

Quantum Riemann Surfaces: II. The Discrete Series[★]

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(Received: 6 January 1992)

Abstract. We continue our study of noncommutative deformations of two-dimensional hyperbolic manifolds which we initiated in Part I. We construct a sequence of C^* -algebras which are quantizations of a compact Riemann surface of genus g corresponding to special values of the Planck constant. These algebras are direct integrals of finite-dimensional C^* -algebras.

Mathematics Subject Classification (1991). 81C05.

1. Introduction

In this paper we continue our study of quantum deformations of Riemann surfaces [4, 5]. Quantum Riemann surfaces can be thought of as nontrivial, noncommutative manifolds of lowest dimension. In [4], we constructed a one-parameter quantum deformation of the universal covering U of hyperbolic Riemann surfaces. To be more specific, we constructed a family $C_\mu(\bar{U})$ of C^* -algebras, where $0 < \mu < 1$ is a deformation parameter. These C^* -algebras can be thought of as the C^* -algebras of ‘continuous functions having limits at infinity’ on the quantum unit disc. We have shown that $C_\mu(\bar{U})$ is isomorphic with the C^* -algebra $\mathcal{T}^*(\bar{U})$ generated by a class of Toeplitz operators. It can be shown that these Toeplitz operators arise naturally in the process of geometric quantization of U [2, 9, 10].

Recall that a quantum deformation (see [4] for relevant references) of a manifold M is a fibration (\mathcal{A}, π, S) over a parameter space S with a fixed base point O . Each fiber $\mathcal{A}_s = \pi^{-1}(s)$ is a C^* -algebra with the property that $\mathcal{A}_O = C(M)$, a C^* -algebra of continuous functions on M . A deformation map D is a connection in (\mathcal{A}, π, S) . If $\gamma: [0, 1] \rightarrow S$ is a curve in S starting at O , then by $D_{\gamma(t)}(f)$ we denote its lift to \mathcal{A} starting at $f \in C(M)$. The crucial requirement is that for f, g

[★] Supported by DOE under Grant DE-FG02-88ER25065.

smooth the limit

$$\lim_{t \rightarrow 0} \frac{1}{t} (D_{\gamma(t)}(f)D_{\gamma(t)}(g) - D_{\gamma(t)}(fg)) \quad (1.1)$$

exists. As a consequence,

$$\{f, g\}_\gamma := \lim_{t \rightarrow 0} \frac{1}{t} [D_{\gamma(t)}(f), D_{\gamma(t)}(g)] \quad (1.2)$$

is a Poisson bracket on M . One often refers to $\{\cdot, \cdot\}_\gamma$ as a direction of deformation. In the situation of [4], $S = [0, 1)$, $O = 0$ and the deformation map is given by the Toeplitz operator $D_t(f) = T^r(f)$, where $r = 1 + t^{-1}$.

Let now $S = U/\Gamma$ be a compact Riemann surface arising as the quotient of U by a Fuchsian group Γ . Consider the family of \mathbb{C}^* -algebras $\{\mathcal{F}_\Gamma^r(U)\}$ of Toeplitz operators with symbols invariant under Γ . This family is a quantum deformation of the algebra of continuous functions on S . In this Letter, we study the structure of $\mathcal{F}_\Gamma^r(U)$ for the discrete sequence of values of the deformation parameter, $r = (2g - 2)^{-1}n$, where g is the genus of S , and where $n = 1, 2, \dots$. The algebra $\mathcal{F}_\Gamma^r(U)$ is a direct integral of finite-dimensional \mathbb{C}^* -algebras. These finite-dimensional \mathbb{C}^* -algebras are the full matrix algebras on a family of Hilbert spaces of automorphic forms. This property is a unique feature of the discrete series of values of the deformation parameter. In fact, the values $r = (2g - 2)^{-1}n$ are precisely the values of deformation parameter obtained for geometric quantization.

The paper is organized as follows. In Section 2 we summarize our results and fix notation. In Section 3 we review the technique of geometric quantization as applied to compact Riemann surfaces. Sections 4 and 5 contain the proofs of our main results, Theorems A and B of Section 2. The proofs involve rather detailed estimates on the r -dependence of automorphic forms of weight r .

2. Main Results

Let S be a compact Riemann surface of genus $g \geq 2$ and let Γ_0 be a Fuchsian group uniformizing it. In other words, Γ_0 is a discrete subgroup of $\text{PSU}(1, 1)$ acting fix point free on the unit disc U , and $S \cong U/\Gamma_0$. By $\Gamma \subset \text{SU}(1, 1)$, we denote a subgroup of $\text{SU}(1, 1)$ which covers Γ_0 . Let $\pi: \Gamma \rightarrow \Gamma_0$ denote the covering homomorphism. For

$$\gamma = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \Gamma$$

we denote

$$\gamma(\zeta) = \frac{a\zeta + b}{b\zeta + a}, \quad \zeta \in U. \quad (2.1)$$

Then the derivative of (2.1) is $\gamma'(\zeta) = (\bar{b}\zeta + \bar{a})^{-2}$. We set

$$\gamma'(\zeta)^{1/2} := (\bar{b}\zeta + \bar{a})^{-1}. \tag{2.2}$$

Let $r > 0$. Recall (see, e.g. [6]) that $v: \Gamma \rightarrow \mathbb{C}$ is a multiplier for the group Γ and the weight r , if

$$|v(\gamma)| = 1, \tag{2.3}$$

$$\gamma'_1(\gamma_2(\zeta))^{-r/2} \gamma'_2(\zeta)^{-r/2} v(\gamma_1)v(\gamma_2) = (\gamma_1\gamma_2)'(\zeta)^{-r/2} v(\gamma_1\gamma_2). \tag{2.4}$$

Here $\gamma'(\zeta)^{-r/2}$ is defined as $\exp\{-r \log \gamma'(\zeta)^{1/2}\}$ and $\log \alpha := \log |\alpha| + i \arg \alpha$ where $-\pi < \arg \alpha \leq \pi$. It is well known [7] that a multiplier exists if and only if $r = n(2g - 2)^{-1}$, $n = 1, 2, \dots$. If v_1 and v_2 are multipliers then, as a consequence of (2.4), their ratio is a character of Γ . As a consequence, it is natural to identify the set $\mathcal{M}(\Gamma, r)$ of multipliers for Γ and r with the set $\hat{\Gamma}$ of characters of Γ . Since Γ has the presentation $A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} = \pm I$, $\hat{\Gamma}$ is a $2g$ -dimensional torus which we normalize so that it can be identified with $\text{Jac}(S)$, the Jacobian of S . In other words, as sets, $\mathcal{M}(\Gamma, r) \cong \text{Jac}(S)$.

Recall that a holomorphic function $\phi: U \rightarrow \mathbb{C}$ is called an automorphic form of weight $r > 0$ and multiplier v if

$$\phi(\gamma(\zeta)) = v(\gamma)\gamma'(\zeta)^{-r/2}\phi(\zeta), \tag{2.5}$$

for each $\gamma \in \Gamma$. Let $\mathcal{H}^r(\Gamma, v)$ denote the complex vector space of automorphic forms of weight r and multiplier v . There is a natural definition of an inner product on $\mathcal{H}^r(\Gamma, v)$ due to Petersson [8]. For $\phi, \psi \in \mathcal{H}^r(\Gamma, v)$ we set

$$(\phi, \psi) := \int_D \overline{\phi(\zeta)}\psi(\zeta) d\mu^r(\zeta), \tag{2.6}$$

where D is a fundamental domain of Γ , and where

$$d\mu^r(\zeta) := \frac{r-1}{\pi} (1 - |\zeta|^2)^{r-2} d^2\zeta. \tag{2.7}$$

As a consequence of (2.5), (ϕ, ψ) is independent of the choice of D . For the future convenience we will assume that $0 \in D$. The space $\mathcal{H}^r(\Gamma, v)$ thus becomes a (finite-dimensional) Hilbert space.

For $\phi: U \rightarrow \mathbb{C}$ holomorphic and bounded we set

$$\Theta_r^v \phi(\zeta) := \sum_{\gamma \in \Gamma} v(\gamma)^{-1} \gamma'(\zeta)^{r/2} \phi(\gamma(\zeta)). \tag{2.8}$$

Classical theorems going back to Poincaré (see, e.g., [6]) state that the series (2.8) converges almost uniformly in U and that $\Theta_r^v \phi$ is an automorphic form (whenever there is no danger of confusion, we will write $\Theta \phi$ rather than $\Theta_r^v \phi$). The series (2.8) is called the Poincaré theta series of ϕ .

Of particular importance will be the following theta series:

$$K_{\Gamma,v}^r(\zeta, \eta) := \sum_{\gamma \in \Gamma} v(\gamma)^{-1} \gamma'(\zeta)^{r/2} K^r(\gamma(\zeta), \eta), \quad \eta \in U, \quad (2.9)$$

where

$$K^r(\zeta, \eta) := (1 - \zeta\bar{\eta})^{-r}. \quad (2.10)$$

Recall that $K^r(\zeta, \eta)$ is the reproducing kernel for the Hilbert space \mathcal{H}^r of functions holomorphic in U and square integrable with respect to $d\mu^r$. This implies that $K_{\Gamma,v}^r(\zeta, \eta)$ is the reproducing kernel for $\mathcal{H}^r(\Gamma, v)$. Furthermore,

$$K_{\Gamma,v}^r(\zeta, \eta)^* = K_{\Gamma,v}^r(\eta, \zeta). \quad (2.11)$$

Let $L_{\Gamma,v}^2(D, d\mu^r)$ denote the Hilbert space of functions on U satisfying (2.5) and square integrable on D with respect to $d\mu^r$. Then, $K_{\Gamma,v}^r(\zeta, \eta)$ is the integral kernel of the orthogonal projection $P: L_{\Gamma,v}^2(D, d\mu^r) \rightarrow \mathcal{H}^r(\Gamma, v)$,

$$(P\phi)(\zeta) = \int_D K_{\Gamma,v}^r(\zeta, \eta) \phi(\eta) d\mu^r(\eta). \quad (2.12)$$

Let $C_\Gamma(U)$ denote the Banach space of bounded continuous functions on U which are invariant under Γ . For $f \in C_\Gamma(U)$ we define $T_{\Gamma,v}^r(f) := PM(f)$, where $M(f)$ denotes multiplication by f , or explicitly

$$(T_{\Gamma,v}^r(f)\phi)(\zeta) = \int_D K_{\Gamma,v}^r(\zeta, \eta) f(\eta) \phi(\eta) d\mu^r(\eta). \quad (2.13)$$

Then $T_{\Gamma,v}^r(f)$ is a linear operator on $\mathcal{H}^r(\Gamma, v)$ called the Toeplitz operator with symbol f . As $\dim \mathcal{H}^r(\Gamma, v) < \infty$, the \mathbb{C}^* -algebra generated by all Toeplitz operators is the full operator algebra $\text{End}(\mathcal{H}^r(\Gamma, v))$.

In Section 3 we will explain how the spaces $\mathcal{H}^r(\Gamma, v)$ arise as a result of geometric quantization of S . Our main concern in this Letter is to study the properties of the mapping $C(S) \ni f \rightarrow T_{\Gamma,v}^r(f)$ (we have identified a continuous function f with its lift to a Γ -invariant function on U). Our first main result is

THEOREM A. *Let $f \in C_\Gamma(U)$. Then*

$$\|T_{\Gamma,v}^r(f)\| \leq \|f\|_\infty \leq \|T_{\Gamma,v}^r(f)\| + o(1), \quad \text{as } r \rightarrow \infty, \quad (2.14)$$

where $\|f\|_\infty$ is the sup-norm of f , and where $\|T_{\Gamma,v}^r(f)\|$ is the operator norm on $\mathcal{H}^r(\Gamma, v)$. In particular,

$$\lim_{r \rightarrow \infty} \|T_{\Gamma,v}^r(f)\| = \|f\|_\infty. \quad (2.15)$$

Let $C_\Gamma^\infty(U)$ denote the Fréchet space of smooth Γ -invariant functions on U such that

$$\|f\|_{p,\infty} := \sum_{j+k \leq p} \sup_{\zeta \in D} |\partial^j \bar{\partial}^k f(\zeta)| < \infty \quad (\text{in particular, } \|f\|_\infty = \|f\|_{0,\infty}).$$

Our second main result is

THEOREM B. *Let $f, g \in C^\infty(U)$. Then, for r sufficiently large,*

$$\begin{aligned} & \|r(T_{\Gamma,v}^r(f)T_{\Gamma,v}^r(g) - T_{\Gamma,v}^r(fg)) + T_{\Gamma,v}^r((1 - |\zeta|^2)^2 \partial f \bar{\partial} g)\| \\ & \leq Cr^{-1/2} \|f\|_{4,\infty} \|g\|_{4,\infty}, \end{aligned} \tag{2.16}$$

where C is a constant.

The above theorem states that the algebra (of operators on the finite-dimensional Hilbert space $\mathcal{H}^r(\Gamma, v)$) generated by all $T_{\Gamma,v}^r(f)$ with smooth f is a deformation of the algebra of smooth functions on S in the sense explained in the Introduction. The deformed product is given by $f \cdot_r g = fg - (1/r)(1 - |\zeta|^2)^2 \partial f \bar{\partial} g + \dots$. Furthermore, the corollary below states that this algebra is a deformation of $C^\infty(S)$ regarded as the Poisson algebra with the Poisson structure induced by the Poincaré symplectic form,

$$\{f, g\} := i(1 - |\zeta|^2)^2 [\partial f(\zeta) \bar{\partial} g(\zeta) - \bar{\partial} f(\zeta) \partial g(\zeta)]. \tag{2.16}$$

COROLLARY TO THEOREM B. *Let $f, g \in C^\infty(U)$. Then*

$$\left\| \frac{r}{i} [T_{\Gamma,v}^r(f), T_{\Gamma,v}^r(g)] - T_{\Gamma,v}^r(\{f, g\}) \right\| \leq Cr^{-1/2} \|f\|_{4,\infty} \|g\|_{4,\infty}, \tag{2.17}$$

for r sufficiently large.

We will prove Theorems A and B in Sections 4 and 5, respectively.

3. Geometric Quantization

In this section, we show that the Hilbert spaces $\mathcal{H}^r(\Gamma, v)$ introduced in Section 2 arise naturally in the process of geometric quantization (see, e.g., [2, 9, 10]) of S .

We choose the Poincaré Hermitian metric on the universal covering U of S , $ds^2 = (1 - |\zeta|^2)^2 d\zeta \otimes d\bar{\zeta}$. Its curvature,

$$\omega := -2(1 - |\zeta|^2)^2 d\zeta \wedge d\bar{\zeta}, \tag{3.1}$$

coincides with the Poincaré symplectic form on U . Let $L \rightarrow S$ be a holomorphic line bundle over S whose Chern number is $n \in \mathbb{Z}$. The space of sections of L can naturally be identified with the space of smooth functions $s: U \rightarrow \mathbb{C}$ such that for $\gamma \in \Gamma$,

$$\gamma^*s(\zeta) = v(\gamma)\gamma'(\zeta)^{\lambda/2}s(\zeta), \tag{3.2}$$

where v is a multiplier of Γ and where $\lambda = (2g - 2)^{-1}n$.

Let $h \in \mathbb{R}$. The first step in geometric quantization is to construct a holomorphic line bundle $L \rightarrow S$ such that $c_1(L) = (i/2\pi)h^{-1}\omega \in H^2(S, \mathbb{Z})$. Since $\int_D \omega = 2\pi i(2g - 2)$, it follows that such an L

exists if and only if

$$h = \lambda^{-1} = (2g - 2)/n. \quad (3.3)$$

This is the quantization condition of geometric quantization. Hence, for each h given by (3.3), we obtain a family $\{L_{h,v}\}$ of holomorphic line bundles parametrized by the set of multipliers of Γ .

The second step is to construct a complex connection ∇ in $L_{h,v}$ such that $\text{curv}(\nabla) = h^{-1}\omega$. Writing $\nabla = d + \alpha(\zeta) d\zeta$ with α such that

$$\gamma^*\alpha(\zeta) = \gamma'(\zeta)^{-1/2}\alpha(\zeta) + h^{-1}\gamma'(\zeta)^{-1}\gamma''(\zeta),$$

and using $\text{curv}(\nabla) = -\partial/\partial\bar{\zeta}\alpha(\zeta) d\zeta \wedge d\bar{\zeta}$, we find that

$$\alpha(\zeta) = h^{-1} \frac{\zeta}{1 - |\zeta|^2}. \quad (3.4)$$

Next, we find a ∇ -invariant hermitian structure $\langle \cdot, \cdot \rangle$ on $L_{h,v}$. In other words, we require that $\langle \cdot, \cdot \rangle$ satisfies

$$X\langle s, t \rangle(\zeta) = \langle \nabla_X s, t \rangle(\zeta) + \langle s, \nabla_X t \rangle(\zeta), \quad (3.5)$$

for each real vector field X . Writing $\langle s, t \rangle(\zeta) = \chi(\zeta)\overline{s(\zeta)}t(\zeta)$ with $\chi(\zeta) > 0$ we find that (3.6) is equivalent to

$$\frac{\partial}{\partial\zeta}\chi(\zeta) = \alpha(\zeta)\chi(\zeta), \quad \frac{\partial}{\partial\bar{\zeta}}\chi(\zeta) = \overline{\alpha(\zeta)}\chi(\zeta).$$

Solving this system of equations yields $\chi(\zeta) = C(1 - |\zeta|^2)^{1/h}$, where $C > 0$. As a consequence, $\langle s, t \rangle(\zeta) = C(1 - |\zeta|^2)^{1/h}\overline{s(\zeta)}t(\zeta)$.

The next step of geometric quantization is to construct the prequantum Hilbert space. This is the completion of $\Gamma(L_{h,v})$, the space of sections of $L_{h,v}$, in the norm induced by the following inner product:

$$(s, t) := \int_D \langle s, t \rangle(\zeta) d\mu_0(\zeta), \quad (3.6)$$

where $d\mu_0(\zeta) = 2(1 - |\zeta|^2)^{-2} d^2\zeta$ is the Poincaré measure.

The final step of geometric quantization is to choose a polarization on S . Clearly, S has a Kähler polarization, namely the vector field $\partial/\partial\bar{\zeta}$. The corresponding subspace of the prequantum Hilbert space consists of holomorphic sections of $L_{h,v}$ and coincides with $H^{1/h}(\Gamma, v)$ if $0 < h < 1$, and if we choose the constant C in $\chi(\zeta)$ to be $(2\pi)^{-1}(h^{-1} - 1)$.

4. Proof of Theorem A

The proof of Theorem A is based on two technical lemmas which we now formulate. For $\eta \in U$ we set

$$\gamma_\eta := (1 - |\eta|^2)^{-1/2} \begin{pmatrix} 1 & \eta \\ \bar{\eta} & 1 \end{pmatrix} \in \text{SU}(1, 1). \quad (4.1)$$

The lemmas are concerned with the properties of the series

$$\sum_{\gamma \in \Gamma} |(\gamma_\eta^{-1} \gamma \eta)'(\zeta)|^{r/2}, \quad \zeta \in U,$$

as η varies over a compact subset $K \subset U$ and $r \rightarrow \infty$. They improve slightly on the classical results, see e.g. [3, 6].

LEMMA 4.1. *Let $K \subset U$ be compact. There exist constants $A > 0$, $C > 1$, $\delta > 0$ and $r_0 > 2$ such that*

$$\sum_{\Gamma \ni \gamma \neq \pm I} |(\gamma_\eta^{-1} \gamma \eta)'(\zeta)|^{r/2} \leq AC^{-r}, \tag{4.2}$$

for $\eta \in K$, $|\zeta| \leq \delta$ and $r \geq r_0$.

LEMMA 4.2. *There exist constants $B > 0$ and $r_0 > 2$ such that*

$$\sum_{\gamma \in \Gamma} |(\gamma_\eta^{-1} \gamma \eta)'(\zeta)|^{r/2} \leq Br(1 - |\zeta|^2)^{-r/2}, \tag{4.3}$$

for $\eta \in U$, $\zeta \in U$ and $r \geq r_0$.

We will prove these lemmas after we have completed the main line of the argument.

The idea of the proof of the theorem is to construct a family of unit vectors $\phi_\eta = \phi_\eta^{(r,v)} \in \mathcal{H}^r(\Gamma, v)$ such that

$$\sup_{\zeta \in D} |f(\eta) - (\phi_\eta^{(r,v)}, T_{\Gamma,v}^r(f)\phi_\eta^{(r,v)})| = 0, \quad \text{as } r \rightarrow \infty. \tag{4.4}$$

Having done so, we obtain

$$\begin{aligned} \sup_{\zeta \in D} |f(\eta)| &\leq |(\phi_\eta, T_{\Gamma,v}^r(f)\phi_\eta)| + \sup_{\zeta \in D} |f(\eta) - (\phi_\eta, T_{\Gamma,v}^r(f)\phi_\eta)| \\ &\leq \|T_{\Gamma,v}^r(f)\| + o(1), \end{aligned}$$

and the claim follows. The vectors $\phi_\eta^{(r,v)}$ are defined as follows:

$$\phi_\eta^{(r,v)}(\eta) := K_{\Gamma,v}^r(\eta, \eta)^{-1/2} K_{\Gamma,v}^r(\zeta, \eta). \tag{4.5}$$

To prove (4.5) we write

$$\begin{aligned} f(\eta) - (\phi_\eta, T_{\Gamma,v}^r(f)\phi_\eta) &= \int_D |\phi_\eta(\zeta)|^2 (f(\eta) - f(\zeta)) \, d\mu^r(\zeta) \\ &= K_{\Gamma,v}^r(\eta, \eta)^{-1} \int_U K_{\Gamma,v}^r(\zeta, \eta)(f(\eta) - f(\zeta))K_{\Gamma,v}^r(\zeta, \eta) \, d\mu^r(\zeta), \end{aligned}$$

where, in the second equality we have used (2.9). Substituting $\zeta \rightarrow \gamma_\eta(\zeta)$ we write this as

$$\begin{aligned} & K_{\Gamma, \nu}^r(\eta, \eta)^{-1} \int_U \overline{K_{\Gamma, \nu}^r(\gamma_\eta(\zeta), \eta)} (f(\gamma_\eta(\zeta)) \\ & \quad - f(\gamma_\eta(0))) K^r(\gamma_\eta(\zeta), \gamma_\eta(0)) |\gamma'_\eta(\zeta)|^r \, d\mu^r(\zeta) \\ &= (\gamma'_\eta(0))^{r/2} K_{\Gamma, \nu}^r(\eta, \eta)^{-1} \int_U \gamma'_\eta(\zeta)^{r/2} K_{\Gamma, \nu}^r(\gamma_\eta(\zeta), \eta) (f(\gamma_\eta(\zeta)) \\ & \quad - f(\gamma_\eta(0))) \, d\mu^r(\zeta). \end{aligned} \quad (4.6)$$

But

$$\begin{aligned} & \gamma'_\eta(\zeta)^{r/2} K_{\Gamma, \nu}^r(\gamma_\eta(\zeta), \eta) \\ &= \sum_{\gamma \in \Gamma} \frac{1}{v(\gamma)} \gamma'_\eta(\zeta)^{r/2} \gamma'(\gamma_\eta(\zeta))^{r/2} K^r(\gamma\gamma_\eta(\zeta), \eta) \\ &= \sum_{\gamma \in \Gamma} \frac{1}{v(\gamma)} (\gamma\gamma_\eta)'(\zeta)^{r/2} K^r(\gamma_\eta(\gamma_\eta^{-1}\gamma\gamma_\eta(\zeta)), \gamma_\eta(0)) \\ &= \gamma'_\eta(0)^{-r/2} \sum_{\gamma \in \Gamma} \frac{1}{v(\gamma)} (\gamma\gamma_\eta)'(\zeta)^{r/2} \gamma'_\eta(\gamma_\eta^{-1}\gamma\gamma_\eta(\zeta))^{-r/2} \\ &= \gamma'_\eta(0)^{-r/2} \sum_{\gamma \in \Gamma} \frac{1}{v(\gamma)} (\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)^{r/2}, \end{aligned}$$

and so (4.6) can be written as

$$F_\eta^{(r, \nu)}(0)^{-1} \int_U \overline{F_\eta^{(r, \nu)}(\zeta)} (f(\gamma_\eta(\zeta)) - f(\gamma_\eta(0))) \, d\mu^r(\zeta), \quad (4.7)$$

where

$$F_\eta^{(r, \nu)}(\zeta) := \sum_{\gamma \in \Gamma} \frac{1}{v(\gamma)} (\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)^{r/2}. \quad (4.8)$$

To bound (4.7) we first observe that, as a consequence of Lemma 4.1, $F_\eta^{(r, \nu)}(0) \leq \text{Const}$, uniformly in $\eta \in D$, and $r \geq r_0$. Then we write the integral in (4.7) as a sum of two integrals: over $|\zeta| \leq \delta$ and over $\delta < |\zeta| < 1$, where δ will be chosen later. Since D is compact,

$$|\gamma_\eta(\zeta) - \gamma_\eta(\zeta')| \leq \frac{1+d}{1-d} |\zeta - \zeta'|, \quad (4.9)$$

where $d = \max_{\eta \in D} |\eta| < 1$. Therefore,

$$\begin{aligned} & \left| \int_{|\zeta| < \delta} \overline{F_\eta^{(r, \nu)}(\zeta)} (f(\gamma_\eta(\zeta)) - f(\gamma_\eta(0))) \, d\mu^r(\zeta) \right| \\ & \leq \left(2 + \sup_{|\zeta| < \delta} \sup_{\eta \in D} \sum_{\Gamma \ni \gamma \neq +I} |(\gamma_\eta^{-1}\gamma\gamma_\eta)'(\zeta)^{r/2}| \right) \sup_{|\zeta - \eta| \leq \delta} |f(\zeta) - f(\eta)|, \end{aligned}$$

where $\delta' := 2\delta(1+d)(1-d)^{-1}$. Let $\varepsilon > 0$ be given. We choose δ as in Lemma 4.1. Since f is Γ invariant with D compact, it is uniformly continuous on U . Making δ smaller, if necessary, we bound (4.10) by $\varepsilon/2$.

On the other hand, using Lemma 4.2,

$$\begin{aligned} & \left| \int_{\delta < |\zeta| < 1} \overline{F_n^{(r,\varepsilon)}(\zeta)} (f(\gamma_n(\zeta)) - f(\gamma_n(0))) \, d\mu^r(\zeta) \right| \\ & \leq 2B \|f\|_\infty \frac{r(r-1)}{\pi} \int_{\delta < |\zeta| < 1} (1-|\zeta|^2)^{r/2-2} \, d^2\zeta \\ & \leq \text{const. } r(1-\delta^2)^{r/2-1} \rightarrow 0, \quad \text{as } r \rightarrow \infty. \end{aligned}$$

As a consequence, for r large enough, the integral over $\delta < |\zeta| < 1$ is less than $\varepsilon/2$ and the theorem is proven.

Proof of Lemma 4.1. We have

$$\begin{aligned} 1 &= \int_U d\mu^\rho(\zeta) = \sum_{\gamma \in \Gamma} \int_{\gamma(D)} d\mu^\rho(\zeta) \\ &= \sum_{\gamma \in \Gamma} \int_D |(\gamma_n^{-1}\gamma\gamma_n)'(\zeta)|^\rho \, d\mu^\rho(\zeta) \geq \mu^\rho(D)(1-d)^{-\rho} \sum_{\gamma \in \Gamma} |(\gamma_n^{-1}\gamma\gamma_n)'(0)|^\rho, \quad (4.10) \end{aligned}$$

for $\rho > 1$, where we have denoted $d := \max_{\gamma \in D} |\zeta|$. Denoting $r_0 = 2\rho$, we thus find that the series

$$\sum_{\gamma \in \Gamma} |(\gamma_n^{-1}\gamma\gamma_n)'(0)|^{r_0/2}$$

is convergent. Therefore, there is $\kappa(\eta) > 0$ such that $|(\gamma_n^{-1}\gamma\gamma_n)'(0)| \leq (1 + \kappa(\eta))^{-1}$, for $\eta \neq \pm e$. Since η ranges over a compact set K , we can choose $\kappa(\eta)$ so that $\kappa := \min_{\eta \in K} \kappa(\eta) > 0$. In other words,

$$|(\gamma_n^{-1}\gamma\gamma_n)'(0)| \leq (1 + \kappa)^{-1}, \quad \gamma \neq e,$$

uniformly in $\eta \in D$. Hence, for $r > r_0$

$$\begin{aligned} & \sum_{\gamma \neq \pm I} |(\gamma_n^{-1}\gamma\gamma_n)'(\zeta)|^{r/2} \\ & \leq (1-|\zeta|)^{-r/2} \sum_{\gamma \neq \pm I} |(\gamma_n^{-1}\gamma\gamma_n)'(0)|^{r/2} \\ & \leq (1-|\zeta|)^{-r/2} (1+\kappa)^{-(r-r_0)/2} \sum_{\gamma \neq \pm I} |(\gamma_n^{-1}\gamma\gamma_n)'(0)|^{r/2}. \end{aligned}$$

We now choose δ so that $(1-\delta)(1+\kappa) > 1$ and (4.2) follows with

$$A := (1+\kappa)^{r_0/2} (1-d)^{r_0/2} \mu_{r_0/2}(D)^{-1} \quad \text{and} \quad C := (1-\delta)(1+\kappa). \quad \square$$

Proof of Lemma 4.2. We first show that there are $B > 0$ and $r_0 > 2$ such that

$$\int_D |K^r(\zeta, \eta)|^2 d\mu^r(\zeta) \geq (Br)^{-1}, \quad (4.11)$$

uniformly in $\eta \in U$ and $r \geq r_0$. Indeed,

$$|K^r(\zeta, \eta)| \geq (1 + |\zeta|)^{-r}, \quad (4.12)$$

and so

$$\begin{aligned} & \int_D |K^r(\zeta, \eta)|^2 d\mu^r(\zeta) \\ & \geq \frac{r-1}{\pi} \int_D \frac{(1-|\zeta|^2)^r}{(1+|\zeta|)^{2r}} (1-|\zeta|^2)^{-2} d^2\zeta \\ & = \frac{r-1}{2\pi} \int_D \exp\{-r\delta(0, \zeta)\} d\mu_0(\zeta), \end{aligned}$$

where $d\mu_0(\zeta) = 2(1-|\zeta|^2)^{-2} d^2\zeta$ is the Poincaré measure on U , and where

$$\delta(\zeta, \eta) := \log \frac{|1 - \zeta\bar{\eta}| + |\zeta - \eta|}{|1 - \zeta\bar{\eta}| - |\zeta - \eta|}$$

is the geodesic distance on U (see, e.g., [1]). Choose r_0 so that D contains the disc $B_{r_0} := \{\zeta : \delta(0, \zeta) \leq r_0^{-1}\}$. Then, for all $r \geq r_0$,

$$\begin{aligned} & \int_D \exp\{-r\delta(0, \zeta)\} d\mu_0(\zeta) \\ & \geq \int_{B_r} \exp\{-r\delta(0, \zeta)\} d\mu_0(\zeta) \\ & \geq e^{-1} \mu_0(B_r) = 4\pi e^{-1} \frac{(\operatorname{th}(2r)^{-1})^2}{1 - (\operatorname{th}(2r)^{-1})^2} \\ & \geq O(1)r^{-2}, \end{aligned}$$

and (4.11) follows.

To prove (4.3) we write

$$\begin{aligned} 1 &= (1-|\zeta|^2)^r \int_U |K^r(\zeta, \theta)|^2 d\mu^r(\theta) \\ &= (1-|\zeta|^2)^r \sum_{\gamma \in \Gamma} \int_{(G_\eta^{-1}\gamma\gamma_\eta)^{-1}(D)} |K^r(\zeta, \theta)|^2 d\mu^r(\theta) \\ &= (1-|\zeta|^2)^r \sum_{\gamma \in \Gamma} |(G_\eta^{-1}\gamma\gamma_\eta)'(\zeta)|^r \int_D |K^r((G_\eta^{-1}\gamma\gamma_\eta)(\zeta), \theta)|^2 d\mu^r(\theta). \end{aligned}$$

Using (4.11), we obtain from the above identity that

$$1 \geq (Br)^{-1} (1-|\zeta|^2)^r \sum_{\gamma \in \Gamma} |(G_\eta^{-1}\gamma\gamma_\eta)'(\zeta)|^r$$

and the claim follows if we replace r by $r/2$. \square

5. Proof of Theorem B

The idea of the proof is to produce an asymptotic expansion of $(\phi, T_{\Gamma,v}^r(f)T_{\Gamma,v}^r(g)\psi)$, $\phi, \psi \in \mathcal{H}^r(\Gamma, v)$, as $r \rightarrow \infty$. The first two orders in r^{-1} of this expansion will be computed explicitly while the remainder will be bounded nonperturbatively.

We have

$$\begin{aligned} & (\phi, T_{\Gamma,v}^r(f)T_{\Gamma,v}^r(g)\psi) \\ &= \int_{D^2} \overline{\phi(\zeta)} K_{\Gamma,v}^r(\zeta, \eta) f(\zeta) g(\eta) \psi(\eta) \, d\mu^r(\zeta) \, d\mu^r(\eta) \\ &= \int_{D \times U} \overline{\phi(\zeta)} K^r(\zeta, \eta) f(\zeta) g(\eta) \psi(\eta) \, d\mu^r(\zeta) \, d\mu^r(\eta). \end{aligned} \tag{5.1}$$

Substituting $\eta = \gamma_\zeta(\theta)$, ($\zeta = \gamma_\zeta(0)$, see (4.1)) we can rewrite this as

$$\begin{aligned} & (\phi, T_{\Gamma,v}^r(f)T_{\Gamma,v}^r(g)\psi) = \\ &= \int_{D \times U} \overline{\phi(\zeta)} f(\zeta) g(\gamma_\zeta(\theta)) \psi(\gamma_\zeta(\theta)) (\gamma'_\zeta(\theta)/\gamma'_\zeta(0))^{r/2} \, d\mu^r(\zeta) \, d\mu^r(\theta). \end{aligned} \tag{5.2}$$

From Taylor's theorem,

$$\begin{aligned} g(\gamma_\zeta(\theta)) &= g(\zeta) + (1 - |\zeta|^2)[\partial g(\zeta)\theta + \bar{\partial} g(\zeta)\bar{\theta}] \\ &\quad + (1 - |\zeta|^2)[- \zeta \partial g(\zeta) + \frac{1}{2}(1 - |\zeta|^2) \partial^2 g(\zeta)]\theta^2 \\ &\quad \times (1 - |\zeta|^2)[- \zeta \bar{\partial} g(\zeta) + \frac{1}{2}(1 - |\zeta|^2) \bar{\partial}^2 g(\zeta)]\bar{\theta}^2 \\ &\quad + (1 - |\zeta|^2)^2 \partial \bar{\partial} g(\zeta)\theta\bar{\theta} + G(\theta, \zeta), \end{aligned} \tag{5.3}$$

where $G(\theta, \zeta)$ is the remainder. In the formula above, we have extracted the second-order contributions, even though we are not interested in evaluating the second-order term in the expansion of (5.1). The reason is that our nonperturbative bound on the remainder in the asymptotic expansion would not give sufficient smallness in r^{-1} , had we kept these terms in the remainder.

LEMMA 5.1. *There is a constant C such that for r sufficiently large and for all $f, g \in C^\infty(U)$ and $\phi, \psi \in \mathcal{H}^r(\Gamma, v)$,*

$$\begin{aligned} & \left| \int_{D \times U} \overline{\phi(\zeta)} f(\zeta) G(\theta, \zeta) \psi(\gamma_\zeta(\theta)) (\gamma'_\zeta(\theta)/\gamma'_\zeta(0))^{r/2} \, d\mu_r(\zeta) \, d\mu_r(\theta) \right| \\ & \leq Cr^{-3/2} \|f\|_{4,\infty} \|g\|_{4,\infty} \|\phi\| \|\psi\|. \end{aligned} \tag{5.4}$$

We will prove this lemma later. Substituting into (5.2) and using

$$\int_U \bar{\zeta}^n u(\zeta) \, d\mu_r(\zeta) = \frac{\Gamma(r)}{\Gamma(r+n)} \partial^n u(0) \tag{5.5}$$

(valid for u holomorphic in U), we obtain

$$\begin{aligned}
& (\phi, T_{\Gamma, \nu}^r(f)T_{\Gamma, \nu}^r(g)\psi) \\
&= \int_D \overline{\phi(\zeta)} f(\zeta) g(\zeta) \psi(\zeta) \, d\mu^r(\zeta) \\
&+ \frac{1}{r} \int_D \overline{\phi(\zeta)} f(\zeta) (1 - |\zeta|^2) \bar{\partial} g(\zeta) \frac{\partial}{\partial \theta} \{ \psi_\zeta(\theta) (\gamma'_\zeta(\theta) / \gamma'_\zeta(0))^{r/2} \} |_{\theta=0} \, d\mu^r(\zeta) \\
&+ \frac{1}{r} \int_D \overline{\phi(\zeta)} f(\zeta) (1 - |\zeta|^2)^2 \partial \bar{\partial} g(\zeta) \psi(\zeta) \, d\mu^r(\zeta) \\
&+ \frac{1}{r(r+1)} \int_D \overline{\phi(\zeta)} f(\zeta) (1 - |\zeta|^2) [-\zeta \bar{\partial} g(\zeta) + \frac{1}{2} (1 - |\zeta|^2) \bar{\partial}^2 g(\zeta)] \\
&\times \frac{\partial^2}{\partial \theta^2} \{ \psi_\zeta(\theta) (\gamma'_\zeta(\theta) / \gamma'_\zeta(0))^{r/2} \} |_{\theta=0} \, d\mu^r(\zeta) + R, \tag{5.6}
\end{aligned}$$

where R denotes the integral of Lemma 5.1. Observe that the first term on the right hand-side of (5.6) is $(\phi, T_{\Gamma, \nu}^r(fg)\psi)$, as required. Using the formula

$$\frac{\partial}{\partial \theta} \{ \gamma'(\theta)^{r/2} \psi(\gamma_\zeta(\theta)) \} = \frac{1 - |\zeta|^2}{1 + \bar{\zeta}\theta} \gamma'_\zeta(\theta)^{-r/2} \frac{\partial}{\partial \zeta} \{ \gamma'_\zeta(\theta)^r \psi(\gamma_\zeta(\theta)) \}, \tag{5.7}$$

we write the second and the third terms on the right-hand side of (5.6) as

$$\begin{aligned}
& \frac{1}{r} \int_D \overline{\phi(\zeta)} f(\zeta) \bar{\partial} g(\zeta) (1 - |\zeta|^2)^{-r+2} \frac{\partial}{\partial \zeta} \{ (1 - |\zeta|^2)^r \psi(\zeta) \} \, d\mu^r(\zeta) \\
&+ \frac{1}{r} \int_D \overline{\phi(\zeta)} f(\zeta) (1 - |\zeta|^2)^2 \partial \bar{\partial} g(\zeta) \psi(\zeta) \, d\mu^r(\zeta). \tag{5.8}
\end{aligned}$$

Integrating the first term in (5.8) by parts (note that the boundary term vanishes, as the one-form $(1 - |\zeta|^2)^r \overline{\phi(\zeta)} \psi(\zeta) f(\zeta) \bar{\partial} g(\zeta) \, d\bar{\zeta}$ is invariant under Γ), we obtain

$$\begin{aligned}
& -r^{-1} \int_D \overline{\phi(\zeta)} \psi(\zeta) (1 - |\zeta|^2)^2 \partial f(\zeta) \bar{\partial} g(\zeta) \psi(\zeta) \, d\mu^r(\zeta) \\
&= -\frac{1}{r} (\phi, T_{\Gamma, \nu}^r((1 - |\zeta|^2)^2 \partial f \bar{\partial} g)\psi),
\end{aligned}$$

as required.

We now show that the fourth term on the right-hand side of (5.6) can be bounded by $O(1)r^{-2} \|f\|_p \|g\|_p \|\phi\| \|\psi\|$. Indeed, applying (5.7) twice we replace θ -derivatives by ζ -derivatives. Integrating by parts we eliminate terms containing $\partial \psi(\zeta)$ and $\partial^2 \psi(\zeta)$ (this does not produce terms containing derivatives of $\overline{\phi(\zeta)}$ as $\phi(\zeta)$ is anti-holomorphic). The result can be written as

$$\frac{1}{r(r+1)} \int_D F(\zeta) \overline{\phi(\zeta)} \psi(\zeta) \, d\mu^r(\zeta)$$

with $F(\zeta)$ continuous on D . This in turn can be bounded by

$$r^{-2} \max_{\zeta \in D} |F(\zeta)| \|\phi\| \|\psi\| \leq O(1) r^{-2} \|f\|_{4,\infty} \|g\|_{4,\infty} \|\phi\| \|\psi\|,$$

as claimed. It remains to prove Lemma 5.1.

Proof of Lemma 5.1. We write $G(\theta, \zeta)$ as

$$G(\theta, \zeta) = \sum_{0 \leq j \leq 3} g_j(\theta, \zeta) \theta^j \bar{\theta}^{3-j}, \quad (5.9)$$

where

$$g_j(\theta, \zeta) := \frac{1}{2} \binom{3}{j} \int_0^1 (1-s)^2 \partial^j \bar{\partial}^{3-j} \gamma_\zeta^* g(s\theta) ds. \quad (5.10)$$

Here, as usually, $\gamma_\zeta^* g(\theta) = g(\gamma_\zeta(\theta))$. We verify as in [4] that

$$(i) \quad |g_j(\theta, \zeta)| \leq O(1)(1-|\zeta|^2)(1-|\theta|)^{-9} \|g\|_{3,\infty}, \quad (5.11)$$

$$(ii) \quad \left| \frac{\partial}{\partial \zeta} [(1-|\zeta|^2)^{-1} g_j(\theta, \zeta)] \right| + \left| \frac{\partial}{\partial \bar{\zeta}} [(1-|\zeta|^2)^{-1} g_j(\theta, \zeta)] \right| \\ \leq O(1)(1-|\theta|)^{-12} \|g\|_{4,\infty}, \quad (5.12)$$

$$(iii) \quad \left| \frac{\partial}{\partial \theta} g_j(\theta, \zeta) \right| + \left| \frac{\partial}{\partial \bar{\theta}} g_j(\theta, \zeta) \right| \leq O(1)(1-|\zeta|^2)(1-|\theta|)^{-12} \|g\|_{4,\infty}. \quad (5.13)$$

Let us now estimate the contribution of the term with $j=0$ to (5.4). The remaining cases are similar and we refer to Section VI of [4] for more details. Using the formula

$$\bar{\theta}(1-|\theta|^2)^{r-2} = \frac{1}{r-1} \frac{\partial}{\partial \theta} (1-|\theta|^2)^{r-1}, \quad (5.14)$$

and integrating by parts with respect to θ we obtain (observe that for r sufficiently large, which is just what we assume, there are no boundary terms):

$$\int_{D \times U} \overline{\phi(\zeta)} f(\zeta) g_0(\theta, \zeta) \bar{\theta}^3 \psi(\gamma_\zeta(\theta)) (\gamma'_\zeta(\theta) / \gamma'_\zeta(0))^{r/2} d\mu^r(\theta) \\ = -\frac{1}{r-1} \int_{D \times U} \overline{\phi(\zeta)} f(\zeta) \frac{\partial}{\partial \theta} g_0(\theta, \zeta) \bar{\theta}^2 \psi(\gamma_\zeta(\theta)) (\gamma'_\zeta(\theta) / \gamma'_\zeta(0))^{r/2} \\ \times (1-|\theta|^2) d\mu^r(\zeta) d\mu^r(\theta) \\ -\frac{1}{r-1} \int_{D \times U} \overline{\phi(\zeta)} f(\zeta) g_0(\theta, \zeta) \bar{\theta}^2 \frac{\partial}{\partial \theta} \{ \psi(\gamma_\zeta(\theta)) (\gamma'_\zeta(\theta) / \gamma'_\zeta(0))^{r/2} \} \\ \times (1-|\theta|^2) d\mu^r(\zeta) d\mu^r(\theta). \quad (5.15)$$

The first term on the right hand side in (5.15) can be bounded by

$$\begin{aligned}
& 2^{11}(r-1)^{-1} \sup_{\zeta \in D} |f(\zeta)| \sup_{(\zeta, \theta) \in D \times U} \left| (1-|\zeta|^2)^{-1} (1-|\theta|^2)^{12} \frac{\partial}{\partial \theta} g(\theta, \zeta) \right| \\
& \quad \times \int_{D \times U} |\phi(\zeta)| (1-|\zeta|^2) |\psi(\gamma_\zeta(\theta))| |\gamma'_\zeta(\theta)/\gamma'_\zeta(0)|^{r/2} |\theta|^2 (1-|\theta|^2)^{-11} d\mu^r(\zeta) d\mu^r(\theta) \\
& \leq O(1)r^{-1} \|f\|_{4,\infty} \|g\|_{4,\infty} \left\{ \int_D (1-|\zeta|^2)^{2-r} d\mu^r(\zeta) \right\}^{1/2} \\
& \quad \times \left\{ \int_U |\theta|^4 (1-|\theta|^2)^{24} d\mu^r(\theta) \right\}^{1/2} \\
& \quad \times \left\{ \int_{D \times U} |\phi(\zeta)|^2 |\psi(\gamma_\zeta(\theta))|^2 |\gamma'_\zeta(\theta)|^r (1-|\theta|^2)^2 d\mu^r(\zeta) d\mu^r(\theta) \right\}^{1/2}. \tag{5.16}
\end{aligned}$$

But

$$\int_D (1-|\zeta|^2)^{2-r} d\mu^r(\zeta) \leq O(1)r, \tag{5.17}$$

$$\int_D |\theta|^4 (1-|\theta|^2)^{-24} d\mu^r(\theta) \leq O(1)r^{-2}. \tag{5.18}$$

Furthermore,

$$\begin{aligned}
& \int_{D \times U} |\phi(\zeta)|^2 |\psi(\gamma_\zeta(\theta))|^2 |\gamma'_\zeta(\theta)|^r (1-|\theta|^2)^2 d\mu^r(\zeta) d\mu^r(\theta) \\
& = \int_{D \times U} |\phi(\zeta)|^2 |\psi(\theta)|^2 (1-|\theta|^2)^2 |\gamma'_\zeta(\theta)|^{-2} d\mu^r(\zeta) d\mu^r(\theta) \\
& \leq 2 \max_{\zeta \in D} (1-|\zeta|^2)^{-2} \int_D |\phi(\zeta)|^2 d\mu^r(\zeta) \int_U (1-|\theta|^2)^2 |\psi(\theta)|^2 d\mu^r(\theta). \tag{5.13}
\end{aligned}$$

We claim that

$$\int_U (1-|\theta|^2)^2 |\psi(\theta)|^2 d\mu^r(\theta) \leq O(1) \int_D |\psi(\theta)|^2 d\mu^r(\theta). \tag{5.20}$$

Indeed,

$$\begin{aligned}
& \int_U (1-|\theta|^2)^2 |\psi(\theta)|^2 d\mu^r(\theta) \\
& = \sum_{\gamma \in \Gamma} \int_{\gamma^{-1}(D)} (1-|\theta|^2)^2 |\psi(\theta)|^2 d\mu^r(\theta) \\
& = \int_D \left\{ \sum_{\gamma \in \Gamma} (1-|\gamma(\theta)|^2)^2 \right\} |\psi(\theta)|^2 d\mu^r(\theta) \\
& \leq \sup_{\theta \in D} \sum_{\gamma \in \Gamma} (1-|\gamma(\theta)|^2)^2 \int_D |\psi(\theta)|^2 d\mu^r(\theta),
\end{aligned}$$

and the claim follows as [3]

$$\sum_{\gamma \in \Gamma} (1 - |\gamma(\theta)|^2)^2 \leq O(1)(1 - |\theta|^2)^{-2}, \quad \theta \in U. \tag{5.21}$$

As a consequence of (5.17–20), the right-hand side of (5.16) can be estimated by $O(1)r^{-3/2}$, which completes the analysis of the first term on the right-hand side of (5.15).

Remark. The above analysis is almost identical with the analogous analysis in Section VI of [4] except for one important point. The analog of the left hand side of (5.16) in [4] was

$$\begin{aligned} & 2^r(r-1) \sup_{\gamma \in D} |f(\zeta)| \sup_{(\zeta, \theta) \in U \times U} \left| (1 - |\zeta|^2)^{-1} (1 - |\theta|^2)^8 \frac{\partial}{\partial \theta} g_0(\theta, \zeta) \right| \\ & \times \int_{U \times U} |\phi(\zeta)(1 - |\zeta|^2)| |\psi(\gamma_\zeta(\theta))| |\gamma'_\zeta(\theta)/\gamma'_\zeta(0)|^{r/2} |\theta|^2 (1 - |\theta|^2)^{-7} d\mu^r(\zeta) d\mu^r(\theta), \end{aligned}$$

which we could directly estimate by

$$\begin{aligned} & O(1)r^{-1} \|f\|_{4,\infty} \|g\|_{4,\infty} \\ & \times \left\{ \int_U (1 - |\zeta|^2)^{2-r} d\mu^r(\zeta) \right\}^{1/2} \left\{ \int_U |\theta|^2 (1 - |\theta|^2)^{-14} d\mu^r(\theta) \right\}^{1/2} \\ & \times \left\{ \int_U |\phi(\zeta)|^2 d\mu^r(\zeta) \right\}^{1/2} \left\{ \int_U |\psi(\theta)|^2 d\mu^r(\theta) \right\}^{1/2} \end{aligned}$$

Here, we need to transfer a power of $1 - |\theta|^2$ in order to be able to use (5.20). Less significantly, the powers of $1 - |\theta|$ occurring in (5.11–13) and (consequently) in (5.16), are different than those of [4].

To estimate the second term in (5.15) we first use (5.7) and then integrate by parts with respect to ζ . Applying the same technique as above we estimate the resulting expression by $O(1)r^{-3/2} \|f\|_{4,\infty} \|g\|_{4,\infty} \|\phi\| \|\psi\|$.

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