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Supersymmetry and the Spectral Condition on a Cylinder★

ARTHUR JAFFE, ANDRZEJ LESNIEWSKI, and
CHRISTIAN WIECZERKOWSKI★★
Harvard University, Cambridge, MA 02138, U.S.A.

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Abstract. We study the spectral condition for supersymmetric Wess–Zumino field theories on a cylindrical spacetime. This condition is preserved under certain ultraviolet cutoff procedures and also leads to analyticity of HKR regularized field operators in a complex neighborhood of spacetime.

1. The Spectral Conditions on a Cylinder

In this Letter, we establish the spectral condition for a class of field theory models on a two-dimensional cylindrical spacetime $S^1 \times \mathbb{R}$, where S^1 denotes a circle in the spatial direction. The positive-energy condition $0 \leqslant H$ is well-known to follow from the assumption of the supersymmetry algebra

$$\{Q_\alpha, \overline{Q}_\beta\} = 2 \sum_\mu \gamma^\mu_{\alpha\beta} P_\mu . \tag{1.1}$$

The spectrum condition

$$\pm P \leqslant H , \tag{1.2}$$

also follows, independent of Lorentz symmetry. The inequality (1.2) also carries over to cutoff models. Our contribution here is to establish (1.2) on a mathematical level for the Wess–Zumino models on a cylinder constructed in [1–3].

In (1.1), the operators Q_α are assumed selfadjoint and $\overline{Q}_\alpha = \Sigma_\beta Q_\beta \gamma^0_{\beta\alpha}$. We take

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1.3}$$

It then follows that

$$Q_1^2 = H + P , \qquad Q_2^2 = H - P , \tag{1.4}$$

where P is the (one-dimensional) spatial momentum. Thus $\pm P \leqslant H$ and the spectral condition holds.

In order to verify that these relations hold in concrete field theory models, it is

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necessary to introduce cutoffs in a particular way. In particular, we wish to write

$$Q_\alpha(\kappa) = Q_{\alpha,0} + Q_{\alpha,I}(\kappa), \tag{1.5}$$

where $Q_{\alpha,0}$ is a free-field supercharge (without cutoff) and $Q_{\alpha,I}(\kappa)$ is the cutoff supercharge perturbation. To see how this works, let us take a particular example, the $N = 1, 2$ models studied in [1–3].

THEOREM 1.1. *The $N = 1, 2$ models constructed in* [1, 2] *satisfy the inequality*

$$\pm P \leqslant H(\kappa), \tag{1.6}$$

for all $0 \leqslant \kappa \leqslant \infty$.

Proof. In order to carry out the analytic proof, we require the formal algebraic relations. We consider the case $N = 1$, where ψ is a two-component, self-adjoint (Majorana) field. Let us introduce

$$Q_1 = \int_{S^1} \{\psi_2(x)\,\partial_0\varphi(x) + \psi_1(x)\,\partial_1\varphi(x) - \psi_1(x):V'(\varphi(x)):\}\,\mathrm{d}x \tag{1.7}$$

and

$$Q_2 = \int_{S^1} \{\psi_1(x)\,\partial_0\varphi(x) + \psi_2(x)\,\partial_1\varphi(x) - \psi_2(x):V'(\varphi(x)):\}\,\mathrm{d}x, \tag{1.8}$$

where V is a polynomial. These forms formally obey the algebra

$$\{Q_1, Q_2\} = 0, \qquad Q_1^2 = H + P, \qquad Q_2^2 = H - P, \tag{1.9}$$

where H_0 and P_0 are the free field energy-momentum operators and

$$H = H_0 + \frac{1}{2}\int_{S^1} \{:V'(\varphi(x)):^2 - i\psi_2(x):V''(\varphi(x)):\,\psi_1(x)\}\,\mathrm{d}x, \tag{1.10}$$

$$P = P_0. \tag{1.11}$$

We claim that the bilinear forms (1.7), (1.8) uniquely determine self-adjoint operators. We introduce a cutoff function into Q_i and establish these properties of Q_i as limits. Let

$$\varphi_\kappa(x) = \chi_\kappa \circ \varphi(x)$$

and

$$Q_1(\kappa) = \int_{S^1} \{\psi_2(x)\,\partial_0\varphi(x) + \psi_1(x)\,\partial_1\varphi(x) - \psi_1(x):V'(\varphi_\kappa(x)):\}\,\mathrm{d}x, \tag{1.12}$$

and similarly for $Q_2(\kappa)$. These operators satisfy

$$\{Q_1(\kappa), Q_2(\kappa)\} = 0, \qquad Q_1(\kappa)^2 = H(\kappa) + P, \qquad Q_2(\kappa)^2 = H(\kappa) - P, \tag{1.13}$$

and $Q_j(\kappa)$, $H(\kappa)$ and P are self-adjoint. Furthermore, letting $U(x) = \exp(ixP)$, we have

$$U(x)Q_j(\kappa) = Q_j(\kappa)U(x),$$
$$U(x)H(\kappa) = H(\kappa)U(x),$$

from which we infer that $Q_j(\kappa)$ and $H(\kappa)$ commute with P. Thus, $H(\kappa) \pm P$ is positive and

$$\exp(-H(\kappa) \pm P) = \exp(-H(\kappa)) \exp(\pm P) .$$

Since by Theorem III.1 of [3] $\exp(-H(\kappa))$ is norm-convergent as $\kappa \to \infty$, we conclude that the operators $\exp(-H(\kappa) \pm P)$ converge strongly. Furthermore, the limiting operator is self-adjoint and bounded so the desired positivity extends to the limiting generators $H \pm P$.

As a consequence of the spectral condition, heat kernel regularized fields are C^∞ functions of (x, t) which, in fact, extend to operator-valued functions analytic in an appropriate domain.

THEOREM 1.2. *The operator-valued function*

$$F_t(x) = e^{-ixP - tH(\kappa)}, \quad 0 \leqslant \kappa \leqslant \infty ,$$

is analytic for $|\mathrm{Im}\, x| < \mathrm{Re}\, t$ *and bounded for* $|\mathrm{Im}\, x| \leqslant \mathrm{Re}\, t$.

Proof. The analytically follows from the positivity of $H(\kappa)$, the commutativity of P and $H(\kappa)$ and Theorem 1.1.

A similar argument can be given for the case $N = 2$ of the complex boson field φ interacting with a Dirac fermion ψ. In this case, the powers φ^n have zero expectation in the Gaussian measure. The form of the supercharges $Q_j(\kappa)$ are similar,

$$Q_1(\kappa) = \int_{S^1} \{ \psi_2\, \partial_0 \varphi^* + \psi_2^*\, \partial_0 \varphi - \psi_1\, \partial_1 \varphi^* - \psi_1^*\, \partial_1 \varphi - i\psi_1 V'(\varphi_\kappa(x)) + \\ + i\psi_1^* V'(\varphi_\kappa(x))^* \}\, \mathrm{d}x \tag{1.14}$$

and

$$Q_2(\kappa) = \int_{S^1} \{ \psi_1\, \partial_0 \varphi^* + \psi_1^*\, \partial_0 \varphi - \psi_2\, \partial_1 \varphi^* - \psi_2^*\, \partial_1 \varphi - i\psi_2 V'(\varphi_\kappa(x)) + \\ + i\psi_2^* V'(\varphi_\kappa(x))^* \}\, \mathrm{d}x . \tag{1.15}$$

The algebra (1.9) again follows. Cutoffs can be introduced and the above arguments are applicable.

2. Analyticity of Fields

Let A be a bilinear form on $\mathscr{H}$ which has a strong heat kernel regularization (HKR) in the sense of [4, 5]. Thus, $A \in \mathscr{L}(\mathscr{H}_\varepsilon, \mathscr{H}_{-\varepsilon})$ for any $\varepsilon > 0$, where $\mathscr{H}_\varepsilon = \mathrm{Dom}(e^{\varepsilon H})$. Consider

$$A(x, t) = e^{i(tH - xP)} A e^{-i(tH - xP)} . \tag{2.1}$$

Assuming that H and P commute, then $A(x, t)$ also has a strong HKR. Furthermore, the family $\{A(x, t)\} \subset \mathscr{L}(\mathscr{H}_\varepsilon, \mathscr{H}_{-\varepsilon})$ is continuous in (x, t) for real x, t. In fact, it follows from the spectral theorem that we have the following corollary.

THEOREM 2.1. *The family $A(x, t) \subset \mathscr{L}(\mathscr{H}_\varepsilon, \mathscr{H}_{-\varepsilon})$ extends to an analytic function of (x, t) in the domain*

$$|\mathrm{Im}\, x| + |\mathrm{Im}\, t| < \varepsilon . \tag{2.2}$$

Imaginary time operators are given for $|t| < \varepsilon$, and (x, t) real by

$$A_E(x, t) = A(x, -it) \in \mathscr{L}(\mathscr{H}_\varepsilon, \mathscr{H}_{-\varepsilon}) . \tag{2.3}$$

These fields extend to analytic functions of (x, t) for $|\mathrm{Im}\, x| + |\mathrm{Re}\, t| < \varepsilon$. Thus $A_E(x, t)$ has a HKR (rather than a strong HKR). Furthermore, if in addition, $\mathrm{Re}\, t < -|\mathrm{Im}\, x|$, then the field $A_E(x, t)$ is an unbounded operator on the domain $\mathscr{H}_\varepsilon$. In any case, if $A(x, t)$ satisfies a partial differential equation, then $A_E(x, t)$ satisfies an analytically continued partial differential equation, see [5].

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