Tensor products of representations of $C(\mathrm{SU}_q(2))$

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The tensor products of the irreducible representations of the algebra of continuous functions on the quantum group $\mathrm{SU}_q(2)$ were studied. Using the methods of q-calculus it was proven that the tensor product of two infinite dimensional irreducible representations may be decomposed into a direct integral of irreducible representations.

I. INTRODUCTION

The algebra of continuous functions $C(\mathrm{SU}_q(2))$ (Refs. 1 and 2) on the quantum group $\mathrm{SU}_q(2)$ is defined as the universal enveloping $\mathbb{C}^*$-algebra of the unital $*$-algebra generated by four elements a, $\bar{a}$, b, $\bar{b}$ satisfying the following relations:

$$ab=q^{1/2}ba, \quad a\bar{b}=q^{1/2}\bar{b}a, \quad b\bar{b}=\bar{b}b,$$

$$a\bar{a}-\bar{a}a=(q^{-1/2}-q^{1/2})\bar{b}b, \quad a\bar{a}+q^{1/2}\bar{b}b=I. \tag{1.1}$$

We assume that $-1<q^{1/2}<1$, $q^{1/2}\neq 0$. The involution is defined by $a^*:=\bar{a}$ and $b^*:=\bar{b}$. $C(\mathrm{SU}_q(2))$ is a Hopf $\mathbb{C}^*$-algebra with the coproduct Δ given by

$$\Delta(a)=a\otimes a-b\otimes\bar{b}, \quad \Delta(\bar{a})=\bar{a}\otimes\bar{a}-\bar{b}\otimes b,$$

$$\Delta(b)=a\otimes b+b\otimes\bar{a}, \quad \Delta(\bar{b})=\bar{a}\otimes\bar{b}+\bar{b}\otimes a, \tag{1.2}$$

the counit ϵ given by

$$\epsilon(a)=\epsilon(\bar{a})=1, \quad \epsilon(b)=\epsilon(\bar{b})=0, \tag{1.3}$$

and the antipode S given by

$$S(a)=\bar{a}, \quad S(\bar{a})=a, \quad S(b)=-q^{-1/2}b, \quad S(\bar{b})=-q^{1/2}\bar{b}. \tag{1.4}$$

The irreducible $*$-representations of $C(\mathrm{SU}_q(2))$ were classified in Refs. 1 and 2. It was found that $C(\mathrm{SU}_q(2))$ has two distinct families of irreducible representations, one dimensional and infinite dimensional (see Sec. II for details). Since $C(\mathrm{SU}_q(2))$ is a bialgebra, the category of its representations comes naturally equipped with a tensor product. In this paper, we compute the tensor products of irreducible representations of $C(\mathrm{SU}_q(2))$. Our main result is that the tensor product of two infinite dimensional irreducible representations is equivalent to a direct integral of infinite dimensional irreducible representations (see, e.g., Ref. 3 for the definition of a direct integral).

The proof of this result involves a number of combinatorial identities of q-calculus.[4,5] Throughout the paper we use the standard notation of q-calculus as explained, e.g., in Ref. 5. In particular, for $|q|<1$, $a\in\mathbb{C}$, we set $(a;q)_0:=1$, $(a;q)_n:=\Pi_{0<j<n-1}(1-aq^j)$, if $n\geqslant 1$, and $(a;q)_\infty:=\Pi_{j>0}(1-aq^j)$.

II. TENSOR PRODUCTS OF IRREDUCIBLE REPRESENTATIONS OF $C(SU_q(2))$

Let $\pi : C(SU_q(2)) \to \mathcal{L}(\mathcal{H})$ be an irreducible $*$-representation of $C(SU_q(2))$ in the $\mathbb{C}^*$-algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. According to Refs. 1 and 2, π is unitarily equivalent to one of the following two families of representations.

One dimensional representations ρ_φ. The Hilbert space is $\mathcal{H} = \mathbb{C}$ and

$$\rho_\varphi(a) = e^{i\varphi}, \quad \rho_\varphi(b) = 0,$$
$$\rho_\varphi(\bar{a}) = e^{-i\varphi}, \quad \rho_\varphi(\bar{b}) = 0, \tag{2.1}$$

where $0 \leqslant \varphi < 2\pi$.

Infinite dimensional representations π_θ. The Hilbert space is $\mathcal{H} = l^2(\mathbb{Z}_+)$, where $\mathbb{Z}_+$ is the set of non-negative integers. Let $\{\phi_n\}_{n \in \mathbb{Z}}$ be the standard orthonormal basis for $\mathcal{H}$ and let $E_{m,n} \in \mathcal{L}(\mathcal{H})$ be defined by $E_{m,n}\phi_k = \delta_{n,k}\phi_m$. We set for $0 \leqslant \theta < 2\pi$,

$$\pi_\theta(a) = \sum_{n \geqslant 1}(1 - q^n)^{1/2} E_{n-1,n}, \quad \pi_\theta(b) = e^{i\theta}\sum_{n \geqslant 0}q^{(1/4)(2n+1)}E_{n,n},$$
$$\pi_\theta(\bar{a}) = \sum_{n \geqslant 0}(1 - q^{n+1})^{1/2}E_{n+1,n}, \quad \pi_\theta(\bar{b}) = e^{-i\theta}\sum_{n \geqslant 0}q^{(1/4)(2n+1)}E_{n,n}. \tag{2.2}$$

Our concern in this paper is the study of tensor product of representations. Recall that if $\pi : C(SU_q(2)) \to \mathcal{L}(\mathcal{H})$ and $\rho : C(SU_q(2)) \to \mathcal{L}(\mathcal{G})$ are representations of $C(SU_q(2))$, then their tensor product $\pi \otimes \rho$ is defined as the composition of morphisms

$$C(SU_q(2)) \xrightarrow{\Delta} C(SU_q(2)) \hat{\otimes} C(SU_q(2)) \to \mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{L}(\mathcal{G}),$$

where $\hat{\otimes}$ denotes the spatial tensor product of $\mathbb{C}^*$-algebras [since $C(SU_q(2))$ is nuclear,[1] the choice of $\mathbb{C}^*$-norm on the tensor product $C(SU_q(2)) \otimes C(SU_q(2))$ is, in fact, optional]. The goal of this paper is to prove the following theorem.

Main Theorem: The following unitary equivalences hold:

$$\text{(i)} \quad \rho_\varphi \otimes \rho_\theta \cong \rho_{\varphi+\theta}, \tag{2.3}$$

$$\text{(ii)} \quad \rho_\varphi \otimes \pi_\theta \cong \pi_{\varphi+\theta}, \tag{2.4}$$

$$\text{(iii)} \quad \pi_\varphi \otimes \rho_\theta \cong \pi_{\varphi-\theta}, \tag{2.5}$$

$$\text{(iv)} \quad \pi_\varphi \otimes \pi_\theta \cong \int_{S^1}^{\oplus} \pi_\alpha \, d\alpha, \tag{2.6}$$

where $\varphi + \theta$ is defined modulo 2π, and where $d\alpha$ is the normalized Lebesque measure on S^1.

Beginning of the proof of the Main Theorem: Since Δ is a $*$-homomorphism of $\mathbb{C}^*$-algebras, it is enough to verify the isomorphisms (i)–(iv) when applied to a and b. Verifying (i) is easy and we omit the details. To prove (ii) we note that

$$(\rho_\varphi \otimes \pi_\theta)(a) = e^{i\varphi}\sum_{n \geqslant 1}(1 - q^n)^{1/2}E_{n-1,n},$$

$$(\rho_\varphi \otimes \pi_\theta)(b) = e^{i(\varphi+\theta)}\sum_{n \geqslant 0}q^{(1/4)(2n+1)}E_{n,n}.$$

Define $U:l^2(\mathbf{Z}_+)\to l^2(\mathbf{Z}_+)$ by $U\phi_n=e^{-in\varphi}\phi_n$. Then U is unitary and

$$U^{-1}(\rho_\varphi\otimes\pi_\theta)(a)U=\sum_{n>1}(1-q^n)^{1/2}E_{n-1,n},$$

$$U^{-1}(\rho_\varphi\otimes\pi_\theta)(b)U=e^{i(\varphi+\theta)}\sum_{n\geqslant0}q^{(1/4)(2n+1)}E_{n,n}.$$

The claim follows. The proof of (iii) is similar and we omit the details.

To prove (iv) we observe that, as a consequence of (i)–(iii),

$$\pi_\varphi\otimes\pi_\theta\cong\rho_{\varphi-\theta}\otimes(\pi_0\otimes\pi_0),$$

and so we need to show that

$$\pi_0\otimes\pi_0\cong\int_{S^1}^\oplus\pi_\alpha\,d\alpha,\tag{2.7}$$

and

$$\rho_\lambda\otimes\int_{S^1}^\oplus\pi_\alpha\,d\alpha\simeq\int_{S^1}^\oplus\pi_\alpha\,d\alpha.\tag{2.8}$$

We will prove Eqs. (2.7) and (2.8) in the following section.

III. PLANCHEREL THEOREM

In this section we reduce the proof of Eq. (2.7) to the proof of a Plancherel-type theorem. This theorem will be proven in Sec. IV.

We begin by defining a sequence $\{\omega_n\}_{n\in\mathbf{Z}_+}$ of $l^2(\mathbf{Z}_+)\otimes l^2(\mathbf{Z}_+)$-valued distribution on S^1. We set

$$\omega_0(\alpha)=\sum_{p,r\in\mathbf{Z}_+}\omega_0^{p,r}(\alpha)\phi_p\otimes\phi_r,\quad 0\leqslant\alpha<2\pi,\tag{3.1}$$

where the coefficients $\omega_0^{p,r}(\alpha)$ are defined as follows:

$$\omega_0^{p,r}(\alpha)=\left(\frac{(q;q)_\infty}{2\pi}\right)^{1/2}\frac{q^{(1/2)pr}}{(q;q)_p^{1/2}(q;q)_r^{1/2}}e^{i(p-r)\alpha}.\tag{3.2}$$

For each $n\geqslant1$ we set

$$\omega_n(\alpha)=(q;q)_n^{-1/2}\Delta(\bar{a})^n\omega_0(\alpha),\tag{3.3}$$

with the obvious action of $\Delta(\bar{a})^n$ on $\omega_0(\alpha)$.

Proposition III 1: With the above definitions,

$$\Delta(a)\omega_n(\alpha)=\begin{cases}0,&n=0,\\(1-q^n)^{1/2}\omega_{n-1}(\alpha),&n\geqslant1,\end{cases}\tag{3.4}$$

$$\Delta(\bar{a})\omega_n(\alpha)=(1-q^{n+1})^{1/2}\omega_{n+1}(\alpha),\tag{3.5}$$

$$\Delta(b)\omega_n(\alpha)=e^{i\alpha}q^{(1/4)(2n+1)}\omega_n(\alpha),\tag{3.6}$$

$$\Delta(\bar{b})\omega_n(\alpha)=e^{-i\alpha}q^{(1/4)(2n+1)}\omega_n(\alpha),\tag{3.7}$$

Proof: The proof is a simple verification. For example,

$$\Delta(a)\omega_0(\alpha)=\sum_{p,r\in\mathbb{Z}_+}\omega_0^{p,r}(\alpha)\{\pi_0(a)\phi_p\otimes\pi_0(a)\phi_r-\pi_0(b)\phi_p\otimes\pi_0(\bar{b})\phi_r\}$$

$$=\sum_{p,r\in\mathbb{Z}_+}\{(1-q^{p+1})^{1/2}(1-q^{r+1})^{1/2}\omega_0^{p+1,r+1}(\alpha)-q^{(1/2)(p+r+1)}\omega_0^{p,r}(\alpha)\}\phi_p\otimes\phi_r$$

$$=0.$$

$\square$

Proposition III 2: As Hilbert space valued distributions on S^1,

$$(\omega_n(\alpha),\omega_m(\beta))=\delta_{n,m}\delta(\alpha-\beta).\tag{3.8}$$

Proof: We have

$$(\omega_0(\alpha),\omega_0(\beta))=\sum_{p,r\in\mathbb{Z}_+}\omega_0^{p,r}(\alpha)^*\omega_0^{p,r}(\beta)$$

$$=\frac{(q;q)_\infty}{2\pi}\sum_{p,r\geqslant0}\frac{q^{pr}}{(q;q)_p(q;q)_r}e^{i(p-r)(\beta-\alpha)}$$

$$=\frac{(q;q)_\infty}{2\pi}\sum_{k\in\mathbb{Z}}e^{ik(\beta-\alpha)}\sum_{r\geqslant0}\frac{q^{r(r+k)}}{(q;q)_r(q;q)_{r+k}}.\tag{3.9}$$

Using Eq. (2.2.8) of Ref. 1 with $z=q^{k+1}$ yields

$$\sum_{r\geqslant0}\frac{q^{r(r+k)}}{(q;q)_r(q;q)_{r+k}}=\frac{1}{(q;q)_k}\sum_{r\geqslant0}\frac{q^{r^2-r}(q^{k+1})^r}{(q;q)_r(q^{k+1};q)_r}=\frac{1}{(q;q)_k(q^{k+1};q)_\infty}=\frac{1}{(q;q)_\infty},$$

and so

$$(\omega_0(\alpha),\omega_0(\beta))=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}e^{ik(\beta-\alpha)}=\delta(\alpha-\beta),\tag{3.10}$$

as claimed. This establishes Eq. (3.8) for $n=m=0$. Let $n>0$. Then, by Eqs. (3.3) and (3.4),

$$(\omega_n(\alpha),\omega_0(\beta))=(1-q^n)^{-1/2}(\Delta(\bar{a})\omega_{n-1}(\alpha),\omega_0(\beta))=(1-q^n)^{-1/2}(\omega_{n-1}(\alpha),\Delta(a)\omega_0(\beta))=0.\tag{3.11}$$

For the general case, it is no loss of generality to assume that $n=m+k$, $k\geqslant0$. Then, as a consequence of Eqs. (3.3)–(3.5),

$$(\omega_{m+k}(\alpha),\omega_m(\beta))=\left\{\prod_{j=1}^m(1-q^{k+j})(1-q)^j\right\}^{-1/2}(\omega_k(\alpha),\Delta(a)^m\Delta(\bar{a})^m\omega_0(\beta)).$$

If $k>0$, then this is zero as a consequence of Eq. (3.11) and the fact that $\Delta(a)^m\Delta(\bar{a})^m$ acts diagonally on $\omega_n(\beta)$. If $k=0$, then

$$(\omega_0(\alpha), \Delta(a)^m \Delta(\bar{a})^m \omega_0(\beta)) = (q;q)_m(\omega_0(\alpha), \omega_0(\beta)),$$

and the claim follows from (3.10). $\square$

Theorem III 3 (Plancherel Theorem): For $f \in l^2(\mathbf{Z}_+) \otimes l^2(\mathbf{Z}_+)$,

$$\|f\|^2 = \sum_{n \geqslant 0} \int_{S^1} |(\omega_n(\alpha), f)|^2 \, d\alpha. \tag{3.12}$$

We are now prepared to complete the proof of the main theorem.

End of the proof of the Main Theorem: We define a map

$$U : \int_{S^1}^{\oplus} \mathscr{H}_\alpha \, d\alpha \to \mathscr{H} \otimes \mathscr{H}, \tag{3.13}$$

where $\mathscr{H}_\alpha = \mathscr{H} = l^2(\mathbf{Z}_+)$, in the following way. For $x_n \in \mathscr{C}^\infty(S^1)$ and any N we set

$$U\left(\sum_{n \leqslant N} x_n(\cdot)\phi_n\right) := \sum_{n \leqslant N} \int_{S^1} x_n(\beta)\omega_n(\beta)d\beta. \tag{3.14}$$

Then, as a consequence of Eq. (3.8),

$$\left\| U\left(\sum_{n \leqslant N} x_n(\cdot)\phi_n\right) \right\|^2 = \sum_{n \leqslant N} \int_{S^1} |x_n(\beta)|^2 \, d\beta = \left\| \sum_{n \leqslant N} x_n(\cdot)\phi_n \right\|^2,$$

and so U is an isometry. To prove that U is onto we define a map

$$V : \mathscr{H} \otimes \mathscr{H} \to \int_{S^1}^{\oplus} \mathscr{H}_\alpha \, d\alpha \tag{3.15}$$

by

$$(Vf)(\alpha) := \sum_{n \geqslant 0} (\omega_n(\alpha), f)\phi_n. \tag{3.16}$$

Then by Eq. (3.12),

$$\|(Vf)(\cdot)\|^2 = \sum_{n \geqslant 0} \int_{S^1} |(\omega_n(\alpha), f)|^2 \, d\alpha = \|f\|^2,$$

and so V is an isometry. Furthermore,

$$(VUx_n(\cdot)\phi_n)(\alpha) = \sum_{m \geqslant 0} (\omega_m(\alpha), U(x_n(\cdot)\phi_n))\phi_m = \sum_{m \geqslant 0} \int_{S^1} (\omega_m(\alpha), \omega_n(\beta))x_n(\beta)d\beta \, \phi_m$$

$$= x_n(\alpha)\phi_n,$$

and so $VU = I$. Therefore, V is bijective and its right inverse is U. As a consequence, U is the inverse of V, and so Eq. (3.13) is an isometric isomorphism. Now,

$$(U^{-1}(\pi_0 \otimes \pi_0)(a)U\phi_n)(\alpha) = \sum_{m \geqslant 0} (\omega_m(\alpha), (\pi_0 \otimes \pi_0)(a)U\phi_n)\phi_m$$

$$= \sum_{m \geqslant 0} \int_{S^1} (\omega_m(\alpha), \Delta(a)\omega_n(\beta))d\beta \, \phi_m.$$

This equals 0, if $n=0$, and

$$(1-q^n)^{1/2} \sum_{m \geqslant 0} \int_{S^1} (\omega_m(\alpha),\omega_{n-1}(\beta))d\beta \, \phi_m = (1-q^n)^{1/2} \phi_{n-1},$$

if $n > 0$. Similarly,

$$(U^{-1}(\pi_0 \otimes \pi_0)(b) U\phi_n)(\alpha) = \sum_{m \geqslant 0} \int_{S^1} (\omega_m(\alpha),\Delta(b)\omega_n(\beta))d\beta \, \phi_m$$

$$= q^{(1/4)(2n+1)} \sum_{m \geqslant 0} \int_{S^1} (\omega_m(\alpha),\omega_n(\beta))e^{i\beta} \, d\beta \, \phi_m$$

$$= e^{i\alpha} q^{(1/4)(2n+1)} \phi_n.$$

The proof of Eq. (2.7) is complete.

To prove Eq. (2.8) we set for $x(\cdot) = \Sigma_{n \geqslant 0} x_n(\cdot)\phi_n \in \int_{S^1}^\oplus \mathcal{H}_\alpha \, d\alpha$,

$$(T_\lambda x)(\alpha) := \sum_{n \geqslant 0} x_n(\alpha-\lambda)e^{in\lambda}\phi_n. \tag{3.17}$$

Then T_λ is a unitary operator on $\int_{S^1}^\oplus \mathcal{H}_\alpha \, d\alpha$ and

$$T_\lambda^{-1}(\pi_0 \otimes \pi_0)(a)T_\lambda x = e^{i\lambda}(\pi_0 \otimes \pi_0)(a)x,$$

$$T_\lambda^{-1}(\pi_0 \otimes \pi_0)(b)T_\lambda x = e^{i\lambda}(\pi_0 \otimes \pi_0)(b)x, \tag{3.18}$$

where we have identified $\pi_0 \otimes \pi_0$ with the corresponding representation on $\int_{S^1}^\oplus \mathcal{H}_\alpha \, d\alpha$. This proves the equivalence Eq. (2.8). $\square$

IV. PROOF OF THE PLANCHEREL THEOREM

In this section we prove Theorem III 3. The proof is a rather tedious computation using a variety of q-calculus identities. In order not to interrupt the main line of computation, we defer the proofs of three crucial combinatorial identities to Sec. V.

We need to show that for all $p,r,s,t \in \mathbb{Z}_+$,

$$\sum_{n \geqslant 0} \int_{S^1} (\omega_n(\alpha),\phi_p \otimes \phi_r)(\phi_s \otimes \phi_t, \omega_n(\alpha))d\alpha = \delta_{ps}\delta_{rt}. \tag{4.1}$$

Denoting by $\omega_n^{p,r}(\alpha)$ the Fourier coefficient of $\omega_n(\alpha)$ with respect to the basis $\phi_p \otimes \phi_r$, we can write Eq. (4.1) as

$$\sum_{n \geqslant 0} \int_{S^1} \omega_n^{p,r}(\alpha)^* \omega_n^{s,t}(\alpha)d\alpha = \delta_{ps}\delta_{rt}. \tag{4.2}$$

Lemma IV 1: For any $n \geqslant 0$ we have

$$\omega_n^{p,r}(\alpha) = (-1)^n \left(\frac{(q;q)_\infty}{2\pi} \right)^{1/2} (q;q)_n^{1/2} (q;q)_p^{1/2} (q;q)_r^{1/2}$$

$$\times \sum_{0 \leqslant k \leqslant n} (-1)^k \frac{q^{(1/2)\{(p-k)(r-k)+(n-k)(p+r-2k+1)\}}}{(q;q)_{p-k}(q;q)_{r-k}(q;q)_{n-k}(q;q)_k} e^{i(p-r)\alpha}. \qquad (4.3)$$

Proof: We have

$$\omega_n(\alpha) = (q;q)_n^{-1/2} (\bar{a} \otimes \bar{a} - \bar{b} \otimes b)^n \omega_0(\alpha) \qquad (4.4)$$

(to simplify the formulas we have suppressed all the π_0's). Recall that for x and y obeying the algebra

$$xy = qyx \qquad (4.5)$$

we have the following binomial formula:

$$(x+y)^n = \sum_{0 \leqslant k \leqslant n} \binom{n}{k}_q y^k x^{n-k}, \qquad (4.6)$$

where $\binom{n}{k}_q := (q,q)_n / (q;q)_k (q;q)_{n-k}$ is the q-deformed binomial coefficient. Since $x = -\bar{b} \otimes b$ and $y = \bar{a} \otimes \bar{a}$ obey Eq. (4.5), Eq. (4.6) yields

$$\omega_n(\alpha) = (-1)^n (q;q)_n^{1/2} \sum_{0 \leqslant k \leqslant n} (-1)^k \frac{\omega_0^{p,r}(\alpha)}{(q;q)_k (q;q)_{n-k}} \bar{a}^k \bar{b}^{n-k} \phi_p \otimes \bar{a}^k b^{n-k} \phi_r.$$

Using Eq. (2.2) and Eq. (3.2) we easily obtain Eq. (4.3). $\qquad \square$

We now substitute (4.3) into the left-hand side of (4.2). The α integration produces $\delta_{p-r,s-k}$. We set $a := p - r = s - t$ and assume in the following that $a \geqslant 0$. The case $a \leqslant 0$ is similar and we omit the details. Introducing the notation

$$A := q^{(1/2)(pr+st)} (q;q)_p^{1/2} (q;q)_r^{1/2} (q;q)_s^{1/2} (q;q)_t^{1/2}, \qquad (4.7)$$

we rewrite the left hand side of Eq. (4.2) as

$$A(q;q)_\infty \sum_{n \geqslant 0} \sum_{0 \leqslant k \leqslant n} \sum_{0 \leqslant l \leqslant n} (-1)^{k+l} (q;q)_n$$

$$\times \frac{q^{(3/2)k^2 - k/2 - (2r+a)k + (3/2)l^2 - l/2 - (2t+a)l} q^{n(r+t+a-k-l+1)}}{(q;q)_{r-k}(q;q)_{r+a-k}(q;q)_k(q;q)_{n-k}(q;q)_{t-l}(q;q)_{t+a-l}(q;q)_l(q;q)_{n-l}}. \qquad (4.8)$$

Using the obvious identity

$$\frac{1}{(q;q)_{n-k}} = \frac{(q^{n-k+1};q)_k}{(q;q)_n}, \qquad (4.9)$$

we note that the constraints $k \leqslant n$ and $l \leqslant n$ in Eq. (4.8) may be dropped. We can thus interchange the order of summations and perform the summation over n first. Using the q-binomial formula

$$(a;q)_k = \sum_{0 \leqslant i \leqslant k} (-1)^i \binom{k}{i}_q q^{(1/2)i(i-1)} a^i,$$ (4.10)

we obtain

$$\sum_{n \geqslant 0} \frac{q^{n(r+t+a-k-l+1)}}{(q;q)_n} (q^{n-k+1};q)_k (q^{n-l+1};q)_l$$

$$= \sum_{0 \leqslant i \leqslant k} \sum_{0 \leqslant j \leqslant l} (-1)^{i+j} q^{-ki-lj+(1/2)i(i+1)+(1/2)j(j+1)}$$

$$\times \binom{k}{i}_q \binom{l}{j}_q \sum_{n \geqslant 0} \frac{q^{n(r+t+a-k-l+i+j+1)}}{(q;q)_n}.$$ (4.11)

We now use the familiar formula

$$\sum_{n \geqslant 0} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1,$$ (4.12)

with $z = q^{r+t+a-k-l+i+j+1}$ (observe that $|z| < 1$, as $r+t+a-k-l \geqslant 0$) to obtain

$$\sum_{n \geqslant 0} \frac{q^{n(r+t+a-k-l+i+j+1)}}{(q;q)_n} = \frac{1}{(q^{r+t+a-k-l+i+j+1};q)_\infty} = \frac{(q;q)_{r+t+a-k-l+i+j}}{(q;q)_\infty}.$$

As a consequence of these manipulations, Eq. (4.8) becomes

$$A \sum_{k \geqslant 0} \sum_{l \geqslant 0} \sum_{0 \leqslant i \leqslant k} \sum_{0 \leqslant j \leqslant l} q^{(3/2)k^2-k/2-(2r+a)k+(3/2)l^2-l/2-(2t+a)l+(1/2)i(i+1)+(1/2)j(j+1)-ik-jl}$$

$$\times \frac{(-1)^{i+j+k+l}(q;q)_{r+t+a-k-l+i+j}}{(q;q)_{r-k}(q;q)_{r+a-k}(q;q)_i(q;q)_{k-i}(q;q)_{l-l}(q;q)_{t+a-l}(q;q)_j(q;q)_{l-j}}.$$ (4.13)

We now perform the summation over i.

Lemma IV 2: For a non-negative integer ρ,

$$\sum_{0 \leqslant i \leqslant k} (-1)^i q^{(1/2)i(i+i)-ik} \binom{k}{i}_q (q;q)_{\rho+i-k} = (q;q)_{\rho-k} q^{k(\rho-k+1)}.$$ (4.14)

Using the above identity with $\rho = r+t+a-l+j$ we rewrite Eq. (4.13) as

$$A \sum_{k \geqslant 0} \sum_{l \geqslant 0} \sum_{0 \leqslant j \leqslant l} (-1)^{j+k+l} q^{(1/2)k(k+1)+k(t+j-l-r)+(3/2)l^2-l/2-(2t+a)l+(1/2)j(j+1)-jl}$$

$$\times \frac{(q;q)_{r+t+a-l+j-k}}{(q;q)_k(q;q)_{r-k}(q;q)_{r+a-k}(q;q)_{t-l}(q;q)_{t+a-l}(q;q)_j(q;q)_{l-j}}.$$ (4.15)

We now perform the summation over k in the above expression.

Lemma IV 3: For $u,v \in \mathbb{Z}_+$ such that $u-v \geqslant 0$,

$$\sum_{0 \leqslant k \leqslant r} (-1)^k q^{(1/2)k(k+1)+k(u-v-r)} \binom{r}{k}_q \frac{(q;q)_{u+r-k}}{(q;q)_{v+r-k}} = (q;q)_{u-v} \binom{u}{v+r}_q.$$ (4.16)

Using the above identity with $u=t+a-l+j$, $v=a$, we rewrite Eq. (4.15) in the form

$$A \sum_{l \geqslant 0} \sum_{0 \leqslant j \leqslant l} \frac{(-1)^{l-j} q^{(3/2)l^2 - l/2 - l(2t+a) + (1/2)j(j+1) - jl} (q;q)_{t+a-l+j} (q;q)_{t-l+j}}{(q;q)_r (q;q)_{t-l} (q;q)_{t+a-l} (q;q)_{l-j} (q;q)_j (q;q)_{r+a} (q;q)_{t-r-l+j}}. \quad (4.17)$$

Observe again that the constraint $j \leqslant l$ may be dropped (at the expense of adding zeroes to the sum). Substituting $l-j \rightarrow l$ in Eq. (4.17) we rewrite it as

$$A \frac{1}{(q;q)_r (q;q)_{r+a}} \sum_{l \geqslant 0} (-1)^l \frac{q^{(3/2)l^2 - l/2 - l(2t+a)}}{(q;q)_l (q;q)_{t-r-l}} \sum_{j \geqslant 0} q^{j^2 + 2lj - j(2t+a)} \binom{t-l}{j}_q \frac{(q;q)_{t+a-l}}{(q;q)_{t+a-l-j}}. \quad (4.18)$$

To compute the sum over j we use the following combinatorial identity.
   Lemma IV 4: For a non-negative integer a,

$$\sum_{0 \leqslant j \leqslant k} q^{(k-j)^2 + a(k-j)} \binom{k}{j}_q \frac{(q;q)_{k+a}}{(q;q)_{k+a-j}} = 1. \quad (4.19)$$

Using Eq. (4.19) with $k = t - l$ we write Eq. (4.18) in the form

$$A \frac{1}{(q;q)_r (q;q)_{r+a} (q;q)_{t-r}} q^{-t^2 - ta} \sum_{l \geqslant 0} (-1)^l q^{(1/2)l(l-1)} \binom{t-r}{l}_q. \quad (4.20)$$

Using Eq. (4.10) we obtain

$$\sum_{0 \leqslant l \leqslant t-r} (-1)^l q^{(1/2)l(l-1)} \binom{t-r}{l}_q = (1;q)_{t-r} = \delta_{t-r,0}.$$

As a consequence, Eq. (4.20) is equal to δ_{tr}, as claimed.

V. PROOF OF THE COMBINATORIAL IDENTITIES

In this section we establish the combinatorial identities (4.14), (4.16), and (4.19) used in the previous section. The proofs follow a standard pattern familiar from q calculus.
   Proof of Lemma IV 2: We denote the left hand side of Eq. (4.14) by $A(\rho,k)$ and observe that

$$A(\rho,0) = (q;q)_\rho. \quad (5.1)$$

Now, using the well-known identity for the q-deformed binomial coefficients,

$$\binom{n+1}{m}_q = \binom{n}{m}_q q^m + \binom{n}{m-1}_q, \quad (5.2)$$

it is easy to verify that

$$A(\rho,k+1) = A(\rho-1,k) - q^{-k} A(\rho,k). \quad (5.3)$$

This recursion relation has a unique solution satisfying Eq. (5.2) for all ρ. Setting $A(\rho,k) := (q;q)_{\rho-k} q^{k(\rho-k+1)}$ we see that this is the required solution to Eq. (5.3). □
   Proof of Lemma IV 3: We denote the left hand side of Eq. (4.16) by $A(u,v,r)$ and observe that

$$A(u,v,0) = \frac{(q;q)_u}{(q;q)_v}. \tag{5.4}$$

As a consequence of Eq. (5.3), $A(u,v,r)$ obeys the following recursion

$$A(u,v,r+1) = A(u+1,v+1,r) - q^{u-v-r}A(u,v,r). \tag{5.5}$$

This recursion relation has a unique solution satisfying Eq. (5.4). Setting $A(u,v,r):= (q;q)_{u-v}\binom{u}{v+r}_q$ we verify that Eq. (5.5) holds and so this is the required solution. □

Proof of Lemma IV 4: Substituting $k-j \to j$ we rewrite Eq. (4.19) in the following slightly simpler form:

$$\sum_{0<j<k} q^{j(j+a)} \binom{k}{j}_q \frac{(q;q)_{k+a}}{(q;q)_{j+a}} = 1. \tag{5.6}$$

Let $A(a,k)$ denote the left hand side of Eq. (5.6). Then

$$A(a,0) = 1, \tag{5.7}$$

for all a. Using the identity

$$\binom{n+1}{m}_q = \binom{n}{m}_q + \binom{n}{m-1}_q q^{n+1-m}, \tag{5.8}$$

we find that

$$A(a,k+1) = A(a,k) + q^{k+a+1}(A(a+1,k) - A(a,k)). \tag{5.9}$$

This recursion has a unique solution satisfying the initial condition (5.8). Clearly $A(a,k) = 1$ is the required solution. □

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