A remark on the Casimir elements of Lie superalgebras and quantized Lie superalgebras

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It is shown that the second order Casimir operators C and C_q of the Lie superalgebra osp(1|2) and the quantized Lie superalgebra $\mathrm{osp}_q(1|2)$, respectively, possess natural square roots. © *1995 American Institute of Physics.*

I. INTRODUCTION

The second order Casimir operator of a Lie algebra can be interpreted as a Laplacian on the underlying Lie group or, in physical terms, a quantum mechanical Hamiltonian of a particle moving on the group. Similarly, the second order Casimir operator of a Lie superalgebra can be thought of as a quantum mechanical Hamiltonian of a particle moving on a Lie supergroup. A characteristic feature of supersymmetric quantum mechanics is that the Hamiltonian is a square of the supercharge.

In this note, we establish an analogous fact for the Casimir element C of the simplest Lie superalgebra, namely, osp(1|2): $C=Q^2$, where Q is a simple expression in the generators of osp(1|2). The "supercharge" Q is not in the center of the universal enveloping algebra of osp(1|2): while it commutes with the even elements, it anticommutes with the odd ones. Remarkably, these properties of the Casimir element are shared by the Casimir element of the quantized universal enveloping algebra $\mathrm{U}_q(\mathrm{osp}(1|2))$. In our discussion we assume that q is not a root of unity.

Our analysis relies on straightforward computations. The question arises, whether the existence of a square root of the Casimir element is a special property of osp(1|2) or whether it holds for all Lie superalgebras and quantized Lie superalgebras.

The note is organized as follows. In Sec. II, we construct and discuss the properties of the square root of the Casimir element of osp(1|2). An analogous analysis of the quantized case is presented in Sec. III.

II. THE LIE SUPERALGEBRA osp(1|2)

Recall[1,2] that the Lie superalgebra osp(1|2) is generated by five elements, L_+, L_-, L_3, G_+, and G_-, with the following parity assignments,

$$p(L_\pm)=p(L_3)=0, \quad p(G_\pm)=1, \tag{2.1}$$

and with the following relations,

$$\begin{gathered}[L_3, L_\pm]=\pm L_\pm, \quad [L_+, L_-]=2L_3, \quad [L_3, G_\pm]=\pm\tfrac{1}{2}G_\pm, \\ [G_+, G_-]=-\tfrac{1}{2}L_3, \quad [L_\pm, G_\mp]=G_\pm, \quad [L_\pm, G_\pm]=0, \quad [G_\pm, G_\pm]=\pm\tfrac{1}{2}L_\pm .\end{gathered} \tag{2.2}$$

The second order Casimir element in the universal enveloping superalgebra U(osp(1|2)) is given by

$$C=\tfrac{1}{2}(L_+L_-+L_-L_+)+L_3^2+G_+G_- -G_-G_+ +\tfrac{1}{16}, \tag{2.3}$$

where, for convenience, we have added the constant 1/16 to the usual expression.

Finite dimensional, irreducible representations of osp(1|2) are classified by the half-integers $s\in\{0,1/2,1,3/2,...\}$ and have the following structure. The vector space W_s carrying the irreducible representation labeled by the half-integer s is a direct sum,

$$W_s=V_s\oplus V_{s-1/2}, \tag{2.4}$$

where V_l, $l\in\{s,s-1/2\}$, carries the spin l representation of sl(2). The dimension of W_s is $4s+1$. The decomposition (2.4) defines a $\mathbb{Z}_2$-grading on W_s with the even subspace $W_s^0=V_s$ and the odd subspace $W_s^1=V_{s-1/2}$. In the space W_s, one can find a basis $|\sigma,m\rangle\in W_s^\sigma$, where $\sigma\in\{0,1\}$, $m\in\{-(s-\sigma/2),(s-\sigma/2)+1,\ldots,s-\sigma/2-1,s-\sigma/2\}$, such that the action of the generators of osp(1|2) on the elements of this basis is given by

$$L_3|\sigma,m\rangle=m|\sigma,m\rangle,$$

$$L_\pm|\sigma,m\rangle=\sqrt{(s-\sigma/2\mp m)(s-\sigma/2+1\pm m)}|\sigma,m\pm 1\rangle,$$

$$G_\pm|0,m\rangle=\mp\tfrac{1}{2}\sqrt{s\mp m}|1,m\pm 1/2\rangle,$$

$$G_\pm|1,m\rangle=-\tfrac{1}{2}\sqrt{s+1/2\pm m}|0,m\pm 1/2\rangle. \tag{2.5}$$

Note that this representation respects the $\mathbb{Z}_2$-grading on osp(1|2) introduced above. The eigenvalue of C on W_s is $(s+1/4)^2$.

The remarkable fact about the Casimir operator (2.3) is that it possesses a square root which is a quadratic expression in the generators of osp(1|2). Namely, consider the following element of U(osp(1|2)):

$$Q=2(G_-G_+-G_+G_-)+1/4. \tag{2.6}$$

Proposition II.1: Q satisfies the following properties:
(i) it is a square root of C:

$$Q^2=C; \tag{2.7}$$

(ii) it commutes with the even generators and anticommutes with the odd ones:

$$QG_\pm=-G_\pm Q,\quad QL_3=L_3Q,\quad QL_\pm=L_\pm Q; \tag{2.8}$$

(iii) it is diagonal in the basis $|\sigma,m\rangle$:

$$Q|\sigma,m\rangle=(-1)^\sigma(s+1/4)|\sigma,m\rangle. \tag{2.9}$$

Proof: (i) We use the following identities in U(osp(1|2)):

$$L_3=-2(G_+G_-+G_-G_+),\quad L_\pm=\pm 4G_\pm^2, \tag{2.10}$$

to compute:

$$
\begin{aligned}
Q^2&=(4G_-G_++L_3)^2+2G_-G_++1/2L_3+1/16\\
&=16G_-G_+G_-G_++4G_-G_+L_3+4L_3G_-G_++L_3^2+2G_-G_++1/2L_3+1/16\\
&=-16G_-^2G_+^2-8G_-L_3G_++8G_-G_+L_3+L_3^2+2G_-G_++1/2L_3+1/16\\
&=L_-L_+-4G_-G_++L_3^2+2G_-G_++1/2L_3+1/16\\
&=1/2(L_-L_++L_+L_-)-L_3+L_3^2+G_+G_--G_-G_++1/2L_3+1/2L_3+1/16=C.
\end{aligned}
$$

(ii) It is sufficient to prove the first of the relations (2.8), as the remaining two follow from (2.10). We compute

$$
\begin{aligned}
QG_\pm+G_\pm Q&=2(G_-G_+G_\pm-G_+G_-G_\pm+G_\pm G_-G_+-G_\pm G_+G_-)+1/2G_\pm\\
&=\pm 2(G_\mp G_\pm G_\pm-G_\pm G_\pm G_\mp)+1/2G_\pm\\
&=1/2(G_\mp L_\pm-L_\pm G_\mp)+1/2G_\pm=-1/2G_\pm+1/2G_\pm=0.
\end{aligned}
$$

(iii) This is a straightforward consequence of (2.5). □

III. THE QUANTIZED UNIVERSAL ENVELOPING ALGEBRA $U_q(\mathrm{osp}(1|2))$

The quantized universal enveloping algebra $U_q(\mathrm{osp}(1|2))$ (Ref. 3) is generated by three elements L_3, G_+, and G_-, with the parity assignments

$$p(L_3)=0, \quad p(G_\pm)=1, \tag{3.1}$$

and with the following relations,

$$L_3G_\pm-G_\pm L_3=\pm\tfrac{1}{2}G_\pm, \quad G_+G_-+G_-G_+=-\frac{\sinh \eta L_3}{\sinh 2\eta}, \tag{3.2}$$

where η is related to the usual deformation parameter q by $q=\exp(-\eta/2)$. The "second order" Casimir element in $U_q(\mathrm{osp}(1|2))$ is given by

$$
\begin{aligned}
C_q=&\sinh^2\eta(L_3+1/4)-2\sinh(\eta/2)\sinh 2\eta\cosh\eta(L_3+1/2)\ G_-G_+\\
&-(2\cosh(\eta/4)\sinh 2\eta)^2G_-^2G_+^2.
\end{aligned}\tag{3.3}
$$

Note that $C_q=\eta^2C+O(\eta^4)$, with C given by (2.3).

Finite dimensional, irreducible representations of $U_q(\mathrm{osp}(1|2))$ have a structure analogous to that of the irreducible representations of osp(1|2) and were constructed in Ref. 3. In the vector space W_s carrying the irreducible representation labeled by the half-integer s, we can choose a basis $|\sigma,m\rangle$, where $\sigma\in\{0,1\}$, $m\in\{-(s-\sigma/2),(s-\sigma/2)+1,\ldots,s-\sigma/2-1,s-\sigma/2\}$, such that the action of the generators of $U_q(\mathrm{osp}(1|2))$ on the elements of this basis is given by

$$
\begin{aligned}
&L_3|\sigma,m\rangle=m|\sigma,m\rangle,\\
&G_\pm|0,m\rangle=\mp\sqrt{\frac{\sinh \eta(s\mp m)/2\ \cosh \eta(s\pm m+1/2)/2}{\cosh(\eta/4)\sinh 2\eta}}\ |1,m\pm 1/2\rangle,\\
&G_\pm|1,m\rangle=-\sqrt{\frac{\sinh \eta(s\pm m+1/2)/2\ \cosh \eta(s\mp m)/2}{\cosh(\eta/4)\sinh 2\eta}}\ |0,m\pm 1/2\rangle.
\end{aligned}\tag{3.4}
$$

The space W_s is naturally $\mathbb{Z}_2$-graded. Note that the above representation respects the $\mathbb{Z}_2$-grading of $U_q(\mathrm{osp}(1|2))$. The eigenvalue of C_q on W_s is $\sinh^2 \eta(s+1/4)$.

Consider the following element of $U_q(\mathrm{osp}(1|2))$:

$$Q_q = \cosh(\eta/4)\sinh 2\eta(G_-G_+ - G_+G_-) + \sinh(\eta/4)\cosh \eta L_3. \tag{3.5}$$

Note that $Q_q = \eta Q + O(\eta^2)$, with Q given by (2.6).

Proposition III.1: Q satisfies the following properties:

(i) it is a square root of C_q:

$$Q_q^2 = C_q; \tag{3.6}$$

(ii) it commutes with L_3 and anticommutes with $G_\pm$:

$$Q_q G_\pm = -G_\pm Q_q,$$

$$Q_q L_3 = L_3 Q_q; \tag{3.7}$$

(iii) it is diagonal in the basis $|\sigma,m\rangle$:

$$Q_q|\sigma,m\rangle = (-1)^\sigma \sinh \eta(s+1/4)|\sigma,m\rangle. \tag{3.8}$$

Proof: (i) From the first of the relations (3.2) we infer

$$f(L_3)G_\pm = G_\pm f(L_3 \pm 1/2), \tag{3.9}$$

for any formal power series f. Hence

$$\begin{aligned}
Q_q^2 &= (2\cosh(\eta/4)\sinh 2\eta)^2 G_-G_+G_-G_+ + \sinh^2 \eta(L_3+1/4) \\
&\quad + 4\cosh(\eta/4)\sinh 2\eta \sinh \eta(L_3+1/4)\, G_-G_+ \\
&= -(2\cosh(\eta/4)\sinh 2\eta)^2 G_-^2 G_+^2 - 4\cosh^2(\eta/4)\sinh 2\eta\, G_- \sinh \eta L_3\, G_+ \\
&\quad + \sinh^2 \eta(L_3+1/4) + 4\cosh(\eta/4)\sinh 2\eta \sinh \eta(L_3+1/4)\, G_-G_+ \\
&= \sinh^2 \eta(L_3+1/4) - (2\cosh(\eta/4)\sinh 2\eta)^2 G_-^2 G_+^2 + 4\cosh(\eta/4) \\
&\quad \times \sinh 2\eta\, (\sinh \eta(L_3+1/4) - \sinh \eta(L_3+1/2)\cosh(\eta/4))\, G_-G_+ = C_q,
\end{aligned}$$

where we have also used the addition formulas for the hyperbolic functions.

(ii) It is sufficient to prove the first of the relations (3.7), as the second one follows from (3.9). We compute

$$\begin{aligned}
Q_qG_+ + G_+Q_q &= 2\cosh(\eta/4)\sinh 2\eta\, (G_-G_+G_+ + G_+G_-G_+) + \sinh \eta(L_3+1/4)\, G_+ \\
&\quad + G_+ \sinh \eta(L_3+1/4) \\
&= (-2\cosh(\eta/4)\sinh \eta L_3 + \sinh \eta(L_3+1/4) + \sinh \eta(L_3-1/4))\, G_+ = 0.
\end{aligned}$$

The calculation with G_+ replaced by G_- is analogous.

(iii) This is a straightforward consequence of (3.4). □

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