delta(0, \zeta)\} \, \mathrm{d}\mu_0(\zeta) \\
&\quad \geqslant \int_{B_r} \exp\{-r\delta(0, \zeta)\} \, \mathrm{d}\mu_0(\zeta) \\
&\quad \geqslant \mathrm{e}^{-1}\mu_0(B_r) = 4\pi \, \mathrm{e}^{-1} \frac{(\mathrm{th}(2r)^{-1})^2}{1-(\mathrm{th}(2r)^{-1})^2} \\
&\quad \geqslant \mathrm{O}(1) r^{-2},
\end{aligned}$$

and (4.11) follows.

To prove (4.3) we write

$$\begin{aligned}
1 &= (1-|\zeta|^2)^r \int_U |K^r(\zeta, \theta)|^2 \, \mathrm{d}\mu^r(\theta) \\
&= (1-|\zeta|^2)^r \sum_{\gamma \in \Gamma} \int_{(\gamma_\eta^{-1}\gamma\gamma_\eta)^{-1}(D)} |K^r(\zeta, \theta)|^2 \, \mathrm{d}\mu^r(\theta) \\
&= (1-|\zeta|^2)^r \sum_{\gamma \in \Gamma} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)|^r \int_D |K^r((\gamma_\eta^{-1}\gamma\gamma_\eta)(\zeta), \theta)|^2 \, \mathrm{d}\mu^r(\theta).
\end{aligned}$$

Using (4.11), we obtain from the above identity that

$$1 \geqslant (Br)^{-1}(1-|\zeta|^2)^r \sum_{\gamma \in \Gamma} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)|^r$$

and the claim follows if we replace r by $r/2$. □

5. Proof of Theorem B

The idea of the proof is to produce an asymptotic expansion of $(\phi, T^r_{\Gamma,v}(f)T^r_{\Gamma,v}(g)\psi)$, $\phi, \psi \in \mathscr{H}^r(\Gamma, v)$, as $r \to \infty$. The first two orders in r^{-1} of this expansion will be computed explicitly while the remainder will be bounded nonperturbatively.

We have

$$
\begin{aligned}
&(\phi, T^r_{\Gamma,v}(f)T^r_{\Gamma,v}(g)\psi)\\
&\quad= \int_{D^2} \overline{\phi(\zeta)}K^r_{\Gamma,v}(\zeta,\eta)f(\zeta)g(\eta)\psi(\eta)\,\mathrm{d}\mu^r(\zeta)\,\mathrm{d}\mu^r(\eta)\\
&\quad= \int_{D\times U} \overline{\phi(\zeta)}K^r(\zeta,\eta)f(\zeta)g(\eta)\psi(\eta)\,\mathrm{d}\mu^r(\zeta)\,\mathrm{d}\mu^r(\eta).
\end{aligned}
\tag{5.1}
$$

Substituting $\eta = \gamma_\zeta(\theta)$, ($\zeta = \gamma_\zeta(0)$, see (4.1)) we can rewrite this as

$$
\begin{aligned}
&(\phi, T^r_{\Gamma,v}(f)T^r_{\Gamma,v}(g)\psi) =\\
&\quad= \int_{D\times U} \overline{\phi(\zeta)}f(\zeta)g(\gamma_\zeta(\theta))\psi(\gamma_\zeta(\theta))(\gamma'_\zeta(\theta)/\gamma'_\zeta(0))^{r/2}\,\mathrm{d}\mu^r(\zeta)\,\mathrm{d}\mu^r(\theta).
\end{aligned}
\tag{5.2}
$$

From Taylor's theorem,

$$
\begin{aligned}
g(\gamma_\zeta(\theta)) &= g(\zeta) + (1-|\zeta|^2)[\partial g(\zeta)\theta + \bar\partial g(\zeta)\bar\theta]\\
&\quad+ (1-|\zeta|^2)[-\bar\zeta\,\partial g(\zeta) + \tfrac12(1-|\zeta|^2)\,\partial^2 g(\zeta)]\theta^2\\
&\quad\times (1-|\zeta|^2)[-\zeta\,\bar\partial g(\zeta) + \tfrac12(1-|\zeta|^2)\,\bar\partial^2 g(\zeta)]\bar\theta^2\\
&\quad+ (1-|\zeta|^2)^2\,\partial\bar\partial g(\zeta)\theta\bar\theta + G(\theta,\zeta),
\end{aligned}
\tag{5.3}
$$

where $G(\theta, \zeta)$ is the remainder. In the formula above, we have extracted the second-order contributions, even though we are not interested in evaluating the second-order term in the expansion of (5.1). The reason is that our nonperturbative bound on the remainder in the asymptotic expansion would not give sufficient smallness in r^{-1}, had we kept these terms in the remainder.

LEMMA 5.1. *There is a constant C such that for r sufficiently large and for all $f, g \in C^\infty_\Gamma(U)$ and $\phi, \psi \in \mathscr{H}^r(\Gamma, v)$,*

$$
\begin{aligned}
&\left|\int_{D\times U} \overline{\phi(\zeta)}f(\zeta)G(\theta,\zeta)\psi(\gamma_\zeta(\theta))(\gamma'_\zeta(\theta)/\gamma'_\zeta(0))^{r/2}\,\mathrm{d}\mu_r(\zeta)\,\mathrm{d}\mu_r(\theta)\right|\\
&\quad\leqslant Cr^{-3/2}\|f\|_{4,\infty}\,\|g\|_{4,\infty}\,\|\phi\|\,\|\psi\|.
\end{aligned}
\tag{5.4}
$$

We will prove this lemma later. Substituting into (5.2) and using

$$
\int_U \bar\zeta^n u(\zeta)\,\mathrm{d}\mu_r(\zeta) = \frac{\Gamma(r)}{\Gamma(r+n)}\,\partial^n u(0)
\tag{5.5}
$$

(valid for u holomorphic in U), we obtain

$$
\begin{aligned}
&(\phi, T^r_{\Gamma,v}(f)T^r_{\Gamma,v}(g)\psi)\\
&\quad= \int_D \overline{\phi(\zeta)} f(\zeta)g(\zeta)\psi(\zeta)\,\mathrm{d}\mu^r(\zeta)\\
&\quad+\frac{1}{r}\int_D \overline{\phi(\zeta)} f(\zeta)(1-|\zeta|^2)\,\bar\partial g(\zeta)\frac{\partial}{\partial\theta}\{\psi_\zeta(\theta)(\gamma'_\zeta(\theta)/\gamma'_\zeta(0))^{r/2}\}|_{\theta=0}\,\mathrm{d}\mu^r(\zeta)\\
&\quad+\frac{1}{r}\int_D \overline{\phi(\zeta)} f(\zeta)(1-|\zeta|^2)^2\,\partial\bar\partial g(\zeta)\psi(\zeta)\,\mathrm{d}\mu^r(\zeta)\\
&\quad+\frac{1}{r(r+1)}\int_D \overline{\phi(\zeta)} f(\zeta)(1-|\zeta|^2)[-\zeta\,\bar\partial g(\zeta)+\tfrac{1}{2}(1-|\zeta|^2)\,\bar\partial^2 g(\zeta)]\\
&\quad\times\frac{\partial^2}{\partial\theta^2}\{\psi_\zeta(\theta)(\gamma'_\zeta(\theta)/\gamma'_\zeta(0))^{r/2}\}|_{\theta=0}\,\mathrm{d}\mu^r(\zeta)+R,
\end{aligned}
\tag{5.6}
$$

where R denotes the integral of Lemma 5.1. Observe that the first term on the right hand-side of (5.6) is $(\phi, T^r_{\Gamma,v}(fg)\psi)$, as required. Using the formula

$$
\frac{\partial}{\partial\theta}\{\gamma'(\theta)^{r/2}\psi(\gamma_\zeta(\theta))\}=\frac{1-|\zeta|^2}{1+\bar\zeta\theta}\,\gamma'_\zeta(\theta)^{-r/2}\frac{\partial}{\partial\zeta}\{\gamma'_\zeta(\theta)^r\psi(\gamma_\zeta(\theta))\},
\tag{5.7}
$$

we write the second and the third terms on the right-hand side of (5.6) as

$$
\begin{aligned}
&\frac{1}{r}\int_D \overline{\phi(\zeta)} f(\zeta)\,\bar\partial g(\zeta)(1-|\zeta|^2)^{-r+2}\frac{\partial}{\partial\zeta}\{(1-|\zeta|^2)^r\psi(\zeta)\}\,\mathrm{d}\mu^r(\zeta)\\
&\quad+\frac{1}{r}\int_D \overline{\phi(\zeta)} f(\zeta)(1-|\zeta|^2)^2\,\partial\bar\partial g(\zeta)\psi(\zeta)\,\mathrm{d}\mu^r(\zeta).
\end{aligned}
\tag{5.8}
$$

Integrating the first term in (5.8) by parts (note that the boundary term vanishes, as the one-form $(1-|\zeta|^2)^r\overline{\phi(\zeta)}\psi(\zeta)f(\zeta)\,\bar\partial g(\zeta)\,\mathrm{d}\bar\zeta$ is invariant under Γ), we obtain

$$
\begin{aligned}
&-r^{-1}\int_D \overline{\phi(\zeta)}\psi(\zeta)(1-|\zeta|^2)^2\,\partial f(\zeta)\,\bar\partial g(\zeta)\psi(\zeta)\,\mathrm{d}\mu^r(\zeta)\\
&\quad=-\frac{1}{r}(\phi, T^r_{\Gamma,v}((1-|\zeta|^2)^2\,\partial f\,\bar\partial g)\psi),
\end{aligned}
$$

as required.

We now show that the fourth term on the right-hand side of (5.6) can be bounded by $O(1)r^{-2}\|f\|_p\,\|g\|_p\,\|\phi\|\,\|\phi\|$. Indeed, applying (5.7) twice we replace θ-derivatives by ζ-derivatives. Integrating by parts we eliminate terms containing $\partial\psi(\zeta)$ and $\partial^2\psi(\zeta)$ (this does not produce terms containing derivatives of $\overline{\phi(\zeta)}$ as $\overline{\phi(\zeta)}$ is anti-holomorphic). The result can be written as

$$
\frac{1}{r(r+1)}\int_D F(\zeta)\overline{\phi(\zeta)}\psi(\zeta)\,\mathrm{d}\mu^r(\zeta)
$$

with $F(\zeta)$ continuous on D. This in turn can be bounded by

$$r^{-2} \max_{\zeta \in D} |F(\zeta)| \, \|\phi\| \, \|\psi\| \leqslant O(1) r^{-2} \|f\|_{4,\infty} \|g\|_{4,\infty} \|\phi\| \, \|\psi\|,$$

as claimed. It remains to prove Lemma 5.1.

Proof of Lemma 5.1. We write $G(\theta, \zeta)$ as

$$G(\theta, \zeta) = \sum_{0 \leqslant j \leqslant 3} g_j(\theta, \zeta)\theta^j \bar{\theta}^{3-j}, \tag{5.9}$$

where

$$g_j(\theta, \zeta) := \tfrac{1}{2}\binom{3}{j} \int_0^1 (1-s)^2 \, \partial^j \bar{\partial}^{3-j} \gamma_\zeta^* g(s\theta) \, \mathrm{d}s. \tag{5.10}$$

Here, as usually, $\gamma_\zeta^* g(\theta) = g(\gamma_\zeta(\theta))$. We verify as in [4] that

$$\text{(i)} \quad |g_j(\theta, \zeta) \leq O(1)(1 - |\zeta|^2)(1 - |\theta|)^{-9} \|g\|_{3,\infty}, \tag{5.11}$$

$$\begin{aligned}\text{(ii)} \quad & \left|\frac{\partial}{\partial \zeta}[(1 - |\zeta|^2)^{-1} g_j(\theta, \zeta)]\right| + \left|\frac{\partial}{\partial \bar{\zeta}}[(1 - |\zeta|^2)^{-1} g_j(\theta, \zeta)]\right| \\ & \leqslant O(1)(1 - |\theta|)^{-12} \|g\|_{4,\infty},\end{aligned} \tag{5.12}$$

$$\text{(iii)} \quad \left|\frac{\partial}{\partial \theta} g_j(\theta, \zeta)\right| + \left|\frac{\partial}{\partial \bar{\theta}} g_j(\theta, \zeta)\right| \leqslant O(1)(1 - |\zeta|^2)(1 - |\theta|)^{-12} \|g\|_{4,\infty}. \tag{5.13}$$

Let us now estimate the contribution of the term with $j = 0$ to (5.4). The remaining cases are similar and we refer to Section VI of [4] for more details. Using the formula

$$\bar{\theta}(1 - |\theta|^2)^{r-2} = \frac{1}{r-1} \frac{\partial}{\partial \theta}(1 - |\theta|^2)^{r-1}, \tag{5.14}$$

and integrating by parts with respect to θ we obtain (observe that for r sufficiently large, which is just what we assume, there are no boundary terms):

$$\begin{aligned}&\int_{D \times U} \overline{\phi(\zeta)} f(\zeta) g_0(\theta, \zeta) \bar{\theta}^3 \psi(\gamma_\zeta(\theta))(\gamma_\zeta'(\theta)/\gamma_\zeta'(0))^{r/2} \, \mathrm{d}\mu^r(\theta) \\ &= -\frac{1}{r-1} \int_{D \times U} \overline{\phi(\zeta)} f(\zeta) \frac{\partial}{\partial \theta} g_0(\theta, \zeta) \bar{\theta}^2 \psi(\gamma_\zeta(\theta))(\gamma_\zeta'(\theta)/\gamma_\zeta'(0))^{r/2} \\ &\quad \times (1 - |\theta|^2) \, \mathrm{d}\mu^r(\zeta) \, \mathrm{d}\mu^r(\theta) \\ &\quad - \frac{1}{r-1} \int_{D \times U} \overline{\phi(\zeta)} f(\zeta) g_0(\theta, \zeta) \bar{\theta}^2 \frac{\partial}{\partial \theta} \{\psi(\gamma_\zeta(\theta))(\gamma_\zeta'(\theta)/\gamma_\zeta'(0))^{r/2}\} \\ &\quad \times (1 - |\theta|^2) \, \mathrm{d}\mu^r(\zeta) \, \mathrm{d}\mu^r(\theta).\end{aligned} \tag{5.15}$$

The first term on the right hand side in (5.15) can be bounded by

$$
\begin{aligned}
&2^{11}(r-1)^{-1} \sup_{\zeta \in D} |f(\zeta)| \sup_{(\zeta,\theta) \in D \times U} \left|(1-|\zeta|^2)^{-1}(1-|\theta|)^{12} \frac{\partial}{\partial\theta} g\ (\theta, \zeta)\right| \\
&\quad \times \int_{D \times U} |\phi(\zeta)|(1-|\zeta|^2)|\psi(\gamma_\zeta(\theta))|\, |\gamma_\zeta'(\theta)/\gamma_\zeta'(0)|^{r/2}|\theta|^2(1-|\theta|^2)^{-11}\, \mathrm{d}\mu^r(\zeta)\, \mathrm{d}\mu^r(\theta) \\
&\leqslant \mathrm{O}(1) r^{-1} \|f\|_{4,\infty} \|g\|_{4,\infty} \left\{\int_D (1-|\zeta|^2)^{2-r}\, \mathrm{d}\mu^r(\zeta)\right\}^{1/2} \\
&\quad \times \left\{\int_U |\theta|^4 (1-|\theta|^2)^{24}\, \mathrm{d}\mu^r(\theta)\right\}^{1/2} \\
&\quad \times \left\{\int_{D \times U} |\phi(\zeta)|^2 |\psi(\gamma_\zeta(\theta))|^2 |\gamma_\zeta'(\theta)|^r (1-|\theta|^2)^2\, \mathrm{d}\mu^r(\zeta)\, \mathrm{d}\mu^r(\theta)\right\}^{1/2}. \qquad (5.16)
\end{aligned}
$$

But

$$\int_D (1-|\zeta|^2)^{2-r}\, \mathrm{d}\mu^r(\zeta) \leqslant \mathrm{O}(1) r, \qquad (5.17)$$

$$\int_D |\theta|^4 (1-|\theta|^2)^{-24}\, \mathrm{d}\mu^r(\theta) \leqslant \mathrm{O}(1) r^{-2}. \qquad (5.18)$$

Furthermore,

$$
\begin{aligned}
&\int_{D \times U} |\phi(\zeta)|^2 |\psi(\gamma_\zeta(\theta))|^2 |\gamma_\zeta'(\theta)|^r (1-|\theta|^2)^2\, \mathrm{d}\mu^r(\zeta)\, \mathrm{d}\mu^r(\theta) \\
&= \int_{D \times U} |\phi(\zeta)|^2 |\psi(\theta)|^2 (1-|\theta|^2)^2 |\gamma_\zeta'(\theta)|^{-2}\, \mathrm{d}\mu^r(\zeta)\, \mathrm{d}\mu^r(\theta) \\
&\leqslant 2 \max_{\zeta \in D} (1-|\zeta|^2)^{-2} \int_D |\phi(\zeta)|^2\, \mathrm{d}\mu^r(\zeta) \int_U (1-|\theta|^2)^2 |\psi(\theta)|^2\, \mathrm{d}\mu^r(\theta). \qquad (5.13)
\end{aligned}
$$

We claim that

$$\int_U (1-|\theta|^2)^2 |\psi(\theta)|^2\, \mathrm{d}\mu^r(\theta) \leqslant \mathrm{O}(1) \int_D |\psi(\theta)|^2\, \mathrm{d}\mu^r(\theta). \qquad (5.20)$$

Indeed,

$$
\begin{aligned}
&\int_U (1-|\theta|^2)^2 |\psi(\theta)|^2\, \mathrm{d}\mu^r(\theta) \\
&= \sum_{\gamma \in \Gamma} \int_{\gamma^{-1}(D)} (1-|\theta|^2)^2 |\psi(\theta)|^2\, \mathrm{d}\mu^r(\theta) \\
&= \int_D \left\{\sum_{\gamma \in \Gamma} (1-|\gamma(\theta)|^2)^2\right\} |\psi(\theta)|^2\, \mathrm{d}\mu^r(\theta) \\
&\leqslant \sup_{\theta \in D} \sum_{\gamma \in \Gamma} (1-|\gamma(\theta)|^2)^2 \int_D |\psi(\theta)|^2\, \mathrm{d}\mu^r(\theta),
\end{aligned}
$$

and the claim follows as [3]

$$\sum_{\gamma \in \Gamma} (1 - |\gamma(\theta)|^2)^2 \leqslant O(1)(1 - |\theta|^2)^{-2}, \quad \theta \in U. \tag{5.21}$$

As a consequence of (5.17–20), the right-hand side of (5.16) can be estimated by $O(1)r^{-3/2}$, which completes the analysis of the first term on the right-hand side of (5.15).

Remark. The above analysis is almost identical with the analogous analysis in Section VI of [4] except for one important point. The analog of the left hand side of (5.16) in [4] was

$$2^7(r-1) \sup_{\gamma \in D} |f(\zeta)| \sup_{(\zeta,\theta) \in U \times U} \left|(1-|\zeta|^2)^{-1}(1-|\theta|^2)^8 \frac{\partial}{\partial\theta} g_0(\theta, \zeta)\right|$$
$$\times \int_{U \times U} |\phi(\zeta)(1-|\zeta|^2)|\psi(\gamma_\zeta(\theta))| \, |\gamma'_\zeta(\theta)/\gamma'_\zeta(0)|^{r/2}|\theta|^2(1-|\theta|^2)^{-7} \, d\mu^r(\zeta) \, d\mu^r(\theta),$$

which we could directly estimate by

$$O(1)r^{-1}\|f\|_{4,\infty}\|g\|_{4,\infty}$$
$$\times \left\{\int_U |(1-|\zeta|^2)^{2-r} \, d\mu^r(\zeta)\right\}^{1/2} \left\{\int_U |\theta|^2 \, |(1-|\theta|^2)^{-14} \, d\mu^r(\theta)\right\}^{1/2}$$
$$\times \left\{\int_U |\phi(\zeta)|^2 \, d\mu^r(\zeta)\right\}^{1/2} \left\{\int_U |\psi(\theta)|^2 \, d\mu^r(\theta)\right\}^{1/2}$$

Here, we need to transfer a power of $1 - |\theta|^2$ in order to be able to use (5.20). Less significantly, the powers of $1 - |\theta|$ occurring in (5.11–13) and (consequently) in (5.16), are different than those of [4].

To estimate the second term in (5.15) we first use (5.7) and then integrate by parts with respect to ζ. Applying the same technique as above we estimate the resulting expression by $O(1)r^{-3/2}\|f\|_{4,\infty}\|g\|_{4,\infty}\|\phi\| \, \|\psi\|$.

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