Letters in Mathematical Physics **32:** 45–61, 1994.

Quantum Riemann Surfaces III. The Exceptional Cases

SŁAWOMIR KLIMEK★
Department of Mathematics, IUPUI, Indianapolis, IN 46205 U.S.A.

and

ANDRZEJ LEŚNIEWSKI★★
Department of Physics, Harvard University, Cambridge, MA 02138, U.S.A.

(Received: 9 December, 1993)

Abstract. We discuss quantum deformations of Riemann surfaces whose fundamental groups are Abelian (the exceptional Riemann surfaces). We prove uniformization theorems, state deformation estimates, and study the dependence of quantum Riemann surfaces on the deformation parameter.

Mathematics Subject Classification (1991). 81Q05.

1. Introduction

In recent years, there has been a growing interest in the theory of quantization. Rieffell's paper [17] contains a recent review of this subject and an extensive list of references. Among other developments, the present authors initiated in [13] and [14] a systematic study of Toeplitz quantization of Hermitian manifolds. This approach is building up on Berezin's ideas contained in [1] and its essence consists in the following.

We consider a Kähler manifold (or supermanifold) M with Kähler form ω. Locally we can write

$$\omega = i\partial_j\bar{\partial}_k\phi \, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k,$$

with some Kähler potential ϕ. We then construct the following measures $\mathrm{d}\mu_r$ on M,

$$\mathrm{d}\mu_r(z) = \mathrm{e}^{-r\phi(z)} \det(\partial_j\bar{\partial}_k\phi) \, \mathrm{d}z \, \mathrm{d}\bar{z}$$

and consider the subspace $H^2(M, \mathrm{d}\mu_r)$ of $L^2(M, \mathrm{d}\mu_r)$ spanned by holomorphic functions. Let P denote the orthogonal projection onto $H^2(M, \mathrm{d}\mu_r)$. For a continuous function f on M, let $M(f)$ denote the operator of pointwise multiplication by f. The corresponding Toeplitz operator $T_r(f)$ is the compression $PM(f)P$ of $M(f)$ to $H^2(M, \mathrm{d}\mu_r)$. This construction can be made global if one uses sections of Hermitian

★Supported in part by NSF grant DMS-9206936.
★★Supported in part by DOE grant DE-FG02-88ER25065.

line bundles instead of functions. Furthermore, topological obstructions to the existence of ϕ may impose restrictions on the values of the parameter r. This will be discussed in more detail in Section 4.

The key fact is now that the correspondence $f \mapsto T_r(f)$ is a quantization map. This means that for f continuous and bounded we have the norm limit

$$\lim_{r\to\infty} \| T_r(f) \| = \| f \|_\infty,$$

where $\|\cdot\|$ denotes the operator norm and where $\|\cdot\|_\infty$ denotes the sup-norm. If, moreover, f, g are smooth and at least one of them is compactly supported, then

$$\lim_{r\to\infty} \| r[T_r(f), T_r(g)] + T_r(i\{f, g\}) \| = 0,$$

where $\{f, g\}$ is the Poisson bracket defined by ω.

Substantial progress in implementing this program has been made recently. Deformation estimates were proven for $\mathbb{C}^n$ by Coburn in [8]. The Cartan domains were studied by Borthwick, Leśniewski and Upmeier in [7]. In a series of papers [4–6], Borthwick, Rinaldi, and the present authors further extended these results to include an infinite family of supermanifolds.

Compact Kähler manifolds present additional topological complications mentioned above. In [14], we described the Toeplitz quantization scheme for compact Riemann surfaces. This construction was recently extended to arbitrary compact Kähler manifolds by Bordemann, Meinrenken and Schlichenmeier, [3].

In this Letter we are concerned with the following equivariant version of Toeplitz quantization. Suppose M is a Kähler manifold and let Γ be a discrete group acting holomorphically and without fix points on M. Let us also assume that the Kähler form ω is Γ-invariant. We may then consider the quotient manifold $M_1 = M/\Gamma$ and the pushed down Kähler form $\omega_1 = \pi_* \omega$, where π is the canonical projection $\pi: M \mapsto M_1$. Let $f \mapsto T_r(f)$ and $f \mapsto T_r^1(f)$ be Toeplitz quantizations of M and M_1, respectively, and let $\mathscr{A}_r(M)$ and $\mathscr{A}_r(M_1)$ denote the corresponding $\mathbb{C}^*$-algebras generated by the Toeplitz operators. We say that $\mathscr{A}_r(M)$ is a quantum covering of $\mathscr{A}_r(M_1)$ if there is an isomorphism U of the corresponding Hilbert spaces such that

$$UT_r(f \circ \pi)U^{-1} = T_r^1(f),$$

for all continuous functions f on M_1. In this Letter, we study uniformization theory for quantum Riemann surfaces. The question we address is whether the classical covering spaces become, upon quantization, quantum covering spaces for the corresponding quantum Riemann surfaces.

Let M_1 be an arbitrary Riemann surface (except a sphere) and let M be its universal covering space. Then M is either the unit disk or the complex plane. Toeplitz quantizations of these spaces were discussed [13] and [8]. The following theorem is proven in [15].

THEOREM 1.1. *With the above definitions, the correspondence*

$$f \mapsto T_r(f \circ \pi)$$

is a quantization of M_1.

In this Letter, we assume additionally that M_1 is an exceptional Riemann surface, i.e. the fundamental group $\pi_1(M_1)$ of M_1 is commutative. This is a substantial technical simplification as it allows us to effectively use Fourier analysis on that group. The main result of this Letter is summarized in the following theorem.

THEOREM 1.2. *With the above definitions, the quantum Riemann surfaces* $\mathscr{A}_r(M)$ *are quantum coverings of* $\mathscr{A}_r(M_1)$.

The Letter is organized as follows. In Sections 2 and 3 we study elliptic and hyperbolic Riemann surfaces, respectively. In Section 4, we compare these results with the direct approach based on geometric quantization.

2. Uniformization of Elliptic Quantum Riemann Surfaces

Elliptic Riemann surfaces are precisely those surfaces which have the complex plane $\mathbb{C}$ as the universal covering space. It is well known (see, e.g., [12]) that there are three types of elliptic Riemann surfaces: the plane $\mathbb{C}$, the punctured plane $\mathbb{C}^*$, and the tori $=\mathbb{C}/\mathbb{Z}^2$.

In this section, we study Toeplitz quantizations of these Riemann surfaces. In particular, we discuss quantization maps and the structure of the $\mathbb{C}^*$-algebras they generate. Deformation quantization of $\mathbb{C}$ is a classical subject going back to Heisenberg, Born and Jordan. For a recent discussion from the point of view close to ours see [8] and references therein. There is also substantial literature on the noncommutative tori, see, e.g., [9], [16]. We should also emphasize that some results described in this section had previously been obtained in [2].

Let $d\mu_r$ be the following probabilistic measure on $\mathbb{C}$,

$$d\mu_r = \frac{r}{\pi} e^{-r|\zeta|^2} d^2\zeta. \tag{2.1}$$

Here $r > 0$, and $d^2\zeta$ is the Lebesgue measure on $\mathbb{C}$. Let $H^2(\mathbb{C}, d\mu_r)$ denote the closed subspace of $L^2(\mathbb{C}, d\mu_r)$ consisting of holomorphic functions. Let $K^r(\zeta, \eta)$ be the integral kernel of the orthogonal projection $P: L^2(\mathbb{C}, d\mu_r) \to H^2(\mathbb{C}, d\mu_r)$, i.e. $K^r(\zeta, \eta)$ is the Bergman kernel for $\mathbb{C}$ associated with $d\mu_r$. Explicitly,

$$K^r(\zeta, \eta) = e^{r\zeta\bar{\eta}}. \tag{2.2}$$

For $a \in \mathbb{C}$ we set

$$U^r(a)\phi(\zeta) = e^{r\zeta\bar{a} - (r/2)|a|^2}\phi(\zeta - a). \tag{2.3}$$

Then the operator $U^r(a)$ is unitary in $H^2(\mathbb{C}, \mathrm{d}\mu_r)$. Moreover, $a \to U^r(a)$ defines a projective unitary representation of the additive group $\mathbb{C}$. In fact, for $a, b \in \mathbb{C}$ we have

$$U^r(a)U^r(b) = \mathrm{e}^{ir\mathrm{Im}(\bar{a}b)}U^r(a+b). \tag{2.4}$$

For a bounded, continuous function f on $\mathbb{C}$ we define an operator $T_r(f)$: $H^2(\mathbb{C}, \mathrm{d}\mu_r) \to H^2(\mathbb{C}, \mathrm{d}\mu_r)$ by

$$T_r(f)\phi = PM(f)\phi, \tag{2.5}$$

where $M(f)$ is the pointwise multiplication by f.

The operator $T_r(f)$ is called a Toeplitz operator with symbol f. Explicitly,

$$T_r(f)\phi(\zeta) = \int K^r(\zeta, \eta)f(\eta)\phi(\eta)\,\mathrm{d}\mu_r(\eta). \tag{2.6}$$

It was shown in [8] that the map $f \to T_r(f)$ is a quantum deformation of $\mathbb{C}$ in the sense discussed in [13]. The $\mathbb{C}^*$-algebra $\mathscr{A}_r(\mathbb{C})$ generated by $T_r(f)$ with f vanishing at infinity is isomorphic to the $\mathbb{C}^*$-algebra $\mathscr{K}$ of compact operators in $H^2(\mathbb{C}, \mathrm{d}\mu_r)$.

Let now Γ be the additive group of complex numbers $\{2\pi in \mid n \in \mathbb{Z}\}$. The group Γ acts on $\mathbb{C}$ by translations and $\mathbb{C}/\Gamma \cong \mathbb{C}^* = \mathbb{C} - \{0\}$. The covering map p is given by

$$p\colon \mathbb{C} \ni \zeta \to \mathrm{e}^{\zeta} \in \mathbb{C}^*. \tag{2.7}$$

We will identify functions on $\mathbb{C}^*$ with Γ invariant functions on $\mathbb{C}$. Consequently, a bounded continuous function f on $\mathbb{C}^*$ defines the Toeplitz operator $T_r(f)$ in $H^2(\mathbb{C}, \mathrm{d}\mu_r)$.

THEOREM 2.1 *With the above definitions, the correspondence*

$$f \to T_r(f)$$

is a quantum deformation of $\mathbb{C}^*$.

This theorem is a special case of the general result of [15]. Our next main result, which will be proven in Section 4 is the following theorem.

THEOREM 2.2. *Let* $\mathscr{A}_r(\mathbb{C}^*)$ *be the* $\mathbb{C}^*$*-algebra generated by the Toeplitz operators* $T_r(f)$ *where the symbols* f *are continuous, compactly supported functions on* $\mathbb{C}^*$. *Then*

$$\mathscr{A}_r(\mathbb{C}^*) \cong C(S^1) \otimes \mathscr{K}.$$

Now, we present a preliminary discussion of the structure of Toeplitz operators with Γ-invariant symbols. Set $U = U^r(2\pi i)$. Observe that if f is a Γ-invariant function on $\mathbb{C}$, then we have

$$UT_r(f) = T_r(f)U. \tag{2.8}$$

This relation implies that $T_r(f)$ is diagonal with respect to the spectral decomposition of U.

DEFINITION 2.3. A holomorphic function ϕ on $\mathbb{C}$ is called an automorphic function with respect to Γ and with multiplier $e^{i\theta}$, if

$$U\phi(\zeta) = e^{i\theta}\phi(\zeta).$$

One can easily verify that automorphic functions exist. In fact, the space of automorphic functions with a given multiplier is infinite dimensional (see Section 4). We denote by $H^2_\theta(\mathbb{C}^*)$ the Hilbert space of automorphic functions with respect to Γ and with multiplier $e^{i\theta}$ which satisfy

$$\int_D |\phi(\zeta)|^2 \, d\mu_r(\zeta) < \infty. \tag{2.9}$$

Here D is a fundamental domain for the action of Γ on $\mathbb{C}$. The above expression, defining the norm in $H^2_\theta(\mathbb{C}^*)$, is independent on the choice of D.

PROPOSITION 2.4. *The map* $P\colon H^2(\mathbb{C}, d\mu_r) \to \int^\oplus_{S^1} H^2_\theta(\mathbb{C}^*)\, d\theta$ *given by*

$$P\phi(\zeta, \theta) = \sum_{n\in\mathbb{Z}} e^{-in\theta} U^n \phi(\zeta)$$

is an isomorphism. Furthermore, for $\psi \in \int^\oplus_{S^1} H^2_\theta(\mathbb{C}^*)\, d\theta$,

$$PUP^{-1}\psi(\zeta, \theta) = e^{i\theta}\psi(\zeta, \theta). \tag{2.10}$$

Proof. We first verify that P is an isometry:

$$\begin{aligned}
\|P\phi\|^2 &= \int_{-\pi}^{\pi} \left(\int_D |P\phi(\zeta, \theta)|^2 \, d\mu_r(\zeta) \right) \frac{d\theta}{2\pi} \\
&= \int_{-\pi}^{\pi} \left(\int_D \sum_{n,m} \overline{U^n\phi(\zeta)}\, U^m\phi(\zeta)\, e^{i(n-m)\theta} \, d\mu_r(\zeta) \right) \frac{d\theta}{2\pi} \\
&= \int_D \sum_n |U^n\phi(\zeta)|^2 \, d\mu_r(\zeta) \\
&= \sum_n \int_{D+2\pi in} |\phi(\zeta)|^2 \, d\mu_r(\zeta) \\
&= \int_{\mathbb{C}} |\phi(\zeta)|^2 \, d\mu_r(\zeta).
\end{aligned}$$

Similar calculations show that the inverse of P is given by

$$P^{-1}\psi(\zeta) = \int_{-\pi}^{\pi} \psi(\zeta, \theta)\, d\theta. \tag{2.11}$$

Verification of (2.10) is straightforward. □

The example of

$$\phi_\theta(\zeta) = \exp\left[-\frac{r}{2}\zeta^2 - \zeta\frac{\theta}{2\pi}\right]$$

shows that there are nowhere vanishing automorphic functions. Any other automorphic function differs from ϕ_θ by a periodic factor, that is by a holomorphic function on $\mathbb{C}^*$. We can thus identify $H^2_\theta(\mathbb{C}^*)$ with the space of holomorphic functions on $\mathbb{C}^*$ which are square integrable with respect to an appropriate measure $\mathrm{d}\mu_r^\theta$. This point is further discussed in Section 4.

Remark. Notice that the Γ-invariant functions $\mathrm{e}^\zeta, \mathrm{e}^{\bar{\zeta}}$ generate the ring of continuous functions on $\mathbb{C}^*$ which have limits at the boundary. The corresponding (unbounded) Toeplitz operators satisfy the relation

$$T_r(\mathrm{e}^{\bar{\zeta}})T_r(\mathrm{e}^\zeta) = \mathrm{e}^{1/r}T_r(\mathrm{e}^\zeta)T_r(\mathrm{e}^{\bar{\zeta}}).$$

This relation, written in the form $xy = qyx$, is often referred to as a relation defining 'the quantum plane'.

We will now discuss the quantum tori. Consider the additive group of complex numbers $\Gamma_\tau = \{n + \tau m \mid n, m \in \mathbb{Z}\}$ where $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$. The group Γ_τ acts on $\mathbb{C}$, and the quotient space $\mathbb{C}/\Gamma_\tau$ is a torus. Functions on $\mathbb{C}/\Gamma_\tau$ can be identified with Γ_τ-invariant functions on $\mathbb{C}$. The following theorem was proven in [15] (it also follows from the results of [2]).

THEOREM 2.5. *Let f be a continuous function on $\mathbb{C}/\Gamma_\tau$. The correspondence*

$$f \to T_r(f)$$

is a quantum deformation of $\mathbb{C}/\Gamma_\tau$.

Below we study the structure of the $\mathbb{C}^*$-algebra $\mathscr{A}_r(\mathbb{C}/\Gamma_\tau)$ generated by the Toeplitz operators $T_r(f)$ where the symbols f are continuous functions on $\mathbb{C}/\Gamma_\tau$. Observe that for $a \in \Gamma_\tau$ and a Γ_τ-invariant function f on $\mathbb{C}$, we have

$$U^r(a)T_r(f) = T_r(f)U^r(a). \tag{2.12}$$

As a consequence, the algebra $\mathscr{A}_r(\mathbb{C}/\Gamma_\tau)$ is a subalgebra of the fix point algebra of the projective representation $\Gamma_\tau \ni a \to U^r(a)$ of Γ_τ.

Set $V = U^r(1)$ and $W = U^r(\tau)$. Obviously, V and W generate $U^r(\Gamma_\tau)$. A simple calculation using

$$U^r(a)U^r(b) = \mathrm{e}^{2ir\mathrm{Im}(\bar{a}b)}U^r(b)U^r(a) \tag{2.13}$$

shows that

$$VW = \mathrm{e}^{2ir\mathrm{Im}(\tau)}WV. \tag{2.14}$$

Consider now the following operators:

$$M = U^r\left(\frac{\pi}{r\,\mathrm{Im}(\tau)}\right),$$

$$N = U^r\left(\frac{\pi\tau}{r\,\mathrm{Im}(\tau)}\right).$$

Using the commutation relation (2.13), we verify that both M and N commute with V and W. Furthermore, they satisfy the relation

$$MN = \mathrm{e}^{2\pi^2 i/r\mathrm{Im}(\tau)}NM. \tag{2.15}$$

PROPOSITION 2.6. *The operators M and N are Toeplitz operators with Γ_τ-invariant symbols. Furthermore, the $\mathbb{C}^*$-algebra $\mathscr{A}_r(\mathbb{C}/\Gamma_\tau)$ is generated by M and N.*

Proof. Consider

$$m(\zeta) = \exp\left(\frac{\pi^2}{2r(\mathrm{Im}(\tau))^2}\right)\exp\left(\frac{\pi}{\mathrm{Im}(\tau)}(\zeta - \bar{\zeta})\right),$$

$$n(\zeta) = \exp\left(\frac{\pi^2|\tau|^2}{2r(\mathrm{Im}(\tau))^2}\right)\exp\left(\frac{\pi}{\mathrm{Im}(\tau)}(\bar{\tau}\zeta - \tau\bar{\zeta})\right).$$

We verify directly that $M = T_r(m)$ and $N = T_r(m)$. It is also clear that m and n are Γ_τ-invariant.

Since $\|T_r(f)\| \leqslant \|f\|_\infty$ and since, by the Stone–Weierstrass theorem, the functions m and n generate the algebra of continuous functions on $\mathbb{C}/\Gamma_\tau$, we can conclude that $\mathscr{A}_r(\mathbb{C}/\Gamma_\tau)$ is generated by M and N. □

We claim that the relation (2.15) is the only relation between M and N and that $\mathscr{A}_r(\mathbb{C}/\Gamma_\tau)$ is universal with respect to this relation. More precisely, we have the following theorem.

THEOREM 2.7. *Let $\mathscr{B}_\alpha$ be the universal $\mathbb{C}^*$-algebra generated by two unitary operators M, N obeying the relation*

$$MN = \mathrm{e}^{i\alpha}NM.$$

Then

$$\mathscr{A}_r(\mathbb{C}/\Gamma_\tau) \cong \mathscr{B}_{(2\pi^2/r\mathrm{Im}(\tau))}.$$

Remark. The algebras $\mathscr{B}_\alpha$ are called quantum tori and were studied in detail in [9], [2] and [15]. Theorem 2.7 can be viewed as a uniformization theorem for quantum tori.

Proof. Since V commutes with both M and N, we can perform spectral decomposition of V to obtain invariant subspaces for M and N. Proceeding as in the proof of

Proposition 2.4, we obtain the decomposition

$$H^2(\mathbb{C}, \mathrm{d}\mu_r) \cong \int_{S^1}^{\oplus} H_\theta^2 \, \mathrm{d}\theta,$$

where H_θ^2 is now the Hilbert space of holomorphic functions ϕ on $\mathbb{C}$ such that

$$V\phi(\zeta) = \mathrm{e}^{i\theta}\phi(\zeta) \quad \text{and} \quad \|\phi\|_\theta^2 = \int_D |\phi(\zeta)|^2 \, \mathrm{d}\mu_r(\zeta) < \infty.$$

For definiteness, we choose $D = \{\zeta : 0 \leqslant \operatorname{Re}\zeta < 1\}$. The operators M and N are diagonal with respect to the above decomposition, and they are again given by the same formulas as the corresponding formulas on $H^2(\mathbb{C}, \mathrm{d}\mu_r)$.

Consider now $\phi_\theta \in H_\theta^2$ given by

$$\phi_\theta(\zeta) = \mathrm{e}^{(r/2)\zeta^2 - i\zeta\theta} c_\theta, \tag{2.16}$$

where the constant c_θ is chosen so that $\|\phi_\theta\|_\theta^2 = 1$. Since ϕ_θ is nowhere vanishing, any other element of H_θ^2 is of the form $\psi(\zeta)\phi_\theta(\zeta)$ where ψ is periodic,

$$\psi(\zeta - 1) = \psi(\zeta).$$

Observe that:

$$\begin{aligned} M\phi_\theta(\zeta) &= c_\theta \exp\left(r\zeta\left(\frac{\pi}{r \operatorname{Im}\tau}\right) - \frac{r}{2}\left(\frac{\pi}{r \operatorname{Im}\tau}\right)^2 + \frac{r}{2}\left(\zeta - \frac{\pi}{r \operatorname{Im}\tau}\right)^2 - i\theta\left(\zeta - \frac{\pi}{r \operatorname{Im}\tau}\right)\right) \\ &= \exp\left(i\theta \frac{\pi}{r \operatorname{Im}\tau}\right)\phi_\theta(\zeta). \end{aligned} \tag{2.17}$$

We claim that $e_k = N^k\phi_\theta, k \in \mathbb{Z}$, is an orthonormal basis for H_θ^2. In fact, a computation similar to the one in (2.17) shows that

$$N^k\phi_\theta(\zeta) = \operatorname{const} \mathrm{e}^{-2ik\pi\zeta}\phi_\theta(\zeta), \tag{2.18}$$

where const denotes a factor which is independent of ζ. The collection $\{N^k\phi_\theta(\zeta); k \in \mathbb{Z}\}$ forms a total set in H_θ^2, since the functions $\mathrm{e}^{-2ik\pi\zeta}$ span the space of holomorphic periodic functions on $\mathbb{C}$. Furthermore,

$$\begin{aligned} (N^k\phi_\theta, N^l\phi_\theta) &= (\phi_\theta, N^{l-k}\phi_\theta) \\ &= \operatorname{const} \int_D \mathrm{e}^{2i(k-l)\pi\zeta} \, \mathrm{e}^{(r/2)(\zeta-\bar\zeta)^2 - i\theta(\zeta-\bar\zeta)} \, \mathrm{d}^2\zeta. \end{aligned} \tag{2.19}$$

The integral over $\operatorname{Re}(\zeta)$ in (2.19) vanishes unless $k = l$, in which case the above scalar product is equal to 1. We have

$$Ne_k = e_{k+1},$$

$$Me_k = MN^k\phi_\theta = \exp\left(\frac{2\pi^2 ik}{r\,\mathrm{Im}(\tau)}\right)N^kM\phi_\theta$$

$$= \exp\left(\frac{i\theta\pi}{r\,\mathrm{Im}(\tau)} + \frac{2\pi^2 ik}{r\,\mathrm{Im}(\tau)}\right)N^k\phi_\theta = \exp\left(\frac{i\theta\pi}{r\,\mathrm{Im}(\tau)} + \frac{2\pi^2 ik}{r\,\mathrm{Im}(\tau)}\right)e_k. \tag{2.20}$$

We can therefore identify $H^2_\theta \cong L^2(S^1)$ in such a way that the operators M and N become

$$Nf(\alpha) = \mathrm{e}^{i\alpha}f(\alpha),$$

$$Mf(\alpha) = \exp\left(\frac{i\theta\pi}{r\,\mathrm{Im}(\tau)}\right)f\left(\alpha + \frac{2\pi^2}{r\,\mathrm{Im}(\tau)}\right). \tag{2.21}$$

This concludes the proof since the direct integral $\int^{\oplus}_{S^1} H^2_\theta\,\mathrm{d}\theta$ contains every irreducible representation of $\mathscr{B}_{(2\pi^2/r\mathrm{Im}(\tau))}$ (see [16] and references therein). $\square$

3. Uniformization of Exceptional Hyperbolic Quantum Riemann Surfaces

A Riemann surface M is called hyperbolic if its holomorphic universal covering space is the unit disk $\mathbb{U}$. If, additionally, $\pi_1(M)$ is commutative then M is isomorphic to either $\mathbb{U}$ or the punctured unit disk $\mathbb{U}^* = \mathbb{U}\setminus\{0\}$ or the annulus $\mathbb{U}_\rho = \{\zeta \in \mathbb{C}: \rho < |\zeta| < 1\}$ $0 < \rho < 1$ [12]. Such Riemann surfaces are called exceptional hyperbolic Riemann surfaces.

In this section, we study Toeplitz quantization of these Riemann surfaces. We begin by recalling the Toeplitz quantization of $\mathbb{U}$ [13]. Then we study Toeplitz operators on $\mathbb{U}$ with $\pi_1(M)$-invariant symbols. As it is explained in Section IV, this approach turns out to be related to geometric quantization. Thus, our theorems describing quantum $\mathbb{U}^*$ and $\mathbb{U}_\rho$ can be viewed as uniformization theorems.

Let $\mathrm{d}\mu_r$ be the following probabilistic measure on $\mathbb{U}$:

$$\mathrm{d}\mu_r = \frac{r-1}{\pi}(1-|\zeta|^2)^{r-2}\,\mathrm{d}^2\zeta. \tag{3.1}$$

Let $H^2(\mathbb{U}, \mathrm{d}\mu_r)$ be the closed subspace of $L^2(\mathbb{U}, \mathrm{d}\mu_r)$ consisting of holomorphic functions. For a continuous bounded function f on $\mathbb{U}$, we define the operator $T_r(f)$ in $H^2(\mathbb{U}, \mathrm{d}\mu_r)$ by

$$T_r(f)\phi(\zeta) = \int_{\mathbb{U}} K^r(\zeta,\eta)f(\eta)\phi(\eta)\,\mathrm{d}\mu_r(\eta), \tag{3.2}$$

where $K^r(\zeta,\eta)$ is the integral kernel of the orthogonal projection

$$P: L^2(\mathbb{U}, \mathrm{d}\mu_r) \to H^2(\mathbb{U}, \mathrm{d}\mu_r),$$

so that $T_r(f)$ is the multiplication by f followed by the projection P. Explicitly,

$$K^r(\zeta,\eta) = (1-\zeta\bar{\eta})^{-r}. \tag{3.3}$$

The group of biholomorphisms of $\mathbb{U}$ can be projectively and unitarily represented in $H^2(\mathbb{U}, \mathrm{d}\mu_r)$ by

$$U^r(\gamma)\phi(\zeta) = ((\gamma^{-1})'(\zeta))^{r/2}\phi(\gamma^{-1}\zeta), \quad \gamma \in \mathrm{SU}(1,1). \tag{3.4}$$

It was shown in [13] that the mapping $f \to T_r(f)$ is a quantum deformation of $\mathbb{U}$. Moreover, the $\mathbb{C}^*$-algebra $\mathscr{A}_r(\mathbb{U})$ generated by the Toeplitz operators $T_r(f)$ with symbols f vanishing at $S^1 = \partial\bar{\mathbb{U}}$ was shown to be isomorphic to $\mathscr{K}$, the $\mathbb{C}^*$-algebra of compact operators on a separable infinite-dimensional Hilbert space. The $\mathbb{C}^*$-algebra $\mathscr{A}_r(\bar{\mathbb{U}})$ generated by Toeplitz operators with symbols continuous on the closed unit disk $\bar{\mathbb{U}}$ is an extension of $\mathscr{K}$:

$$0 \to \mathscr{K} \to \mathscr{A}_r(\bar{\mathbb{U}}) \to C(S^1) \to 0. \tag{3.5}$$

To discuss the uniformization, it is convenient to identify $\mathbb{U}$ with the upper half plane $\mathbb{H}$ via the biholomorphic map

$$\begin{aligned} \mathbb{U} \ni \zeta \to j(\zeta) &= i\frac{1+\zeta}{1-\zeta} \in \mathbb{H}, \\ j^{-1}(\eta) &= \frac{\eta - i}{\eta + i}. \end{aligned} \tag{3.6}$$

We set

$$\mathrm{d}v_r = \frac{\pi}{4^{r-1}(r-1)}(\mathrm{Im}\,\zeta)^{r-2}\,\mathrm{d}^2\zeta \tag{3.7}$$

and consider the space $H^2(\mathbb{H}, \mathrm{d}v_r)$ of holomorphic functions on $\mathbb{H}$ square integrable with respect to $\mathrm{d}v_r$. The corresponding Bergman kernel is given by

$$K^r_{\mathbb{H}}(\zeta, \eta) = \left(\frac{i}{2}\right)^r \frac{1}{(\zeta - \bar{\eta})^r} \tag{3.8}$$

For a bounded continuous function f on $\mathbb{H}$ we denote by $T^r_{\mathbb{H}}(f)$ the Toeplitz operator in $H^2(\mathbb{H}, \mathrm{d}v_r)$ with symbol f.

PROPOSITION 3.1. *The map* $J\colon H^2(\mathbb{U}, \mathrm{d}\mu_r) \to H^2(\mathbb{H}, \mathrm{d}v_r)$ *given by*:

$$J\phi(\zeta) = (\zeta + i)^{-r}\phi\left(\frac{\zeta - i}{\zeta + i}\right)$$

is an isomorphism. Furthermore,

$$J^{-1}T^r_{\mathbb{H}}(f)J = T_r(f \circ j).$$

Proof. The claim follows by simple calculations which we do not reproduce here. □

The advantage of working with the upper half plane rather than with the unit disk is that the uniformizing groups are easier to write down explicitly on the former.

The group $SL(2, \mathbb{R})$ of biholomorphisms of $\mathbb{H}$ acts projectively and unitarily on $H^2(\mathbb{H}, dv_r)$ by a formula analogous to (3.4).

Consider the following subgroup Γ of $SL(2, \mathbb{R})$

$$\begin{aligned} \Gamma &= \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, n \in \mathbb{Z} \right\} \\ U(n)\phi(\zeta) &= \phi(\zeta - n). \end{aligned} \tag{3.9}$$

The quotient $\mathbb{H}/\Gamma$ is isomorphic with $\mathbb{U}^*$. The covering map s is given by

$$s: \mathbb{H} \ni \zeta \to e^{2\pi i \zeta} \in \mathbb{U}^* \tag{3.10}$$

Toeplitz quantization of $\mathbb{U}^*$ is, by definition, the correspondence $f \to T^r_{\mathbb{H}}(f)$ where f's are bounded, continuous and Γ-invariant functions on $\mathbb{H}$. The deformation estimates were established in [15]. Here we want to study the structure of the $\mathbb{C}^*$-algebra generated by such Toeplitz operators. To this end, we observe that $UT^r_{\mathbb{H}}(f) = T^r_{\mathbb{H}}(f)U$ where $U := U(1)$ (see formula 3.9) and perform the spectral decomposition of the unitary operator U. Following the method of Section 2, we consider the Hilbert space $H^2_\theta(\mathbb{H})$ of holomorphic functions ϕ on $\mathbb{H}$ such that

$$\begin{aligned} &U\phi(\zeta) = \phi(\zeta - 1) = e^{i\theta}\phi(\zeta), \\ &\int_D |\phi(\zeta)|^2 \, dv_r(\zeta) < \infty, \end{aligned} \tag{3.11}$$

where D is a fundamental domain for the action of Γ on $\mathbb{H}$. Reasoning as in the proof of Proposition 2.4, we obtain the isomorphism

$$H^2(\mathbb{H}, dv_r) \cong \int_{S^1}^{\oplus} H^2_\theta(\mathbb{H}) \, d\theta.$$

Observe next that the function $\zeta \to e^{-i\theta\zeta}$ is a nowhere vanishing element of $H^2_\theta(\mathbb{H})$. Finally, let $H^2(\mathbb{U}^*, d\mu_\theta)$ be the Hilbert space of holomorphic functions on $\mathbb{U}^*$ square integrable with respect to the measure

$$d\mu_\theta = \frac{1}{(r-1)(16\pi)^{r-1}} \left(\log \frac{1}{|\zeta|^2} \right)^{r-2} |\zeta|^{-(\theta/\pi)-2} \, d^2\zeta.$$

As a consequence of the above discussion we obtain the following result.

PROPOSITION 3.2. *The map* $S: H^2(\mathbb{U}^*, d\mu_\theta) \to H^2_\theta(\mathbb{H})$ *given by*

$$S\phi(\zeta) = e^{-i\theta\zeta}\phi(e^{2\pi i\zeta})$$

is an isomorphism of Hilbert spaces. Furthermore, if f is a continuous function on $\mathbb{U}^*$, then

$$S^{-1}T^r_{\mathbb{H}}(f)S = \int_{S^1}^{\oplus} T^r_\theta(f \circ s)\,\mathrm{d}\theta,$$

where $T^r_\theta(f \circ s)$ is the Toeplitz operator in $H^2_\theta(\mathbb{H})$ with symbol $f \circ s$ (see 2.10).

Proof. The claims follow by straightforward computations which we omit. □

This discussion enables us to describe the structure of $\mathbb{C}^*$-algebras generated by Toeplitz operators on $\mathbb{U}^*$. In what follows, we identify functions on $\mathbb{U}^*$ with Γ-invariant functions on $\mathbb{H}$. Let $\mathscr{A}_r(\mathbb{U}^*)$ be the $\mathbb{C}^*$-algebra generated by the Toeplitz operators with symbols $f\colon \mathbb{U}^* \to \mathbb{C}$, vanishing at the boundary of $\mathbb{U}^*$ and let $\mathscr{A}_r(\bar{\mathbb{U}}^*)$ be the $\mathbb{C}^*$-algebra generated by Toeplitz operators on $\bar{\mathbb{U}}^*$, the closure of $\mathbb{U}^*$ in $\mathbb{C}$.

THEOREM 3.3. *We have the exact sequence*

$$0 \to \mathscr{A}_r(\mathbb{U}^*) \to \mathscr{A}_r(\bar{\mathbb{U}}^*) \to C(S^1) \oplus \mathbb{C} \to 0.$$

Furthermore,

$$\mathscr{A}_r(\mathbb{U}^*) \cong C(S^1) \otimes \mathscr{K}.$$

Proof. The $\mathbb{C}^*$-algebra generated by $T^r_\theta(f)$ in $H^2(\mathbb{U}^*, \mathrm{d}\mu_\theta)$, with symbols f continuous on $\bar{\mathbb{U}}^*$, is generated by a single operator $T^r_\theta(\zeta)$. This follows from the Stone–Weierstrass theorem and the estimate

$$\| T^r_\theta(f) \| \leqslant \| f \|_\infty .$$

The operator $T^r_\theta(\zeta)$ is a bilateral shift and so we have the exact sequence [10]:

$$0 \to \mathscr{K} \to \mathscr{A}^\theta_r \to C(S^1) \oplus \mathbb{C} \to 0.$$

Furthermore, $T^r_\theta(f)$ is compact if and only if $f(0) = 0$ and $f(\zeta) = 0$ for every ζ such that $|\zeta| = 1$. In fact, such Toeplitz operators generate all of $\mathscr{K}$. These statements follow from considerations analogous to those in [13]. As a consequence, we have the following short exact sequence

$$0 \to \mathscr{A}_r(\mathbb{U}^*) \to \mathscr{A}_r(\bar{\mathbb{U}}^*) \to C(S^1) \oplus \mathbb{C} \to 0.$$

Observe now that according to Proposition 3.2, $\mathscr{A}_r(\mathbb{U}^*)$ can be identified with a subalgebra of $C(S^1, \mathscr{K}) \cong C(S^1) \otimes \mathscr{K}$. Since $C(S^1, \mathscr{K})$ is type I, and $\mathscr{A}_r(\mathbb{U}^*)$ is rich, we conclude [11] that

$$\mathscr{A}_r(\mathbb{U}^*) \cong C(S^1) \otimes \mathscr{K}. \qquad \square$$

Consider now the Abelian subgroup Γ_λ of $\mathrm{SL}(2, \mathbb{R})$ generated by the following biholomorphism of $\mathbb{H}\colon \zeta \to \lambda\zeta$, $\lambda > 1$. The quotient $\mathbb{H}/\Gamma_\lambda$ is isomorphic to $\mathbb{U}_\rho$, where

$\rho = \exp(-2\pi^2/\log\lambda)$. The covering map $\mathbb{H} \to U_\rho$ is given by [12]

$$\zeta \to \exp\left(2\pi i \frac{\log\zeta}{\log\lambda}\right). \tag{3.12}$$

Let $\mathscr{A}_r(\mathbb{U}_\rho)$ be the $\mathbb{C}^*$-algebra generated by the Toeplitz operators with symbols $f: \mathbb{U}_\rho \to \mathbb{C}$, vanishing at the boundary of $\mathbb{U}_\rho$ and let $\mathscr{A}_r(\bar{\mathbb{U}}_\rho)$ be the $\mathbb{C}^*$-algebra generated by Toeplitz operators on $\bar{\mathbb{U}}_\rho$, the closure of $\mathbb{U}_\rho$ in $\mathbb{C}$. We proved in [15] that the correspondence $f \to T^r_{\mathbb{H}}(f) \in \mathscr{A}_r(\bar{\mathbb{U}}_\rho)$ is a deformation quantization of $\bar{\mathbb{U}}_\rho$. Arguments similar to those used in the proof of Theorem 3.3 lead now to the following statement.

THEOREM 3.4. *We have the exact sequence*

$$0 \to \mathscr{A}_r(\mathbb{U}_\rho) \to \mathscr{A}_r(\bar{\mathbb{U}}_\rho) \to C(S^1) \oplus C(S^1) \to 0.$$

Furthermore, we have the isomorphism

$$\mathscr{A}_r(\mathbb{U}_\rho) \cong C(S^1) \otimes \mathscr{K}.$$

4. Geometric Quantization

It is well known that the quantum deformations of $\mathbb{C}$ and $\mathbb{U}$ used in Sections 2 and 3 are closely related to the geometric quantization prescription with holomorphic polarization, see, e.g., [2]. The purpose of this section is to compare the definitions of exceptional quantum Riemann surfaces given in the previous sections with those of geometric quantization. The statement that the two approaches lead to the same Hilbert space (of physical states) can be viewed as a uniformization principle. In what follows we prove that such uniformization theorems are indeed true, with obvious restrictions that we will now discuss.

The situation is more subtle if M is compact. For one thing, the structure of $\mathscr{A}_r(M)$ strongly depends on r. Geometric quantization is meaningful only for a discrete sequence of values of r, namely those dictated by the quantization condition. In those cases $\mathscr{A}_r(M)$ is a direct integral of finite dimensional full matrix algebras and, as we will see, both quantization methods coincide (at least for the tori). Uniformization method is, at least in principle, more general as it does not require any topological condition on r in $\mathscr{A}_r(M)$.

In this section, we will concentrate on two representative examples: $\mathbb{C}^*$ and tori. The details of the other examples discussed in this Letter, namely $\mathbb{U}^*$ and $\mathbb{U}_\rho$ are essentially identical to those of $\mathbb{C}^*$ and we leave them out.

We begin with a brief discussion of geometric quantization of a complex manifold M in the direction of a symplectic form ω [18]. First, one considers hermitian holomorphic line bundles $L, \langle\,,\rangle$ such that

$$\operatorname{curv}(\nabla) = r\omega, \tag{4.1}$$

where ∇ is the canonical complex connection associated to $\langle\,,\rangle$, and where $\operatorname{curv}(\nabla)$ is the curvature of ∇.

Such bundles exist only if

$$c_1(L) = \frac{i}{2\pi} r\omega \in H^2(M, \mathbb{Z}), \tag{4.2}$$

and are classified by the characters χ of the fundamental group of M. The choice of holomorphic polarization amounts to the following choice of a Hilbert space:

$$\mathscr{H} = \int_{\chi}^{\oplus} H^2(L_\chi, \langle\,,\rangle_\chi \omega^n)\, \mathrm{d}\chi. \tag{4.3}$$

Here $H^2(L_\chi, \langle\,,\rangle_\chi \omega^n)$ is the Hilbert space of holomorphic sections s of L_χ such that

$$\int_M \langle s, s\rangle_\chi(\zeta)\omega^n(\zeta) < \infty,$$

where n is the complex dimension of M. The quantization map is simply the correspondence:

$$f \mapsto \int T_r^\chi(f)\, \mathrm{d}\chi, \tag{4.4}$$

where $T_r^\chi(f)$ are Toeplitz operators on the space $H^2(L_\chi, \langle\,,\rangle_\chi\, \omega^n)$ with symbols $f \in C(M)$.

We will now discuss the geometric quantization procedure for $\mathbb{C}^*$ viewed as the quotient $\mathbb{C}/\{2\pi i n, n \in \mathbb{Z}\}$. Since we deform $\mathbb{C}$ in the direction of the invariant symplectic form

$$\omega = i\, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}, \tag{4.5}$$

we choose the following form on $\mathbb{C}^*$:

$$\omega_1 = \frac{i}{|\zeta|^2} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}. \tag{4.6}$$

Every holomorphic line bundle on $\mathbb{C}^*$ is trivial, and so the quantization condition is void. However, $\pi_1(\mathbb{C}^*) = \mathbb{Z}$, and different Hermitian structures are classified by S^1. Denoting $\langle s, t\rangle(\zeta) = \gamma(\zeta)\bar{s}(\zeta)t(\zeta)$, we obtain the equation

$$\partial\bar{\partial} \log \gamma(\zeta) = -\frac{r}{|\zeta|^2}.$$

Solving the above equation yields, up to equivalence, the following expression for γ:

$$\gamma_\theta(\zeta) = \text{const}\, \exp\!\left(-\frac{r}{2} \log^2 |\eta|^2\right) |\zeta|^{-(\theta/\pi)}.$$

Furthermore, γ_θ is equivalent to $\gamma_{\theta'}$ iff $\theta - \theta' \in 2\pi\mathbb{Z}$. As a consequence, the Hilbert space of geometric quantization is

$$\mathscr{H} = \int_{S^1}^{\oplus} H^2(\mathbb{C}^*, \mathrm{d}\mu_r^\theta)\,\mathrm{d}\theta,$$

where

$$\mathrm{d}\mu_r^\theta = \text{const}\,\exp\left(-\frac{r}{2}\log^2|\eta|^2\right)|\zeta|^{-(\theta/\pi)-2}\,\mathrm{d}^2\zeta.$$

We then construct Toeplitz operators $T_\theta^r(f)$ in the usual fashion.

Recall now that we constructed in Proposition 2.4 an isomorphism

$$P: H^2(\mathbb{C}, \mathrm{d}\mu_r) \cong \int_{S^1}^{\oplus} H_\theta^2(\mathbb{C}^*)\,\mathrm{d}\theta.$$

PROPOSITION 4.1. *The map*

$$R: H^2(\mathbb{C}^*, \mathrm{d}\mu_r^\theta) \mapsto H_\theta^2(\mathbb{C}^*)$$

given by

$$R\phi(\zeta) = \exp\left(-\frac{r}{2}\zeta^2 - \frac{\theta\zeta}{2\pi}\right)\phi(\mathrm{e}^\zeta)$$

is an isomorphism. Furthermore,

$$PT_r(f \circ p)P^{-1} = \int_{S^1}^{\oplus} RT_\theta^r(f)R^{-1}\,\mathrm{d}\theta.$$

Proof. We omit the elementary calculations. □

It is a standard fact in the theory of Toeplitz operators that $T_\theta^r(f)$ is compact if and only if f vanishes at the boundary of $\mathbb{C}^*$ and that such $T_\theta^r(f)$ generate $\mathscr{K}$. This means that $\mathscr{A}_r(\mathbb{C}^*)$ is a subalgebra of $C(S^1) \otimes \mathscr{K}$. But $\mathscr{A}_r(\mathbb{C}^*)$ is clearly rich and so we conclude [11] that

$$\mathscr{A}_r(\mathbb{C}^*) \cong C(S^1) \otimes \mathscr{K}.$$

This proves Theorem 2.2.

Our second example is that of tori. In the following, we use the notation of Section 2. The symplectic form on $\mathbb{C}/\Gamma_\tau$ is given by the same expression as the symplectic form on $\mathbb{C}$, namely (4.5). The quantization condition (4.2) leads to the following restriction on r:

$$\frac{r\,\mathrm{Im}\,\tau}{\pi} \in \mathbb{N}. \tag{4.7}$$

Comparing (4.7) with (2.14) shows that for such r's, the projective representation $U^r(\gamma)$ of Γ_τ in $H^2(\mathbb{C}, d\mu_r)$ is a genuine representation.

Let χ be a character of Γ_τ. The Hilbert space of (4.3) can be identified with the following direct integral

$$\mathscr{H} = \int_\chi^\oplus H_\chi^2(\mathbb{C}/\Gamma_\tau)\, d\chi,$$

where $d\chi$ is the Haar measure on the dual of Γ_τ. Here, $H_\chi^2(\mathbb{C}/\Gamma_\tau)$ is the space of holomorphic functions ϕ on $\mathbb{C}$ such that for every $\gamma \in \Gamma_\tau$,

$$U^r(\gamma)\phi(\zeta) = \chi(\gamma)\phi(\zeta).$$

The norm on this space is given by

$$\|\phi\|^2 = \int_D |\phi(\zeta)|^2\, d\mu_r(\zeta),$$

where D is a fundamental domain for Γ_τ. This construction is well known from the theory of automorphic forms. We denote the Toeplitz operators in $H_\chi^2(\mathbb{C}/\Gamma_\tau)$ by $T_\chi^r(f)$. Finally, let k be the standard projection $\mathbb{C} \mapsto \mathbb{C}/\Gamma_\tau$.

THEOREM 4.2. *The map*

$$K\colon H^2(\mathbb{C}, d\mu_r) \mapsto \mathscr{H}$$

given by

$$K\phi(\chi, \zeta) = \sum_{\gamma\in\Gamma_\tau} \chi(\gamma^{-1})U^r(\gamma)\phi(\zeta)$$

is an isomorphism. Furthermore,

$$KT_r(f \circ k)K^{-1} = \int_\chi^\oplus T_\chi^r(f)\, d\chi.$$

Proof. The proof is a straightforward calculation similar to that leading to Proposition 2.4 and involving Fourier analysis on Γ_τ. □

References

1. Berezin, F. A., General concept of quantization, *Comm. Math. Phys.* **40**, 153–174 (1975).
2. Bordeman, M., Hoppe, J., Schaller, P., and Schlichenmaier, M., gl(∞) and geometric quantization, *Comm. Math. Phys.* **138**, 209–244 (1991).
3. Bordeman, M., Meinrenken, E., and Schlichenmaier, M., Toeplitz quantization of Kähler manifolds and gl(N), N → ∞, Preprint.
4. Borthwick, D., Klimek, S., Lesniewski, A., and Rinaldi, M., Super Toeplitz operators and non-perturbative deformation quantization of supermanifolds, *Comm. Math. Phys.* **153**, 49–76 (1993).
5. Borthwick, D., Klimek, S., Lesniewski, A., and Rinaldi, M., Matrix Cartan superdomains, Super Toeplitz operators and deformation quantization, to appear.
6. Borthwick, D., Klimek, S., Lesniewski, A., and Rinaldi, M., Supersymmetry and Fredholm modules over quantized spaces, to appear.

7. Borthwick, D., Lesniewski, A., and Upmeier, H., Non-perturbative deformation quantization of Cartan domains, *J. Funct. Anal.* **113**, 153–176 (1993).
8. Coburn, L. A., Deformation estimates for the Berezin-Toeplitz quantization, *Comm. Math. Phys.* **149**, 415–424 (1992).
9. Connes, A., Non-commutative differential geometry, *Publ. Math. IHES*, **62**, 94–144 (1986).
10. Conway, J., *Subnormal Operators*, Pitman, London, 1981.
11. Dixmier, J., $\mathbb{C}^*$-*Algebras*, North-Holland, Amsterdam, 1977.
12. Farkas, H. M. and Kra, I., *Riemann surfaces*, Springer, Berlin, 1980.
13. Klimek, S. and Lesniewski, A., Quantum Riemann surfaces, I. The unit disc, *Comm. Math. Phys.* **146**, 103–122 (1992).
14. Klimek, S., and Lesniewski, A., Quantum Riemann surfaces, II. The discrete series, *Lett. Math. Phys.* **24**, 125–139 (1992).
15. Klimek, S. and Lesniewski, A., Quantum Riemann surfaces, IV. Equivariant quantization, to appear.
16. Rieffel, M. A., Non-commutative tori – a case study of non-commutative differentiable manifolds, *Contemp. Math.* **105**, 191–211 (1990).
17. Rieffel, M. A., Quantization and $\mathbb{C}^*$-algebras, preprint.
18. Sniatycki, J., *Geometric Quantization and Quantum Mechanics*, Springer, Berlin, 1980.