The Loop Space $S^1 \to \mathbb{R}$ and Supersymmetric Quantum Fields*

ARTHUR JAFFE, ANDRZEJ LESNIEWSKI, AND JONATHAN WEITSMAN[†]

Harvard University, Cambridge, Massachusetts 02138

Received January 6, 1988

We present index theorey for Dirac operators Q on the loop space $S^1 \to \mathbb{R}$. These Dirac operators are obtained from sypersymmetric quantum field models containing one real Bose field and one real (Majorana) Fermi field. The interactions of the model are described by a real polynomial V. We prove that Q is Fredholm and we compute its index, namely $\pm [(\deg V + 1) \bmod 2]$. © 1988 Academic Press, Inc.

1. INTRODUCTION

In this paper we continue our program, begun in [1–3], of index theory for Dirac operators on loop space. See [4] for a review of our work. We present here the construction of a family of Dirac operators Q on $\Lambda\mathbb{R}$, the space of smooth maps $\varphi: S^1 \to \mathbb{R}$, where S^1 is a circle. This space is an infinite-dimensional manifold, the loop space of $\mathbb{R}$.

The Dirac operators we study are suggested by two-dimensional, supersymmetric quantum field models and are parameterized by a real polynomial V of degree $n \geq 1$. Unlike the case of the complex loop space $\Lambda\mathbb{C}$, the quantum field theories studied here are not ultraviolet finite. This means that the theory is not completely specified by the parameters of V but it requires a "renormalization." This is accomplished by introducing an additional real parameter, the Wick ordering mass $m > 0$. In the physics literature, this class of models is sometimes referred to as the $N = 1$, Wess–Zumino model.

Our main result is the computation of the index $i(Q_+)$ associated with Q (see [4, Sect. III], for the definitions); we find that on $\Lambda\mathbb{R}$, $i(Q_+) = \varepsilon[(\deg V + 1) \bmod 2]$, where $\varepsilon = \pm 1$ is the sign of the highest degree coefficient of V. This result is proved by establishing the existence of a homotopy between the Dirac operator Q and a finite-dimensional Dirac operator, whose index has been computed in [5, 6] (see also [7]).

The technical aspects of the construction given here are somewhat different from those of [1–3]. The main difference in this respect centers about the presence of a Majorana, or self-adjoint, Fermi field in the model. As a consequence, we find that

a Pfaffian appears in the functional integral representation of the heat kernel of Q^2 (rather than a determinant, as was the case for a Dirac fermion). In order to establish convergence estimates, we use the theory of infinite-dimensional Pfaffians developed for this purpose in [8].

Another substantial difference between the quantum fields studied here and those of [1–3] is the possibility of supersymmetry breaking [5, 6]. For deg V odd, the index vanishes; hence it is possible that zero is not an eigenvalue of the Hamiltonian. Vanishing of the index is not sufficient to guarantee the absence of a zero energy state. However, it is known [5–7] that for the quantum mechanics models described by the endpoint $Q(0)$ of the homotopy we construct, the ground state energy is in fact nonzero. As a consequence, the ground state for $H(0) = Q(0)^2$ is degenerate and supersymmetry is said to be broken. As all the models in [1–3] give rise to Q's which have a nonzero index, the quantum field theories presented here give our first candidates for supersymmetry breaking.

The main mathematical object which we construct and then use to establish estimates is a measure on path space over loop space. This is a measure $d\mu$ on the space of functions

$$\{\varphi: B \times S^1 \to \mathbb{C}\}.$$

The set of times B is either $\mathbb{R}$ or S^1. This measure has the general form

$$d\mu = \mathrm{Pf}_3 e^{-\mathscr{A}} \, d\mu. \tag{I.1}$$

Here Pf_3 is a regularized, relative Pfaffian, $\mathscr{A}$ denotes a renormalized action, and $d\mu$ is a Gaussian measure. A precise formulation of this result is found in Theorem III.3. We construct, in fact, three such measures of this form, $d\mu$, $d\mu_p$ and $d\mu_a$, where

$$\int d\mu = \langle \Omega_0, e^{-\beta H} \Omega_0 \rangle, \tag{I.2}$$

$$\int d\mu_a = \mathrm{Tr}(e^{-\beta H}), \tag{I.3}$$

and

$$\int d\mu_p = \mathrm{Str}(e^{-\beta H}), \tag{I.4}$$

where Ω_0 is the Fock vacuum vector, where Tr denotes a trace on the Fock space $\mathscr{H}$, and where Str is a graded (super) trace on $\mathscr{H}$. These measures yield corresponding states on the field algebra.

II. Dirac Operators on Loop Space

In this section we define the Dirac operator Q on the loop space $\Lambda\mathbb{R}$. We first define the loop space, both as maps $\varphi: S^1 \to \mathbb{R}$ and also by its Fock space construction. The need for a spin structure on loop space leads to the introduction of the ferminonic Fock space. We then introduce a polynomial function V on loop space which can be interpreted as a connection (or potential) for the Dirac operator Q. The Q we construct is self-adjoint and its square H is a Laplace operator on loop space. We show that H and Q have compact resolvents and that Q is Fredholm. We then show that there is a continuous family $Q(\kappa)$ of Fredholm operators with compact resolvent of the form $Q_0 + Q_I(\kappa)$, such that $Q_I(\kappa)$ acts on a space of dimension $O(\kappa)$ and such that the index of $Q(\kappa)$ is independent of κ. Using this homotopy, we compute the index of Q.

II.1. *The Hilbert Space*

The Hilbert space $\mathcal{H}$ is the tensor product of a bosonic Hilbert space $\mathcal{H}_b$ and a fermionic Hilbert space $\mathcal{H}_f$. The bosonic space $\mathcal{H}_b$ is a symmetric tensor algebra

$$\mathcal{H}_b = \bigoplus_{k=0}^{\infty} \otimes_S^k K \tag{II.1}$$

over $K = L_2(T_l^1)$. We denote the circle here as the one-torus $T_l^1 = \mathbb{R}/l\mathbb{Z}$, $l > 0$. The symbol $\otimes_S^k$ means the kth symmetric tensor power. The fermionic space $\mathcal{H}_f$ is an antisymmetric tensor algebra over W,

$$\mathcal{H}_f = \bigoplus_{k=0}^{\infty} \bigwedge^k K, \tag{II.2}$$

where $\bigwedge^k$ means the kth exterior power.

Let $\Omega_0^b \in \mathcal{H}_b$, the bosonic Fock vacuum vector, be defined by

$$\Omega_0^b = (1, 0, 0, \ldots). \tag{II.3}$$

Likewise we define $\Omega_0^f \in \mathcal{H}$, as the fermionic Fock vacuum vector. The full Fock vacuum vector is $\Omega_0 = \Omega_0^b \otimes \Omega_0^f$. Let $\mathcal{D}_0$ denote the set of vectors in $\mathcal{H}$ with finitely many nonvanishing components and C^∞ wavefunctions.

On $\mathcal{H}_b$ we define for $p \in \hat{T}_l^1 = (2\pi/l)\,\mathbb{Z}$ the operator $a(p)$ satisfying

$$a(p)\,\Omega_0^b = 0, \tag{II.4}$$

and

$$[a(p), a(q)] = [a(p)^*, a(q)^*] = 0,$$
$$[a(p), a(q)^*] = \delta_{pq}, \tag{II.5}$$

where δ_{pq} is the Kronecker delta. The time zero Bose field is an operator valued distribution defined by

$$\varphi(x) = \frac{1}{\sqrt{l}} \sum_{p \in \hat{T}_l^1} (2\mu(p))^{-1/2} (a(p)^* + a(-p)) e^{-ipx}, \qquad (\text{II}.6)$$

where $\mu(p) = (p^2 + m^2)^{1/2}$, and where $m > 0$ is fixed. The momentum conjugate to φ is given by

$$\pi(x) = \frac{i}{\sqrt{l}} \sum_{p \in \hat{T}_l^1} (\mu(p)/2)^{1/2} (a(p)^* - a(-p)) e^{-ipx}. \qquad (\text{II}.7)$$

We then verify that

$$[\varphi(x), \varphi(y)] = [\pi(x), \pi(y)] = 0,$$
$$[\pi(x), \varphi(y)] = -i\delta(x - y), \qquad (\text{II}.8)$$

where δ is the Dirac measure.

On $\mathscr{H}_f$ we define operators $b(p)$, $p \in \hat{T}_l^1$, satisfying

$$\{b(p), b(q)\} = \{b(p)^*, b(q)^*\} = 0,$$
$$\{b(p), b(q)^*\} = \delta_{pq}, \qquad (\text{II}.9)$$

where $\{\ ,\ \}$ denotes the anticommutator. The time zero, Majorana Fermi fields are defined by

$$\psi_+(x) = \frac{1}{\sqrt{l}} \sum_{p \in \hat{T}_l^1} (2\mu(p))^{-1/2} (v(-p) b(p)^* + v(p) b(-p)) e^{-ipx},$$
$$\psi_-(x) = \frac{i}{\sqrt{l}} \sum_{p \in \hat{T}_l^1} (2\mu(p))^{-1/2} (v(p) b(p)^* - v(-p) b(-p)) e^{-ipx}, \qquad (\text{II}.10)$$

where $v(p) = (p + \mu(p))^{1/2}$. They satisfy the anticommutation relations

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha\beta} \, \delta(x - y), \qquad (\text{II}.11)$$

where $\alpha, \beta \in \{+, -\}$.

By tensoring with the appropriate identity operator we let $\varphi(x)$, $\pi(x)$, and $\psi_\alpha(x)$ act on the whole Hilbert space. As no confusion will arise we will denote these by abuse of notation as $\varphi(x)$, $\pi(x)$, and $\psi_\alpha(x)$. Technically, $\varphi(x)$, $\pi(x)$, and $\psi_\alpha(x)$, $\alpha = \pm$, are real, bilinear forms on the domain $\mathscr{D}_0 \times \mathscr{D}_0$. Averaged with real C_0^∞ test functions, they extend uniquely to self-adjoint operators.

Let N_b denote the bosonic number operator, defined as the unique self-adjoint extension of the operator

$$\sum_{p \in \hat{T}_l^1} a(p)^* a(p) \qquad (\text{II}.12)$$

with domain $\mathcal{D}_0$. Similarly we define the fermionic number operator N_f. The operator

$$\Gamma = \exp(i\pi N_f) \tag{II.13}$$

defines a grading of $\mathcal{H}$ (see [4]), and

$$\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-. \tag{II.14}$$

Let

$$\gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

denote the two-dimensional, Dirac matrices. We define $\bar{\psi} = \psi^* \gamma_0$. As γ_0 is hermitian, the bilinear form $:\bar{\psi}\psi: = :\psi^*\gamma_0\psi:$ is real. Here the colons denote Wick ordering with respect to the mass m.

II.2. The Regularized Dirac Operator with a Polynomial Potential

Let $V(x)$ be a real polynomial of degree $n \geqslant 2$; let $W(x) = V(x) - \frac{1}{2}mx^2$. Let $\chi \in \mathscr{S}(\mathbb{R})$ satisfy

  (i) $\chi \geqslant 0$,
  (ii) $\int \chi(x)\,dx = 1$,
  (iii) $\chi(-x) = \chi(x)$,
  (iv) $\hat{\chi}(p) \geqslant 0$,
  (v) $\operatorname{supp} \hat{\chi}(p) \subset [-1, 1]$, $\hat{\chi}(p) > 0$ for $|p| \leqslant \frac{1}{2}$.

We use the following periodic regularization of the Dirac measure,

$$\chi_\kappa(x) = \kappa \sum_{k \in \mathbb{Z}} \chi(\kappa(x - kl)), \tag{II.15}$$

and define the regularized field operators by

$$\begin{aligned} \varphi_\kappa(x) &= \chi_\kappa * \varphi(x), \\ \psi_{\alpha,\kappa}(x) &= \chi_\kappa * \psi_\alpha(x). \end{aligned} \tag{II.16}$$

The regularized supercharge is defined as a bilnear form of the domain $\mathcal{D}_0 \times \mathcal{D}_0$ by

$$Q_\alpha(\kappa) = Q_{\alpha,0} + Q_{\alpha,l}(\kappa), \tag{II.17}$$

where

$$Q_{\pm,0} = \int_{T_l^1} (\pm\psi_\pm(x)\,\pi(x) - \psi_\mp(x)\,\partial_x\varphi(x) + m\psi_\pm(x)\,\varphi(x))\,dx, \tag{II.18}$$

$$Q_{\pm,l}(\kappa) = \int_{T_l^1} \psi_\pm(x) :W'(\varphi_\kappa(x)): dx. \tag{II.19}$$

Here : : denotes Wick ordering with respect to the mass m. The regularized Dirac operator is defined by

$$Q(\kappa) = \frac{1}{\sqrt{2}} (Q_+(\kappa) + Q_-(\kappa))$$

$$= Q_0 + Q_I(\kappa), \tag{II.20}$$

where

$$Q_0 = \frac{1}{\sqrt{2}} (Q_{+,0} + Q_{-,0}) \quad \text{and} \quad Q_I(\kappa) = \frac{1}{\sqrt{2}} (Q_{+,I}(\kappa) + Q_{-,I}(\kappa)).$$

PROPOSITION II.1. *The form $Q(\kappa)$ uniquely defines an essentially self-adjoint operator with domain $\mathcal{D}_0$, such that (as a form) its square $H(\kappa) = Q(\kappa)^2$ is also essentially self-adjoint on $\mathcal{D}_0$. On $\mathcal{D}_0$,*

$$H(\kappa) = H_0 + \frac{1}{2} \int_{T_l^1} (:W'(\varphi_\kappa(x)):)^2 \, dx + m \int_{T_l^1} \varphi(x): W'(\varphi_\kappa(x)): dx$$

$$+ \frac{1}{2} \int_{T_l^1} \bar{\psi}_\kappa(x) \, \psi(x): W''(\varphi_\kappa(x)): dx. \tag{II.21}$$

Remarks. (1) The last term in $H(\kappa)$ is actually real, without further symmetrization. (2) We use, without danger of confusion, the symbols $Q(\kappa)$ and $H(\kappa)$ to denote the unique self-adjoint extensions of $Q(\kappa)$ and $H(\kappa)$, respectively.

Proof. The statements follow by arguments similar to [1, Sect. II.5; 2, Sect. VI.1].

II.3. *The a priori Estimates*

We state now four basic estimates which are proved in Section III. Let $I_p(\mathcal{H})$ denote the pth Schatten class of operators with the norm $\|T\|_p = \{\mathrm{Tr}(T^*T)^{p/2}\}^{1/p}$.

THEOREM II.2. *For any $\beta > 0$ and $0 \leqslant \kappa < \infty$, $\exp(-\beta H(k)) \in I_2(\mathcal{H})$. There is $C = C(\beta) < \infty$ such that*

$$\|\exp(-\beta H(\kappa))\|_2 \leqslant C, \tag{II.22}$$

uniformly in κ.

Our second estimate states that the semigroup $\exp(-\beta H(\kappa))$ is continuous in κ and has a limit as $k \to \infty$.

THEOREM II.3. *Let $\beta > 0$ be fixed. The map*

$$\kappa \to \exp(-\beta H(\kappa)) \tag{II.23}$$

is continuous from $\mathbb{R}_+$ *to* I_2. *The family* $\{\exp(-\beta H(\kappa))\}$ *converges in* I_2 *as* $\kappa \to \infty$ *to a semigroup* $T(\beta)$, $\beta \geqslant 0$.

In order to express $T(\beta)$ as $\exp(-\beta H)$ with a self-adjoint generator H, we require strong continuity of $T(\beta)$.

THEOREM II.4. *The semigroup* $T(\beta)$ *is strongly continuous at* $\beta = 0$,

$$\text{st} \lim_{\beta \to 0} T(\beta) = I. \tag{II.24}$$

As a corollary we obtain

THEOREM II.2′. *For* $\beta > 0$, $\exp(-\beta H) \in I_2(\mathscr{H})$.

Finally, our last a priori estimate deals with continuity and convergence of $Q(\kappa)$. Let $\delta Q = (Q(\kappa) - Q(\kappa'))^-$, where $^-$ denotes the operator closure.

THEOREM II.5. *Let* $\beta > 0$. *Then* $\text{Range}(\exp(-\beta H(\kappa))) \subset D(\delta Q)$ *and*

$$\|e^{-\beta H(\kappa')} \delta Q e^{-\beta H(\kappa)}\|_2 = o(1), \tag{II.25}$$

as $|\kappa - \kappa'| \to 0$, *and as* $\kappa, \kappa' \to \infty$.

A consequence of these four estimates is the existence of the $\kappa \to \infty$ limit of $Q(\kappa)$.

THEOREM II.6. *The resolvent* $(Q(\kappa) \pm i)^{-1}$ *is norm-continuous in* κ *and norm-convergent as* $\kappa \to \infty$. *The limiting operator is the resolvent of a self-adjoint operator* Q *with* $Q^2 = H$.

The proof of this theorem follows *verbatim* [2, Sect. V]. Write

$$Q(\kappa) = \begin{pmatrix} 0 & Q_-(\kappa) \\ Q_+(\kappa) & 0 \end{pmatrix}, \tag{II.26}$$

according to the decomposition (II.14). Set $Q(\infty) = Q$. Another consequence of the above result is

THEOREM II.7. *For* $0 \leqslant \kappa \leqslant \infty$,

(i) $Q_+(\kappa)$ *is Fredholm*,

(ii) *the index* $i(Q_+(\kappa))$ *is constant in* κ. *In particular*,

$$i(Q_+) = i(Q_+(0)). \tag{II.27}$$

Proof. See [4], Section III.

II.4. *Index Theorem*

In this section we compute the index of Q_+.

THEOREM II.8 (Index Theorem). *Let Q be the Dirac operator corresponding to a polynomial V. Then*

$$i(Q_+) = \varepsilon[(1 + \deg V) \bmod 2], \qquad (\text{II.28})$$

where ε is the sign of the highest order coefficient of V.

Remark. The index $i(Q_+)$ is independent of the Wick ordering mass m. In fact, if $\deg V = n$, then the Wick ordering adds, for $\kappa < \infty$, a polynomial δV of degree $n - 2$ to V. Thus we do not expect that Wick ordering influences the index. The limit $\kappa \to \infty$, however, exists only with the Wick ordering taken into account. In this limit the coefficients of δV diverge (asymptotically as polynomials in $\log \kappa$). Our estimates establish the existence and continuity of the index also at $\kappa = \infty$. Hence the index is independent of κ for all $\kappa \leqslant \infty$.

Proof. Let $\varphi_0 = l^{-1/2}\hat{\varphi}(0)$, $\pi_0 = l^{-1/2}\hat{\pi}(0)$, and $\psi_{\alpha,0} = l^{-1/2}\hat{\psi}_\alpha(0)$. Then $Q(0)$ can be expressed as

$$Q(0) = Q^0(0) \oplus Q_0^\perp, \qquad (\text{II.29})$$

corresponding to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, where $\mathcal{H}_0$ is the subspace spanned by the zero momentum modes. Here, $Q_0^\perp$ is the restriction of the free Dirac operator to $\mathcal{H}_0^\perp$ and it has a unique ground state. Therefore, $i(Q_+(0)) = i(Q_+^0(0))$. An explicit expression for $Q^0(0)$ is

$$Q^0(0) = \frac{1}{\sqrt{2}}((\psi_{-,0} - \psi_{+,0})\pi_0 + (\psi_{-,0} + \psi_{+,0}):V'(\varphi_0):), \qquad (\text{II.30})$$

which is unitarily equivalent to

$$D = \frac{1}{\sqrt{2}}\left(-i\sigma_1\frac{d}{dx} + \sigma_2 v'(x)\right). \qquad (\text{II.31})$$

Here $x = \varphi_0$, $v(x) = :V(x):$, where σ_1 and σ_2 are the usual Pauli matrices and $::$ denotes the linear map defined on monomials by

$$x^j \to c^{j/2}H_j(c^{-1/2}x),$$

where H_j denotes the jth Hermite polynomial, and where $c = \langle \Omega_0, \varphi_0^2\Omega_0 \rangle$ is the Wick ordering constant. This is precisely the Dirac operator corresponding to the $N = 1$ supersymmetric quantum mechanics. Its index is known to be equal to $\varepsilon[(1 + \deg v) \bmod 2]$ [5, 7]. This is our claim, since the highest order terms of $v(x)$ and $V(x)$ are identical.

III. Infinite-Dimensional Pfaffians and the a priori Estimates

In this section we present the proofs of Theorems II.2–5. Many of the proofs in this section are substantially the same as those given for similar arguments in [3]. The basic difference is that Pfaffians replace determinants. To avoid redundancy, we only present arguments that differ from those of [3].

The proofs of Theorems II.2–5 rely on path integral representations of the heat kernel for the approximating Hamiltonians $H(\kappa)$. Such representations were established for the $N = 2$ Wess–Zumino model in [2]. Here we use our theory of infinite-dimensional Pfaffians developed in [8]; we refer the reader to this paper for the definitions and properties of the Pfaffians.

We also establish a path integral representation of the index, an analog of the similar representation of [1]. This formula may be used to give an alternative computation of the index.

III.1. Feynman–Kac Formulas for the Regularized Theory

Let us begin with some notation. We consider a two-dimensional cylindrical or toroidal space-time $B \times T^1_l$, where either $B = \mathbb{R}$ or $B = T^1_\beta$. Let Δ be the Laplacian on $B \times T^1_l$. If $B = \mathbb{R}$, we set $C_l = (-\Delta + m^2)^{-1}$ on the cylinder, while if $B = T^1_\beta$, we set $C_{l,\beta} = (-\Delta + m^2)^{-1}$ on the torus. Let $d\mu_C(\Phi)$ denote the Gaussian measure on $\mathscr{S}'(B \times T^1_l)$ with mean zero and with covariance C, where $C = C_l$ or $C = C_{l,\beta}$.

Let $D = \sum_\mu i\gamma^E_\mu \partial_\mu$ denote the Dirac operator on $B \times T^1_l$, where γ^E_μ are the Euclidean Dirac matrices: $\gamma^E_0 = -i\gamma_0$, $\gamma^E_1 = \gamma_1$, and γ_0, γ_1 are the matrices of Section II.1. We set $S_l = \gamma_0(D + m)^{-1}$ for $B = \mathbb{R}$ and similarly we define $S_{l,\beta}$ for $B = T^1_\beta$. By $\tilde{D}$ we denote the Dirac operator on the torus twisted in the time direction by π; i.e., functions in the domain of $\tilde{D}$ satisfy $f(x_0 + \beta, x_1) = -f(x_0, x_1)$. We let $\tilde{S}_{l,\beta} = \gamma_0(\tilde{D} + m)^{-1}$. Note that all the above "fermionic covariances" are skew-symmetric,

$$S^T = -S, \qquad (III.1)$$

where T means transposition (no complex conjugation!).

For $\Phi \in \mathscr{S}'(\mathbb{R} \times T^1_l)$ we set

$$\Phi_\kappa(x_0, x_1) = \int_{T^1_l} \Phi(x_0, x'_1) \chi_\kappa(x_1 - x'_1) \, dx'_1, \qquad (III.2)$$

and

$$A^{(\kappa)}_l(\Phi) = \int_{[0,\beta] \times T^1_l} ((:W'(\Phi_\kappa(x)):)^2 + m\Phi(x):W'(\Phi_\kappa(x)):) \, dx. \qquad (III.3)$$

Let $\mathscr{K}_{-1/2} = \mathscr{H}_{-1/2} \oplus \mathscr{H}_{-1/2}$, where $\mathscr{H}_\alpha = \mathscr{H}_\alpha(B \times T^1_l)$ is the Sobolev space of order α. Also, let $B^{(\kappa)}_l(\Phi)$ denote the operator on $\mathscr{K}_{-1/2}$, whose integral kernel is given by

$$B^{(\kappa)}_l(\Phi)(x, y) = \tfrac{1}{4}\gamma_0(:W''(\Phi_\kappa(x)): + :W''(\Phi_\kappa(y)):) \chi_{[0,\beta]}(x_0) \chi_{[0,\beta]}(y_0) \chi_\kappa(x_1 - y_1). \qquad (III.4)$$

Here $\chi_{[0,\,\beta]}(t)$ is the characteristic function of the interval $[0, \beta]$. Clearly, $B_l^{(\kappa)}(\Phi)$ is a skew-symmetric operator on $\mathcal{K}_{-1/2}$. A simple estimate (cf. [2, Sect. VI.2]) shows that for almost all Φ, and for $0 \leqslant \kappa < \infty$, the assumptions of Definition III.9 of [8] are fulfilled with $A = S_l$, $B = B_l^{(\kappa)}(\Phi)$, $V = (-d^2/dx_1^2 + m^2)^\varepsilon$, where $\varepsilon > 0$ is sufficiently small. Then the relative Pfaffian

$$\mathrm{Pf}(S_l, B_l^{(\kappa)}(\Phi)) \tag{III.5}$$

is a random variable. The Pfaffian has the property [8] that

$$\mathrm{Pf}(S_l, B_l^{(\kappa)}(\Phi))^2 = \det(I - K_l^{(\kappa)}(\Phi)), \tag{III.6}$$

with

$$K_l^{(\kappa)}(\Phi) = S_l B_l^{(\kappa)}(\Phi). \tag{III.7}$$

Equation (III.6) and its generalizations allow us to reduce estimates to those of [2, 3]. Finally, we define

$$F_l^{(\kappa)}(\Phi) = \mathrm{Pf}(S_l, B_l^{(\kappa)}(\Phi)) \exp(-A_l^{(\kappa)}(\Phi)), \tag{III.8}$$

and the analogous quantities $\tilde{F}_{l,\,\beta}^{(\kappa)}(\Phi)$ and $F_{l,\,\beta}^{(\kappa)}(\Phi)$ obtained by replacing (C_l, S_l) by $(C_{l,\,\beta}, \tilde{S}_{l,\,\beta})$ and $(C_{l,\,\beta}, S_{l,\,\beta})$, respectively. Relation (III.6) and arguments similar to those of [2, Sect. VI.2] show the following: there exists $\alpha > 0$ such that $F^{(\kappa)} \in L_p(d\mu_C)$ for all $1 \leqslant p \leqslant 1 + \alpha$ and for all $0 \leqslant \kappa < \infty$.

Our proof of Theorem II.4 requires a path integral representation for the matrix elements of the heat kernel. Such a representation was established in [2, Proposition VI.8] for the complex case. Let $u_j \in \mathcal{K}_{-1/2}(T_l^1), j = 1, ..., q$, $w_j \in \mathcal{K}_{-1}(T_l^1)$, $j = 1, ..., p$, and set $\xi_j = \psi_{\mu_j}(u_j)$, $j = 1, ..., q$, $\xi_{j-q} = \varphi(w_j)$, $j = 1, ..., p$. For $0 \leqslant s \leqslant \beta$ we define

$$\xi_j(s) = e^{-sH_0} \xi_j e^{sH_0}. \tag{III.9}$$

Let $\alpha_j, j = 1, ..., p + q$, be C^∞ functions with $\mathrm{supp}\,\alpha_j \subset [0, \beta]$. We consider the state

$$\Omega = \int T\left(\prod_{j=1}^{p+q} \xi_j(s_j)\right) \prod_{j=1}^{p+q} \alpha_j(s_j)\, ds\Omega_0, \tag{III.10}$$

where T means "time ordering" defined as follows.

Consider the cone $s_{j_1} < s_{j_2} < \cdots < s_{j_{p+q}}$. In this cone the time ordered product is defined by

$$T\left(\prod_{j=1}^{p+q} \xi_j(s_j)\right) = \varepsilon \prod_{i=1}^{p+q} \xi_{j_i}(s_{j_i}), \tag{III.11}$$

where ε is here the signature of the permutation of the order of the fermionic operators induced by time ordering, compared to the natural order. On the boun-

dary of the cone, fields may occur at equal times. For a subset of fields at equal times, the time ordered product is not unique. Its order is specified by choosing a given cone and letting the time differences tend to zero within that cone. Let Ω' be a second vector of the form (III.10) such that $q + q' = 2k$. We set

$$g_1 = u_q^* \theta_\beta \alpha_q, \ ..., g_q = u_1^* \theta_\beta \alpha_1, \qquad g_{q+1} = u_1' \alpha_1', \ ..., g_{2k} = u_{q'}' \alpha_{q'}',$$

where $(\theta_\beta \alpha)(s) = \alpha(\beta - s)^*$, and relabel correspondingly the spinor indices. Likewise, we define the test functions $f_1, ..., f_{p+p'}$ corresponding to the bosonic operators. Let $\mathrm{Pf}(A, B; g_1, ..., g_{2k})$ be the Pfaffian minor as defined in the Appendix.

PROPOSITION III.1. *With the above definitions,*

$$\langle \Omega, \exp(-\beta H(\kappa)) \Omega' \rangle = \varepsilon \int \mathrm{Pf}(S_l, B_l^{(\kappa)}(\Phi); g_1, ..., g_{2k}) \left(\prod_{j=1}^{p+p'} \Phi(f_j) \right)$$

$$\times \exp(-A_l^{(\kappa)}(\Phi)) \, d\mu_{C_l}(\Phi), \qquad (\text{III.12})$$

where $\varepsilon = \pm 1$.

Proof. The proof of this formula follows the methods of [9, 10, 2]. The only difference is that the formula (VI.41) of [2] is replaced by an expression at strictly time ordered points, $s_1 < s_2 < \cdots < s_{2n}$,

$$\langle \Omega_0^f, \psi_{\mu_1}(x_1) \, e^{-(s_2 - s_1) H_{0, f}} \psi_{\mu_2}(x_2) \cdots e^{-(s_{2n} - s_{2n-1}) H_{0, f}} \psi_{\mu_n}(x_{2n}) \, \Omega_0^f \rangle$$

$$= \mathrm{Pf}\{(\gamma_0 S_l)_{\mu_j \mu_k}(s_j - s_k, x_j - x_k)\}. \qquad (\text{III.13})$$

The Pfaffian (III.13) is defined on the indices $j, k = 1, ..., 2n$. The kernel of $(\gamma_0 S_l)_{\mu\nu}$ is a function defined on space-time.

The proofs of Theorems II.2, 3, and 5 require path integral representations involving "finite temperature states"; see [1–4] for a discussion. Let us state the simplest representation of this sort.

PROPOSITION III.2. *Let* $\Xi_{l, \beta} = \prod_{p \in \hat{T}_l^!} \coth(\beta \mu(p))$. *Then for* $\beta > 0$,

$$\mathrm{Tr}(\exp(-\beta H(\kappa))) = \Xi_{l, \beta} \int \tilde{F}_{l, \beta}^{(\kappa)}(\Phi) \, d\mu_{C_{l, \beta}}(\Phi). \qquad (\text{III.14})$$

Finally, we state a path integral representation of the index of $Q_+(\kappa)$. To this aim we use the well-known heat kernel representation of the index (see, e.g., [4, Sect. III]):

$$i(Q_+(\kappa)) = \mathrm{Tr}(\Gamma \exp(-\beta H(\kappa))). \qquad (\text{III.15})$$

This yields

PROPOSITION III.3.   *For $\beta > 0$,*

$$i(Q_+) = \int F_{l,\beta}^{(\kappa)}(\Phi)\, d\mu_{C_{l,\beta}}(\Phi). \tag{III.16}$$

III.2. *The Limit $\kappa \to \infty$*

The proofs of Theorems II.2–5 are simple applications of the methods used to prove analogous statements in [3] and we do not reproduce them here in full detail. The following remarks should help the reader to bring the methods of [3] into the context of the $N = 1$ case.

1.   To prove Theorem II.2 we use (V.11) of [8] to write

$$\mathrm{Pf}(\tilde{S}_{l,\beta}, B_{l,\beta}^{(\kappa)}(\Phi)) = \mathrm{Pf}_3(\tilde{S}_{l,\beta}, B_{l,\beta}^{(\kappa)}(\Phi)) \exp\{ -\tfrac{1}{2}\,\mathrm{Tr}\,\tilde{K}_{l,\beta}^{(\kappa)}(\Phi) - \tfrac{1}{4}\,\mathrm{Tr}\,\tilde{K}_{l,\beta}^{(\kappa)}(\Phi)^2\}, \tag{III.17}$$

with $\tilde{K}_{l,\beta}^{(\kappa)}(\Phi)$ given by (III.7), and correspondingly

$$\tilde{F}_{l,\beta}^{(\kappa)}(\Phi) = \mathrm{Pf}_3(\tilde{S}_{l,\beta}, B_{l,\beta}^{(\kappa)}(\Phi)) \exp(-\tilde{\mathscr{A}}_{l,\beta}^{(\kappa)}(\Phi)), \tag{III.18}$$

where

$$\tilde{\mathscr{A}}_{l,\beta}^{(\kappa)}(\Phi) = A_{l,\beta}^{(\kappa)}(\Phi) + \tfrac{1}{2}\,\mathrm{Tr}\,\tilde{K}_{l,\beta}^{(\kappa)}(\Phi) + \tfrac{1}{4}\,\mathrm{Tr}\,\tilde{K}_{l,\beta}^{(\kappa)}(\Phi)^2. \tag{III.19}$$

Note the identity (see [8, Eq. (V.1)])

$$\mathrm{Pf}_3(\tilde{S}_{l,\beta}, B_{l,\beta}^{(\kappa)}(\Phi))^2 = \det_3(I - \tilde{K}_{l,\beta}^{(\kappa)}(\Phi)). \tag{III.20}$$

Estimates similar to those of [3] show that the L_p-limit $\tilde{\mathscr{A}}_{l,\beta}(\Phi) = \lim_{\kappa \to \infty} \tilde{\mathscr{A}}_{l,\beta}^{(\kappa)}(\Phi)$ exists for all $1 \leq p < \infty$ and thus $\exp(-\tilde{\mathscr{A}}_{l,\beta}(\Phi))$ is a random variables. Similarly, $\mathrm{Pf}_3(\tilde{S}_{l,\beta}, B_{l,\beta}(\Phi))$ is a random variable, where $B_{l,\beta}(\Phi)$ is defined by (III.4) with $\chi_\kappa(x_1 - y_1)$ replaced by the Dirac measure $\delta(x_1 - y_1)$. Furthermore, we have the following:

THEOREM III.3.   (i)   *The functions $\tilde{F}_{l,\beta}^{(\kappa)}$ converge as $\kappa \to \infty$ in each $L_p(d\mu_{C_{l,\beta}})$, for $1 \leq p < \infty$ to the limit*

$$\tilde{F}_{l,\beta}(\Phi) = \mathrm{Pf}_3(\tilde{S}_{l,\beta}, B_{l,\beta}(\Phi)) \exp(-\tilde{\mathscr{A}}_{l,\beta}(\Phi)). \tag{III.21}$$

(ii)   *Similar statements hold for $F_{l,\beta}^{(\kappa)}(\Phi)$ and $F_l^{(\kappa)}(\Phi)$ with respect to $d\mu_{C_{l,\beta}}$ and $d\mu_{C_l}$, respectively.*

Remark.   This theorem establishes the existence of the measures of the form $d\mu$, $d\mu_p$, $d\mu_a$ discussed in the Introduction.

The proofs of Theorem II.2 and Theorem III.3 are similar to the proof of Theorem II.1 of [3] with $\zeta = 0$. In particular, we use (III.20) to reduce estimates on the Pfaffian to the familiar Fredholm determinant estimates.

2. The proof of Theorem II.3 follows the method of proof of Theorem III.2 of [3]. Set $S = \tilde{S}_{l,\beta}$ and $B(s) = sB_{l,\beta}(\Phi) + (1-s) B_{l,\beta}^{(\kappa)}(\Phi)$, $0 \leqslant s \leqslant 1$. We require an estimate on

$$\frac{d}{ds} \mathrm{Pf}_3(S, B(s)) = -\frac{1}{2} \mathrm{Tr}(B'(s) \, SB(s) \, SB(s)(S^{-1} - B(s))^{-1} \, \mathrm{Pf}_3(S, B(s))). \qquad \text{(III.22)}$$

Let

$$K = \tilde{S}_{l,\beta} B_{l,\beta}(\Phi), \qquad K^{(\kappa)} = \tilde{S}_{l,\beta} B_{l,\beta}^{(\kappa)}(\Phi),$$

and $K(s) = sK + (1-s) K^{(\kappa)}$. Then we can bound (III.22) by

$$\tfrac{1}{2} \|K - K^{(\kappa)}\|_3 \, \|K(s)\|_3^2 \, \|(S^{-1} - B(s))^{-1} \, \mathrm{Pf}_3(S, B(s)) \, S^{-1}\|, \qquad \text{(III.23)}$$

and we need a bound on the last factor in the above product. We have

$$\|(S^{-1} - B(s))^{-1} \, \mathrm{Pf}_3(S, B(s)) \, S^{-1}\|$$

$$= \sup_{\|f\| = \|g\| = 1} |(\mathscr{C}f, (S^{-1} - B(s))^{-1} \, \mathrm{Pf}_3(S, B(s)) \, S^{-1}g)|, \qquad \text{(III.24)}$$

where the norms and the inner product in this formula denote the $\mathscr{K}_{1/2}$ norms and inner product; also $\mathscr{C}$ denotes complex conjugation. Using (A.4) we write the right hand side of (III.24) as

$$\sup_{\|f\| = \|g\| = 1} \left| \det \begin{pmatrix} 0 & (\mathscr{C}f, (I - K(s))^{-1} g) \\ (\mathscr{C}g, (I - K(s))^{-1} f) & 0 \end{pmatrix} \det{}_3(I - K(s)) \right|^{1/2}$$

$$= \sup_{\|f\| = \|g\| = 1} |(\mathscr{C}f \wedge \mathscr{C}g, \wedge^2 (I - K(s))^{-1} f \wedge g)_{\wedge^2 \mathscr{K}_{1/2}} \det{}_3(I - K(s))|_{1/2}$$

$$\leqslant \|\wedge^2 (I - K(s))^{-1} \det{}_3(I - K(s))\|_{\wedge^2 \mathscr{K}_{1/2}}^{1/2},$$

and now we proceed as in [3].

3. The proof of Theorem II.4 does not differ significantly from the corresponding proof in [3]. The expansion (65) [3] for the Fredholm minor is replaced by the corresponding expansion for the Pfaffian minors. The resulting terms are estimated by means of (A.6).

4. The proof of Theorem II.5 follows the proof of Theorem VI of [3] with the difference that the Fredholm minors must be replaced by Pfaffian minors and estimated as explained in Remarks 2 and 3.

Finally, we have the integral representations

THEOREM III.4. *For $\beta > 0$,*

$$\mathrm{Tr}(e^{-\beta H}) = \Xi_{l,\beta} \int \tilde{F}_{l,\beta}(\Phi) \, d\mu_{C_{l,\beta}}(\Phi). \tag{III.25}$$

THEOREM III.5. *For $\beta > 0$,*

$$i(Q_+) = \int F_{l,\beta}(\Phi) \, d\mu_{C_{l,\beta}}(\Phi). \tag{III.26}$$

APPENDIX: PFAFFIAN MINORS

In [8, Sect. IV] we defined the notion of a Pfaffian minor. Let A and B be skew-symmetric, $I_{2n}(\mathcal{H})$ (n odd) operators on a Hilbert space $\mathcal{H}$ such that

$$(A^{-1} - B)^{-1} \text{ exists and is bounded.} \tag{A.1}$$

For $f_1, ..., f_{2k} \in \mathcal{H}$ we set

$$\mathrm{Pf}_n(A, B; f_1, ..., f_{2k}) = \mathrm{Pf}((\mathscr{C}f_j, (A^{-1} - B)^{-1} f_k)) \, \mathrm{Pf}_n(A, B), \tag{A.2}$$

where $\mathscr{C}$ denotes a complex conjugation on $\mathcal{H}$.[1] Note that $\mathrm{Pf}_n^\sigma(A, B)$ of [8] equals $\mathrm{Pf}_n(A, B; e_1, ..., e_{j_{2k}})$, where $\sigma = \{j_1, ..., j_{2k}\}$, and where e_j are elements of a real basis for $\mathcal{H}$. Since by Theorem V.6 of [8],

$$|\mathrm{Pf}_n(A, B; f_1, ..., f_{2k})| \leqslant C_1 \|A\|_{2n}^k \exp(C_2 \|AB\|_n^n) \prod_{j=1}^{2k} \|f_j\|, \tag{A.3}$$

we infer that (A.1) can in fact be relaxed. Furthermore,

$$\mathrm{Pf}_n(A, B; f_1, ..., f_{2k})^2 = \det((\mathscr{C}f_j, (A^{-1} - B)^{-1} f_k)) \det_n(I - AB). \tag{A.4}$$

This identity allows us to conclude

$$|\mathrm{Pf}_n(A, B; f_1, ..., f_{2k})|^2$$
$$= \left| \left(\bigwedge_{j=1}^{2k} \mathscr{C}f_j, \bigwedge^{2k} (I - AB)^{-1} \bigwedge_{j=1}^{2k} Af_j \right)_{\bigwedge^{2k} \mathcal{H}} \det_n(I - AB) \right|, \tag{A.5}$$

and finally

$$|\mathrm{Pf}_n(A, B; f_1, ..., f_{2k})| \leqslant \|\bigwedge^{2k} (I - AB)^{-1} \det_n(I - AB)\|_{\bigwedge^{2k} \mathcal{H}}^{1/2}$$
$$\times \prod_{j=1}^{2k} \|f_j\|^{1/2} \|Af_j\|^{1/2}. \tag{A.6}$$

---

[1] Note that it was essential in [8] to assume that $\mathcal{H}$ comes from the complexification of a real Hilbert space.

The above estimate reduces estimates on Pfaffian minors to familiar estimates on Fredholm minors and is frequently used in the proofs of Section III.2.

Finally, we note the identity

$$\mathrm{Pf}_n(A, B; f_1, ..., f_{2k}) = \mathrm{Pf}_n((V^{-1})^\mathrm{T} A V^{-1}, VBV^\mathrm{T}; Vf_1, ..., Vf_{2n}), \qquad (A.7)$$

which is valid for skew-symmetric $A, B \in I_{2n}(\mathscr{H})$ and bounded V. As in [8] we use (A.7) to extend the definition of $\mathrm{Pf}_n(A, B; f_1, ..., f_{2k})$ to pairs of operators (A, B) which are not necessarily both I_{2n}, but have the property that $(V^{-1})^\mathrm{T} A V^{-1}$, $VBV^\mathrm{T} \in I_{2n}$, for some V. This is useful, if for instance $A \in I_{2n-\varepsilon}$, and we "transfer" the extra regularity to B. Such a transfer of regularity has been used in Section III.

REFERENCES

1. A. JAFFE, A. LESNIEWSKI, AND J. WEITSMAN, *Commun. Math. Phys.* **112** (1987), 75.
2. A. JAFFE, A. LESNIEWSKI, AND J. WEITSMAN, *Commun. Math. Phys.* **114** (1988), 147.
3. A. JAFFE AND A. LESNIEWSKI, *Commun. Math. Phys.* **114** (1988), in press.
4. A. JAFFE AND A. LESNIEWSKI, "Supersymmetric Quantum Fields and Infinite Dimensional Analysis," 1987 Cargèse Lecture Notes, Preprint HUTMP B–210.
5. E. WITTEN, *Nucl. Phys. B* **188** (1981), 513.
6. E. WITTEN, *Nucl. Phys. B* **202** (1982), 253.
7. A. JAFFE, A. LESNIEWSKI, AND M. LEWENSTEIN, *Ann. Phys. (N.Y.)* **178** (1987), 313.
8. A. JAFFE, A. LESNIEWSKI, AND J. WEITSMAN, *J. Funct. Anal.*, in press.
9. E. SEILER AND B. SIMON, *Ann. Phys. (N.Y.)* **97** (1976), 470.
10. L. GROSS, *J. Funct. Anal.* **25** (1977), 162.