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A Two-Parameter Quantum Deformation of the Unit Disc

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We define and study a two-parameter deformation of the unit disc. This deformation is described in terms of a family of type I $\mathbb{C}^*$-algebras $C_{\mu,q}(\bar{U})$. We study the representation theory for $C_{\mu,q}(\bar{U})$ and construct an action of the quantized universal enveloping algebra $U_q(sl(2))$ on $C_{\mu,q}(\bar{U})$.

I. INTRODUCTION

In [KL] we constructed and studied the $\mathbb{C}^*$-algebra $C_\mu(\bar{U})$, $0<\mu<1$, of "continuous functions" on a non-commutative unit disc. The family $\mu \to C_\mu(\bar{U})$ is a quantum deformation of the algebra of continuous functions on the unit disc U in the direction of the $SU(1,1)$-invariant Poisson structure given by the Poincaré symplectic form. Our construction may be viewed as a non-perturbative implementation of the program of deformation quantization proposed in [Be] and [BFFLS] (see [KL] for a more complete list of references).

The present paper extends some of the results of [KL] to the case of a two-parameter quantum deformation of the unit disc. We study a two-parameter family of $\mathbb{C}^*$-algebras $C_{\mu,q}(\bar{U})$, $0 \leqslant \mu < 1$, $0 < q \leqslant 1$, $(\mu, q) \neq (0, 1)$, which are non-commutative deformations of a family of Poisson structures on U. Motivated by the work [SLW] on the quantum

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sphere we show that the $\mathbb{C}^*$-algebras $C_{\mu,q}(\bar{U})$ are extensions of certain standard C^*-algebras. They are not isomorphic for different values of the parameters (μ, q) and fall into three distinct categories depending on whether $\mu < 1-q$, $\mu = 1-q$, or $\mu > 1-q$. This is related to the fact that all $SU(1, 1)$-covariant Poisson structures on U can be subdivided into three classes with different symplectic leaf structures.

All infinite dimensional irreducible representations of $C_{\mu,q}(\bar{U})$ are generated by hyponormal weighted shift operators. These operators admit, in almost all cases, a Bergman space representation. For each (μ, q) we construct a probability measure $d\mu$ on a subset D of U, and consider the Hilbert space $\mathscr{H}(D, d\mu)$ of holomorphic functions square integrable with respect to $d\mu$. The measure $d\mu$ is concentrated on a discrete set of circles contained in U. The associated Bergman kernels can be expressed in terms of certain functions of q-analysis [GR]. The $\mathbb{C}^*$-algebra $C_{\mu,q}(\bar{U})$ is then represented as the $\mathbb{C}^*$-algebra generated by the Toeplitz operators on $\mathscr{H}(D, d\mu)$ with symbols in $C(\bar{D})$. This construction should be important for a non-perturbative analysis of the trajectory of a smooth function on U under the deformation map.

The paper is organized as follows. We begin with a general overview of multiparameter deformations of a smooth manifold (Section II). In particular, we remark that each curve in the parameter space starting at the classical value gives rise to a Poisson structure on the manifold. In Section III we classify, following [LW] and [SLW], the $SU(1, 1)$-covariant Poisson structures on U. In Section IV we define $C_{\mu,q}(\bar{U})$ and study all of its irreducible representations. Section V describes the structure of $C_{\mu,q}(\bar{U})$. In Section VI we construct holomorphic (Bergman space) representations of $C_{\mu,q}(\bar{U})$. In Section VII we show that $C_{\mu,q}(\bar{U})$ admits a natural action of the quantized universal enveloping algebra $U_q(sl(2))$. Formally, $C_{\mu,q}(\bar{U})$ admits an action of the quantum group $SU_q(1, 1)$ but, since $SU_q(1, 1)$ does not seem to admit a $\mathbb{C}^*$-algebraic description, we do not know how to formulate this fact rigorously. In Section VIII we discuss some open questions. Finally, in the Appendix we review certain facts on the structure of $\mathbb{C}^*$-algebras generated by weighted shifts.

II. Non-Commutative Deformations of Smooth Manifolds

A natural method of constructing non-commutative spaces is to "deform" an ordinary (smooth) manifold M. Such a deformation is parametrized by the points of a topological Hausdorff space S. "Continuous functions" on the non-commutative deformation are elements of a non-commutative $\mathbb{C}^*$-algebra $\mathscr{A}_s$, $s \in S$.

To be more specific, let M be a smooth manifold and let $C(M)$ denote a $\mathbb{C}^*$-algebra of continuous functions on (a suitable compactification of)

M. Let S be a Hausdorff space with a base point O. We refer to S as the parameter space. By a non-commutative deformation of M we understand a quadruple (($\mathscr{A}$, S, π, D) satisfying the following conditions D1–D5 (these conditions are very closely related to the conditions defining "strict deformation quantization" of [Ri]).

D1. $\mathscr{A}$ is a Hausdorff space and $\pi\colon \mathscr{A} \to S$ is a continuous surjection.

D2. Each fiber $\mathscr{A}_s := \pi^{-1}(s)$, $s \in S$, is a $\mathbb{C}^*$-algebra.

D3. $\mathscr{A}_O \cong C(M)$.

Let $I = [0, 1]$ and let $\gamma\colon I \to S$ be a continuous path such that $\gamma(0) = O$. By a connection we understand a mapping assigning to each γ a continuous curve $\Gamma\colon I \to \mathscr{A}$ in such a way that $\pi \circ \Gamma = \gamma$. We write $D_{\gamma(t)}(f) := \Gamma(t)$, if $\Gamma(0) = f \in C(M)$.

D4. The mapping $C(M) \ni f \to D_{\gamma(t)}(f) \in \mathscr{A}_{\gamma(t)}$ is linear and continuous.

We do not require, however, that $f \to D_{\gamma(t)}(f)$ be multiplicative; this would lead us to commutative deformations. Let $C^\infty(M) \subset C(M)$ denote a suitable algebra of smooth functions on M.

D5. For $f, q \in C^\infty(M)$, the limit

$$\lim_{t \downarrow 0} \frac{1}{t} \{D_{\gamma(t)}(f)\, D_{\gamma(t)}(g) - D_{\gamma(t)}(fg)\} = h \in C^\infty(M) \tag{II.1}$$

exists.

To be more precise, the last requirement says that for f, $g \in C^\infty(M)$ there is $h \in C^\infty(M)$ such that

$$\lim_{t \downarrow 0} \left\| \frac{1}{t} \{D_{\gamma(t)}(f)\, D_{\gamma(t)}(g) - D_{\gamma(t)}(fg)\} - D_{\gamma(t)}(h) \right\|_{\gamma(t)} = 0, \tag{II.2}$$

where $\|\cdot\|_s$ denotes the norm in $\mathscr{A}_s$.

We should emphasize that the above conditions form a system of natural guidelines for constructing deformations of smooth manifolds rather than a stringent system of axioms. In particular, the choice of $C(M)$ and $C^\infty(M)$ is a matter of convenience.

The proposition below states that a deformation ($\mathscr{A}$, S, π, D) of M defines a family of Poisson structures on M.

PROPOSITION II.1. *For each* γ,

$$\{f, g\}_\gamma := \lim_{t \downarrow 0} \frac{1}{t} [D_{\gamma(t)}(f), D_{\gamma(t)}(g)] \tag{II.3}$$

defines a Poisson bracket on $C^\infty(M)$.

Proof. Clearly, $\{f, g\}_\gamma$ is linear in f and g. Using the Jacobi identity for commutators, the fact that commutators are derivations, and condition D5, it is easy to verify that $\{\cdot, \cdot\}_\gamma$ satisfies the Jacobi identity and Leibniz rule. ▮

The proposition above says, roughly, that each direction in S from which the classical limit may be reached determines a Poisson structure on M.

III. Poisson Structures on the Unit Disc

In this section we discuss the Poisson structures on the Lie group $SU(1, 1)$ and its homogeneous space $U = SU(1, 1)/U(1)$, the unit disc. Our analysis follows the methods of [LW] and [SLW], where the case of $SU(2)$ was considered.

$SU(1, 1)$ is the group of complex 2×2 matrices $\gamma = \{\gamma_{jk}\}$ such that $\bar{\gamma}_{11} = \gamma_{22}$, $\bar{\gamma}_{12} = \gamma_{21}$, and $|\gamma_{11}|^2 - |\gamma_{12}|^2 = 1$. We wish to define a Poisson structure on $SU(1, 1)$, i.e., a Poisson bracket on the algebra $C^\infty(SU(1, 1))$ of smooth functions on $SU(1, 1)$. This algebra is generated by the functions

$$\begin{aligned} a(\gamma) &:= \gamma_{11}, & \bar{a}(\gamma) &:= \gamma_{22}, \\ b(\gamma) &:= \gamma_{12}, & \bar{b}(\gamma) &:= \gamma_{21}, \end{aligned} \tag{III.1}$$

and so it is enough to specify the Poisson brackets between these functions. For $\lambda \in \mathbb{R}$, we set

$$\begin{aligned} \{a, b\} &= \frac{i}{2} \lambda ab, & \{b, \bar{a}\} &= \frac{i}{2} \lambda b\bar{a}, \\ \{a, \bar{b}\} &= \frac{i}{2} \lambda a\bar{b}, & \{b, \bar{b}\} &= 0, \\ \{a, \bar{a}\} &= i\lambda b\bar{b}, & \{\bar{b}, \bar{a}\} &= \frac{i}{2} \lambda \bar{b}\bar{a}. \end{aligned} \tag{III.2}$$

The proof of the following proposition is straightforward.

PROPOSITION III.1. (i) *Equations* (III.2) *define a real Poisson structure on* $SU(1, 1)$; *i.e., they are consistent with the relations defining* $SU(1, 1)$ *and* $\{\bar{f}, \bar{g}\} = \overline{\{f, g\}}$.

(ii) $SU(1, 1)$ *with the bracket* $\{\cdot, \cdot\}$ *defined by* (III.2) *is a Poisson Lie group*; *i.e., the multiplication map* $SU(1, 1) \times SU(1, 1) \to SU(1, 1)$ *induces a homomorphism of Poisson algebras.*

Now let U be the unit disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$ with the usual $SU(1, 1)$-action

$$SU(1,1) \times U \ni (\gamma, \zeta) \to (\gamma_{11}\zeta + \gamma_{12})(\bar{\gamma}_{12}\zeta + \bar{\gamma}_{11})^{-1} \in U. \qquad \text{(III.3)}$$

We wish to determine all real Poisson structures on U such that the action (III.3) is a Poisson map. Note that the set of all such Poisson structures is an affine space modeled on the vector space of $SU(1, 1)$-invariant Poisson structures on U. The latter space is one dimensional; an arbitrary $SU(1, 1)$-invariant Poisson structure is given by

$$\{z, \bar{z}\} = i\mu(1 - z\bar{z})^2, \qquad \mu \in \mathbb{R}, \qquad \text{(III.4)}$$

where $z(\zeta) := \zeta$, $\bar{z}(\zeta) := \bar{\zeta}$. Therefore, in order to determine all $SU(1, 1)$-equivariant Poisson structures on U, it is enough to find one of them.

Observe that $U(1) \subset SU(1, 1)$ (the subgroup of diagonal matrices) is a Poisson Lie subgroup with zero Poisson bracket. Therefore, by [LW], the homogeneous space $SU(1, 1)/U(1) = U$ inherits the Poisson structure of $SU(1, 1)$. Furthermore, the natural projection p: $SU(1, 1) \to U$ and the left action of $SU(1, 1)$ on U are Poisson maps. Now, the projection p is given by

$$p(\gamma) = \gamma_{12}/\bar{\gamma}_{11}, \qquad \text{(III.5)}$$

and it follows that the projection of (III.2) to U has all the desired properties. Explicitly,

$$\begin{aligned} \{z, \bar{z}\} &= \{b\bar{a}^{-1}, \bar{b}a^{-1}\} \\ &= i\lambda[b\bar{a}^{-1}\bar{b}a^{-1} - (b\bar{a}^{-1})^2 (\bar{b}a^{-1})^2] \\ &= i\lambda[z\bar{z} - (z\bar{z})^2]. \end{aligned}$$

We have thus proven the following proposition.

PROPOSITION III.2. *Every real $SU(1, 1)$-equivariant Poisson structure on U is of the form*

$$\{z, \bar{z}\} = i(1 - |z|^2)(\mu + (\lambda - \mu)\,|z|^2), \qquad \text{(III.6)}$$

where $\lambda, \mu \in \mathbb{R}$.

In the following sections we will study a two-parameter deformation of U. It will then become clear that the Poisson structures (III.6) are precisely those of Proposition II.1 with λ, μ parameterizing the direction from which the classical limit is approached.

We also note that the Poisson structures (III.6) have different behaviors for different ranges of the parameters λ and μ. If $=0$, U has two symplectic leaves: $\{0\}$ and $U\setminus\{0\}$. If $\mu\neq 0$, then

$$\{z, \bar{z}\} = i\mu(1-|z|^2)\left(1-\left(1-\frac{\lambda}{\mu}\right)|z|^2\right),$$

and so for $\lambda/\mu<0$, U has the following symplectic leaves: each point of the circle $|\zeta|^2=(1-\lambda/\mu)^{-1}$, the disc $|\zeta|^2<(1-\lambda/\mu)^{-1}$, and the annulus $(1-\lambda/\mu)^{-1}<|\zeta|^2<1$; while for $\lambda/\mu\geqslant 0$, the Poisson structure (III.6) is symplectic.

IV. Deformations of the Unit Disc: Representation Theory

In this section we define a family of $\mathbb{C}^*$-algebras $C_{\mu,q}(\bar{U})$, where $(\mu, q)\in S:=\{(\mu, q): 0\leqslant\mu\leqslant 1,\ 0<q\leqslant 1\}$. We choose the basepoint to be $O=(0, 1)$ and the corresponding algebra $C_{0,1}(\bar{U})=C(\bar{U})$, the $\mathbb{C}^*$-algebra of continuous functions on the closed unit disc $\bar{U}$. Following the usual procedure, we define $C_{\mu,q}(\bar{U})$ as the universal enveloping $\mathbb{C}^*$-algebra of an algebra given in terms of generators and relations. This $\mathbb{C}^*$-algebra has a rather interesting structure which we study in this and the subsequent sections. We classify all irreducible $*$-representations of $C_{\mu,q}(\bar{U})$ and show that they fall into two categories: one-dimensional and infinite-dimensional. All the infinite-dimensional representations are generated by hyponormal weighted shift operators.

The algebra $C_{\mu,q}(\bar{U})$, $(\mu, q)\neq O$, is defined as follows. Let $\mathscr{P}_{\mu,q}$ denote the unital algebra generated by two elements, z and $\bar{z}$, with the following relation

$$qz\bar{z}-\bar{z}z=q-I+\mu(I-z\bar{z})(I-\bar{z}z). \tag{IV.1}$$

Let $\mathscr{H}$ be a Hilbert space and let $\pi\colon \mathscr{P}_{\mu,q}\to\mathscr{L}(\mathscr{H})$ be a representation of $\mathscr{P}_{\mu,q}$ by bounded linear operators in $\mathscr{H}$. π is called a $*$-representation if $\pi(\bar{z})=\pi(z)^*$. Clearly, $*$-representations of $\mathscr{P}_{\mu,q}$ exist, as we can set $\mathscr{H}=\mathbb{C}$ and $\pi(z)=e^{i\theta}$, $\pi(\bar{z})=e^{-i\theta}$, for $0\leqslant\theta<2\pi$.

Proposition IV.1. *Let $(\mu, q)\in S\setminus\{O\}$ and let π be a $*$-representation of $C_{\mu,q}(\bar{U})$. Then*

(I) *if $\mu<1-q$, then $\|\pi(z)\|=1$;*

(II) *if $\mu=1-q$, then either $\pi(z)=0$ or $\|(z)\|=1$;*

(III) *if $\mu>1-q$, then either $\|\pi(z)\|=\sqrt{(\mu+q-1)/\mu}$ or $\|\pi(z)\|=1$.*

Proof. We introduce the notation

$$x := z\bar{z}, \qquad y := \bar{z}z. \tag{IV.2}$$

Note that for $0 \leqslant \mu < 1$ the operator $\mu\pi(x) + 1 - \mu$ is invertible, as $\pi(x) \geqslant 0$. Using (IV.1) we can thus write

$$\pi(y) = \{(q+\mu)\,\pi(x) + 1 - q - \mu\}\{\mu\pi(x) + 1 - \mu\}^{-1}. \tag{IV.3}$$

By the functional calculus, the norm of the right-hand side of (IV.3) is equal to

$$\{(q+\mu)\,\|\pi(x)\| + 1 - q - \mu\}\{\mu\,\|\pi(x)\| + 1 - \mu\}^{-1}.$$

Since $\|\pi(x)\| = \|\pi(y)\| = \|\pi(z)\|^2 =: t$, Eq. (IV.3) yields the quadratic equation

$$\mu t^2 + (1 - 2\mu - q)\,t + q + \mu - 1 = 0. \tag{IV.4}$$

For $\mu = 0$, this equation has only one solution: $t = 1$. For $\mu > 0$, there are two solutions: $t = 1$ and $t = (\mu + q - 1)/\mu$. The claims follow. ∎

Let us now define, for $u \in \mathscr{P}_{\mu,q}$,

$$\|u\| := \sup_{\pi} \|\pi(u)\|, \tag{IV.5}$$

where the supremum is taken over all $*$-representations of $\mathscr{P}_{\mu,q}$. As a consequence of Proposition IV.1, $\|u\| < \infty$. Let $\mathscr{N} = \{u \in \mathscr{P}_{\mu,q} : \|u\| = 0\}$ be the nul-ideal in $\mathscr{P}_{\mu,q}$. We define $C_{\mu,q}(\bar{U})$ as the completion of $\mathscr{P}_{\mu,q}/\mathscr{N}$ in the norm obtained as the projection of (IV.5). By construction, $C_{\mu,q}(\bar{U})$ is a unital $\mathbb{C}^*$-algebra.

Observe that as a consequence of (IV.3),

$$z\bar{z}\bar{z}z = \bar{z}zz\bar{z} \tag{IV.6}$$

in $C_{\mu,q}(\bar{U})$.

The main goal of this section is to classify all irreducible $*$-representations of $C_{\mu,q}(\bar{U})$. As expected from Proposition IV.1, the character of these representations varies depending on the range of (μ, q). We will analyze them case by case. Observe first that for all $(\mu, q) \in S$ we have the following family of one-dimensional $*$-representations $\rho_{1,\theta}$, $0 \leqslant \theta < 2\pi$, of $C_{\mu,q}(\bar{U})$:

$$\rho_{1,\theta}(z) = e^{i\theta}. \tag{IV.7}$$

Case I. $\mu < 1 - q$. Set

$$\lambda_n := \frac{1 - q^n}{1 - R^{-1} q^n}, \qquad n = 0, 1, 2, \ldots, \tag{IV.8}$$

where R^{-1} is defined by

$$R^{-1} := \frac{\mu}{\mu + q - 1}. \tag{IV.9}$$

LEMMA IV.2. *Let π be a $*$-representation of $C_{\mu, q}(\bar{U})$. Then*

$$\begin{aligned} \mathrm{Spec}(\pi(x)) &\subset \{\lambda_n\}_{n \geqslant 0} \cup \{1\}, \\ \mathrm{Spec}(\pi(y)) &\subset \{\lambda_n\}_{n \geqslant 1} \cup \{1\}. \end{aligned} \tag{IV.10}$$

Proof. By Proposition IV.1, $\|\pi(x)\| = \|\pi(y)\| = 1$, and so $\mathrm{Spec}(\pi(x))$, $\mathrm{Spec}(\pi(y)) \subset [0, 1]$. Furthermore,

$$\mathrm{Spec}(\pi(x)) \setminus \{0\} = \mathrm{Spec}(\pi(y)) \setminus \{0\}. \tag{IV.11}$$

Let I_n, $n = 0, 1, 2, \ldots$, be the sequence of open subintervals of $[0, 1]$ defined by

$$I_n := (\lambda_n, \lambda_{n+1}). \tag{IV.12}$$

We claim that $I_n \cap \mathrm{Spec}(\pi(x)) = I_n \cap \mathrm{Spec}(\pi(y)) = \varnothing$. Indeed, (IV.3) implies that

$$\pi(y) \geqslant \frac{1 - \mu - q}{1 - \mu} = \lambda_1,$$

so that $I_n \cap \mathrm{Spec}(\pi(y)) = \varnothing$ and, by (IV.11), $I_n \cap \mathrm{Spec}(\pi(x)) = \varnothing$. We now proceed by induction. Assume that I_j, $0 \leqslant j \leqslant n-1$, do not intersect the spectra of $\pi(x)$ and $\pi(y)$. If $\lambda \in I_n$ is in the spectrum of $\pi(y)$, then by (IV.3), $\lambda = \{(q + \mu)\tilde{\lambda} + 1 - q - \mu\{(\mu\tilde{\lambda} + 1 - \mu)^{-1}$ with $\tilde{\lambda}$ in the spectrum of $\pi(x)$. But $\tilde{\lambda} \in I_{n-1}$, which is a contradiction. Hence, I_n does not intersect $\mathrm{Spec}(\pi(x))$. By (IV.11), it does not intersect $\mathrm{Spec}(\pi(y))$ and the lemma is proven. ∎

THEOREM IV.3. *Let $(\mu, q) \in S$ with $\mu < 1 - q$, and let π be an irreducible $*$-representation of $C_{\mu, q}(\bar{U})$. Then π is unitarily equivalent to one of the following representations:*

(i) $\rho_{1,\theta}$,

(ii) *an infinite dimensional representation defined as follows. Let* $\mathscr{H} = l^2(\mathbb{Z}_+)$ *and let* $\{\phi_n\}_{n \geqslant 0}$ *be the canonical basis for* $\mathscr{H}$. *Then*

$$\begin{aligned} \pi^I(z)\,\phi_n &= \sqrt{\lambda_{n+1}}\,\phi_{n+1}, && n \geqslant 0, \\ \pi^I(\bar{z})\,\phi_n &= \begin{cases} 0 & n = 0, \\ \sqrt{\lambda_n}\,\phi_{n-1}, & n \geqslant 1. \end{cases} \end{aligned} \tag{IV.13}$$

Proof. We write $\mathscr{H} = \operatorname{Ker}(I - \pi(x)) \oplus \mathring{\mathscr{H}}$, where $\mathring{\mathscr{H}}$ is the orthogonal complement of $\operatorname{Ker}(I - \pi(x))$. Observe that both direct summands are invariant under π and so one of them must be zero. If $\mathring{\mathscr{H}} = 0$, then $\pi(z)\,\pi(z)^* = \pi(z)^*\,\pi(z) = I$; i.e., $\pi(z)$ is unitary. As a consequence, $\mathscr{H}$ is one-dimensional and $\pi(z) = e^{i\theta}$ for some $0 \leqslant \theta < 2\pi$.

Now, let $\operatorname{Ker}(I - \pi(x)) = 0$ and let $\mathscr{G}_n$, $n = 0, 1, 2, \ldots$, be the eigenspace of $\pi(x)$ corresponding to the eigenvalue λ_n. By the spectral theorem, $\mathscr{H} = \oplus_{n=1}^{\infty} \mathscr{G}_n$. Since

$$\pi(z)\colon \mathscr{G}_n \to \mathscr{G}_{n+1}, \qquad \pi(\bar{z})\colon \mathscr{G}_{n+1} \to \mathscr{G}_n,$$

and

$$\pi(x) \restriction \mathscr{G}_n = \lambda_n, \qquad \pi(y) \restriction \mathscr{G}_n = \lambda_{n+1},$$

it follows that all $\mathscr{G}_n$ are mutually isomorphic. We claim that they are one-dimensional. Indeed, if $e_1, e_2 \in \mathscr{G}_0$ are non-zero and orthogonal to each other, then the subspaces spanned by $\{\pi(z^n)\,e_j\}_{n \geqslant 0}$, $j = 1, 2$, are non-trivial, invariant under π, and mutually orthogonal. This contradicts the assumption that π is irreducible.

Now let $\phi_0 \in \mathscr{G}_0$ with $\|\phi_0\| = 1$ and let ϕ_n, $n \geqslant 1$, be defined by

$$\phi_n := \|\pi(z)^n\,\phi_0\|^{-1}\,\pi(z)^n\,\phi_0 \in \mathscr{G}_n. \tag{IV.14}$$

Then $\{\phi_n\}_{n \geqslant 0}$ forms an orthonormal basis for $\mathscr{H}$. A simple computation leads to formulas (IV.13). ∎

Case II. $\mu = 1 - q$, $q < 1$. Let $t > 0$ be arbitrary and let

$$\mu_n(t) := (1 + tq^n)^{-1}, \qquad n \in \mathbb{Z}. \tag{IV.15}$$

LEMMA IV.4. *Let* π *be an irreducible* $*$*-representation of* $C_{\mu, 1-\mu}(\bar{U})$. *Then there exists* $t > 0$ *such that* $\operatorname{Spec}(\pi(x))$ *and* $\operatorname{Spec}(\pi(y))$ *are contained in* $\{\mu_n(t)\}_{n \in \mathbb{Z}} \cup \{0, 1\}$.

Proof. There is nothing to prove unless $\mathrm{Spec}(\pi(x)) \cap (0, 1) \neq \varnothing$. Suppose there is $\lambda \in \mathrm{Spec}(\pi(x))$ with $0 < \lambda < 1$. We set $t = \lambda^{-1} - 1$ and define $\mu_n(t)$ by (IV.15). Observe that

$$\mu_n(t) < \mu_{n+1}(t), \tag{IV.16}$$

and

$$\lim_{n \to -\infty} \mu_n(t) = 0, \qquad \lim_{n \to \infty} \mu_n(t) = 1. \tag{IV.17}$$

Using (IV.11) and (IV.3) we verify easily that $\mu_n(t) \in \mathrm{Spec}(\pi(x))$, for all $n \in \mathbb{Z}$. We claim that this is the entire spectrum of $\pi(x)$. Indeed, let $\lambda \to E(\lambda)$ denote the spectral family of the self-adjoint operator $\pi(x)$ and let σ, τ be such that

$$q < \tau < \sigma < 1. \tag{IV.18}$$

Consider the sequence of intervals

$$I_n^{(\sigma, \tau)} := ((1 + \sigma t q^n)^{-1}, (1 + \tau t q^n)^{-1}) \subset (\mu_n(t), \mu_{n+1}(t)),$$

and define $P^{(\sigma, \tau)} := E(\bigcup_{n \in \mathbb{Z}} I_n^{(\sigma, \tau)})$. We will prove first that the closed subspace $P^{(\sigma, \tau)}\mathscr{H}$ of $\mathscr{H}$ is invariant under π. Using the formula ([DS], p. 921)

$$E((a, b)) = s - \lim_{\delta \downarrow 0} s - \lim_{\varepsilon \downarrow 0} (2\pi i)^{-1} \int_{a+\delta}^{b-\delta} [R(\mu - i\varepsilon) - R(\mu + i\varepsilon)] \, d\mu, \tag{IV.19}$$

where $R(\mu) = (\mu - \pi(x))^{-1}$ is the resolvent of $\pi(x)$, and (IV.3), we easily see that

$$\begin{aligned} \pi(z)\, E(I_n^{(\sigma, \tau)}) &= E(I_{n+1}^{(\sigma, \tau)})\, \pi(z), \\ \pi(\bar{z})\, E(I_n^{(\sigma, \tau)}) &= E(I_{n-1}^{(\sigma, \tau)})\, \pi(\bar{z}). \end{aligned} \tag{IV.20}$$

Since E is strongly σ-additive, Eqs. (IV.20) imply that $[\pi(z), P^{(\sigma, \tau)}] = [\pi(\bar{z}), P^{(\sigma, \tau)}] = 0$, and $P^{(\sigma, \tau)}\mathscr{H}$ is invariant under π. Since π is irreducible, either $P^{(\sigma, \tau)} = 0$ or $P^{(\sigma, \tau)} = I$. But $P^{(\sigma, \tau)} = I$ contradicts $\{\mu_n(t)\} \subset \mathrm{Spec}(\pi(x))$. Therefore, $P^{(\sigma, \tau)} = 0$, i.e., $I_n^{(\sigma, \tau)} \cap \mathrm{Spec}(\pi(x)) = \varnothing$, and the claim follows as σ, τ satisfying (IV.18) are arbitrary. ∎

THEOREM IV.5. *The following representations include all unitarily non-equivalent irreducible $*$-representations of $C_{\mu, 1-\mu}(\bar{U})$, $\mu > 0$:*

(i) $\rho_{1, \theta}$;

(ii) *a one-dimensional representation ρ_0 defined by*

$$\rho_0(z) = \rho_0(\bar{z}) = 0; \tag{IV.21}$$

(iii) *infinite-dimensional representations defined as follows. Let* $\mathscr{H} = l^2(\mathbb{Z})$ *and let* $\{\phi_n\}_{n\in\mathbb{Z}}$ *be the canonical basis for* $\mathscr{H}$. *Then*

$$\begin{aligned} \pi_t^{\mathrm{II}}(z)\,\phi_n &= \sqrt{\mu_{n+1}(t)}\,\phi_{n+1}, \\ \pi_t^{\mathrm{II}}(\bar{z})\,\phi_n &= \sqrt{\mu_n(t)}\,\phi_{n-1}. \end{aligned} \tag{IV.22}$$

Proof. Let $(\pi, \mathscr{H})$ be an irreducible $*$-representation of $C_{\mu,1-\mu}(\bar{U})$. We write $\mathscr{H}$ as an orthogonal sum,

$$\mathscr{H} = \mathrm{Ker}(\pi(x)) \oplus \mathrm{Ker}(I - \pi(x)) \oplus \mathring{\mathscr{H}}, \tag{IV.23}$$

and verify that each direct summand in (IV.23) is invariant under π. Using Lemma IV.4 and repeating the steps of the proof of Theorem IV.3 we easily prove the above theorem. ∎

Case III. $\mu > 1-q$, $q<1$. (For the analysis of the $q=1$ case, see [KL].) According to Proposition IV.1, we either have $\|\pi(z)\| = \sqrt{R}$, with $0<R<1$ defined by (IV.9), or $\|\pi(z)\| = 1$. Assume first that $\|\pi(z)\| = \sqrt{R}$. Let

$$\alpha_n := R\,\frac{1-q^n}{1-Rq^n}, \qquad n \geqslant 0. \tag{IV.24}$$

LEMMA IV.6. *Let* π *be a* $*$*-representation of* $C_{\mu,q}$, $\mu > 1-q$, *with* $\|\pi(z)\| = \sqrt{R}$. *Then*

$$\begin{aligned} \mathrm{Spec}(\pi(x)) &\subset \{\alpha_n\}_{n\geqslant 1} \cup \{1\}, \\ \mathrm{Spec}(\pi(y)) &\subset \{\alpha_n\}_{n\geqslant 0} \cup \{1\}. \end{aligned} \tag{IV.25}$$

The proof of this lemma uses similar arguments to those of the proof of Lemma IV.2 and we omit the details.

Assume now that $\|\pi(z)\| = 1$. For $t>0$ we set

$$\beta_n(t) := \frac{1+tRq^n}{1+tq^n}, \qquad n\in\mathbb{Z}. \tag{IV.26}$$

LEMMA IV.7. *Let* π *be an irreducible* $*$*-representation of* $C_{\mu,q}(\bar{U})$, $\mu > 1-q$, $(\mu, q) \in S\setminus\{O\}$. *Then there exists* $t>0$ *such that*

$$\begin{aligned} \mathrm{Spec}(\pi(y)) &\subset \{\beta_n(t)\}_{n\in\mathbb{Z}} \cup \{0, 1\}, \\ \mathrm{Spec}(\pi(y)) &\subset \{\beta_n(t)\}_{n\in\mathbb{Z}} \cup \{0, 1\}. \end{aligned} \tag{IV.27}$$

We omit the proof of this lemma as it is similar to the proof of Lemma IV.4. As a consequence of the last two lemmas we obtain the following theorem.

THEOREM IV.8. *Any irreducible $*$-representation of $C_{\mu,q}(\bar{U})$, $\mu > 1-q$, $(\mu, q) \in S \setminus \{O\}$ is unitarily equivalent to one of the following representations:*

(i) $\rho_{1,\theta}$;

(ii) *a one-dimensional representation $\rho_{R,\theta}$, $0 \leqslant \theta < 2\pi$, defined as follows:*

$$\rho_{R,\theta}(z) = \sqrt{R}\, e^{i\theta}; \tag{IV.28}$$

(iii) *an infinite-dimensional representation defined as follows. Let $\mathscr{H} = l^2(\mathbb{Z}_+)$ and let $\{\phi_n\}_{n \geqslant 0}$ be the canonical basis for $\mathscr{H}$. Then*

$$\begin{aligned} \pi_R^{\mathrm{III}}(\bar{z})\, \phi_n &= \sqrt{\alpha_{n+1}}\, \phi_{n+1}, \qquad n \geqslant 0, \\ \pi_R^{\mathrm{III}}(\bar{z})\, \phi_n &= \begin{cases} 0, & n = 0, \\ \sqrt{\alpha_n}\, \phi_{n-1}, & n \geqslant 1; \end{cases} \end{aligned} \tag{IV.29}$$

(iv) *infinite-dimensional representations defined as follows. Let $\mathscr{H} = l^2(\mathbb{Z})$ and let $\{\phi_n\}_{n \geqslant 0}$ be the canonical basis for $\mathscr{H}$. Then*

$$\begin{aligned} \pi_{1,t}^{\mathrm{III}}(z)\, \phi_n &= \sqrt{\beta_{n+1}(t)}\, \phi_{n+1}, \qquad n \in \mathbb{Z}, \\ \pi_{1,t}^{\mathrm{III}}(\bar{z})\, \phi_n &= \sqrt{\beta_n(t)}\, \phi_{n-1}, \qquad n \in \mathbb{Z}. \end{aligned} \tag{IV.30}$$

V. DEFORMATIONS OF THE UNIT DISC: STRUCTURAL THEORY

In this section we study the structure of the $\mathbb{C}^*$-algebra $C_{\mu,q}(\bar{U})$, for $(\mu, q) \in S \setminus \{O\}$. It turns out that the structure of $C_{\mu,q}(\bar{U})$ depends dramatically on whether $\mu < 1-q$, $\mu = 1-q$, or $\mu > 1-q$.

Let $(\pi, \mathscr{H})$ be a $*$-representation of $C_{\mu,q}(\bar{U})$, and let $\pi(C_{\mu,q}(\bar{U}))$ denote the $\mathbb{C}^*$-algebra of operators on $\mathscr{H}$ generated by $\pi(a)$, $a \in C_{\mu,q}(\bar{U})$. By $\mathscr{K}$ we denote the $\mathbb{C}^*$-algebra of compact operators on $\mathscr{H}$.

PROPOSITION V.1. *Let $(\mu, q) \in S \setminus \{O\}$. We have the following short exact sequences of $\mathbb{C}^*$-algebras:*

(I) *for $\mu < 1-q$,*

$$0 \to \mathscr{K} \to \pi^{\mathrm{I}}(C_{\mu,q}(\bar{U})) \to C(S^1) \to 0; \tag{V.1}$$

(II) *for $\mu = 1-q$,*

$$0 \to \mathscr{K} \to \pi_t^{\mathrm{II}}(C_{\mu,q}(\bar{U})) \to C(S^1) \oplus \mathbb{C} \to 0; \tag{V.2}$$

(III) *for $\mu > 1-q$, $q < 1$,*

$$0 \to \mathscr{K} \to \pi_R^{\mathrm{III}}(C_{\mu,q}(\bar{U})) \to C(S^1) \to 0, \tag{V.3}$$

and

$$0 \to \mathcal{K} \to \pi^{\mathrm{III}}_{1,t}(C_{\mu,q}(\bar{U})) \to C(S^1) \oplus C(S^1) \to 0. \tag{V.4}$$

Proof. Each of the $\mathbb{C}^*$-algebras $\pi(C_{\mu,q}(\bar{U}))$ listed in the above proposition is generated by a unilateral or bilateral weighted shift satisfying the assumptions of Theorem A.1 or Theorem A.2, respectively. ∎

COROLLARY V.2. *$C_{\mu,q}(\bar{U})$ is a type* I *$\mathbb{C}^*$-algebra for all $(\mu, q) \in S \setminus \{O\}$.*

COROLLARY V.3. *Let $a \in C_{\mu,q}(\bar{U})$. Then*

(I) *for $\mu < 1-q$, $0 \leqslant \theta < 2\pi$,*

$$\|\pi^{\mathrm{I}}(a)\| \geqslant \|\rho_{1,\theta}(a)\|; \tag{V.5}$$

(II) *for $\mu = 1-q$, $0 \leqslant \theta < 2\pi$, $t > 0$,*

$$\|\pi^{\mathrm{II}}_t(a)\| \geqslant \max\{\|\rho_{1,\theta}(a)\|, \|\rho_0(a)\|\}; \tag{V.6}$$

(III) *for $\mu > 1-q$, $q < 1$, $0 \leqslant \theta < 2\pi$,*

$$\|\pi^{\mathrm{III}}_R(a)\| \geqslant \|\rho_{R,\theta}(a)\|, \tag{V.7}$$

and for $t > 0$, $0 \leqslant \theta$, $\theta' < 2\pi$,

$$\|\pi^{\mathrm{III}}_{1,t}(a)\| \geqslant \max\{\|\rho_{1,\theta}(a)\|, \|\rho_{R,\theta'}(a)\|\}. \tag{V.8}$$

The structure of $C_{\mu,q}(\bar{U})$ is described by the following theorem.

THEOREM V.4. *For $(\mu, q) \in S \setminus \{0\}$ we have the following short exact sequences of* C**-algebras:*

(I) *for $\mu < 1-q$,*

$$0 \to \mathcal{K} \to C_{\mu,q}(\bar{U}) \to C(S^1) \to 0; \tag{V.9}$$

(II) *for $\mu = 1-q$,*

$$0 \to C(S^1) \otimes \mathcal{K} \to C_{\mu,q}(\bar{U}) \to C(S^1) \oplus \mathbb{C} \to 0; \tag{V.10}$$

(III) *for $\mu > 1-q$, $q < 1$,*

$$0 \to \mathcal{K} \oplus (C(S^1) \otimes \mathcal{K}) \to C_{\mu,q}(\bar{U}) \to C(S^1) \oplus C(S^1) \to 0. \tag{V.11}$$

Proof. For a $\mathbb{C}^*$-algebra $\mathcal{A}$ we have the exact sequence

$$0 \to I \to \mathcal{A} \to C(\sigma) \to 0, \tag{V.12}$$

where I is the commutator ideal of $\mathscr{A}$, and where σ is the spectrum of $\mathscr{A}$ (i.e., the space of all multiplicative functionals on $\mathscr{A}$). Since $C_{\mu,q}(\bar{U})$ is defined in terms of generators, it is easy to determine σ:

$$\sigma = S^1, \qquad \text{if} \quad \mu < 1-q, \tag{V.13}$$

$$\sigma = S^1 \cup \{0\}, \qquad \text{if} \quad \mu = 1-q, \tag{V.14}$$

$$\sigma = S^1 \cup S^1, \qquad \text{if} \quad \mu > 1-q, \quad q<1. \tag{V.15}$$

To prove (V.9) observe that, as a consequence of (V.5), π^{I} is a faithful representation and so $\pi^{\mathrm{I}}(C_{\mu,q}(\bar{U})) \simeq C_{\mu,q}(\bar{U})$. The claim follows from (V.1).

To prove (V.10) note that π_t^{II} and π_s^{II} are unitarily equivalent if and only if $t/s \in q^{\mathbb{Z}}$ with unitary equivalence given by a shift $\phi_n \to \phi_{n+k}$, $n \in \mathbb{Z}$ (indeed, $\mathrm{Spec}(\pi_t^{\mathrm{II}}(x)) = \mathrm{Spec}(\pi_s^{\mathrm{II}}(x))$, if and only if $t/s = q^k$, $k \in \mathbb{Z}$). We can regard the commutator ideal I as a $\mathbb{C}^*$-subalgebra of $C(S^1, \mathscr{K}) \cong C(S^1) \otimes \mathscr{K}$ with the identification given by $I \ni a \to \pi_\cdot^{\mathrm{II}}(a) \in C(S^1, \mathscr{K})$, where $\pi_\cdot^{\mathrm{II}}(a)$ denotes the function $S^1 \ni t \to \pi_t^{\mathrm{II}}(a) \in \pi_t^{\mathrm{II}}(I) = \mathscr{K}$. The map $a \to \pi_\cdot^{\mathrm{II}}(a)$ is injective because, as a consequence of (V.6), $\int_{S^1}^{\oplus} \pi_t^{\mathrm{II}}\, dt$ is a faithful representation of $C_{\mu,q}(\bar{U})$. Now, since $C(S^1, \mathscr{K})$ is type I and $I \subset C(S^1, \mathscr{K})$ is clearly rich, Proposition II.1.6 in [Di] implies that $I \cong C(S^1, \mathscr{K})$.

To prove (V.11) note that $\pi_{1,t}^{\mathrm{III}}$ and $\pi_{1,s}^{\mathrm{III}}$ are unitarily equivalent if and only if $t/s \in s^{\mathbb{Z}}$. Since by (V.7) and (V.8), $\pi_R^{\mathrm{III}} \oplus \int_{S^1}^{\oplus} \pi_{1,t}^{\mathrm{III}}\, dt$ is a faithful representation of $C_{\mu,q}(\bar{U})$, the above arguments show that $I \cong \mathscr{K} \oplus (C(S^1) \otimes \mathscr{K})$ with the first summand coming from π_R^{III} and the second one coming from $\int_{S^1}^{\oplus} \pi_{1,t}^{\mathrm{III}}\, dt$. ∎

VI. Bergman Space Representations

The representations of $C_{\mu,q}(\bar{U})$ constructed in Section IV are equivalent to representations by Toeplitz operators on certain Hilbert spaces of holomorphic functions (Bergman spaces). This implies, in particular, that the hyponormal shifts $\pi(z)$ are, in fact, subnormal.

Let us recall the definition of a Toeplitz operator. Let $D \subset \mathbb{C}$ be a (bounded) domain and let $d\mu$ be a probability measure on D. By $\mathscr{H}(D, d\mu) \subset \mathscr{L}^2(D, d\mu)$ we denote the Hilbert space of holomorphic functions on D which are square integrable with respect to $d\mu$. Then the orthogonal projection $P\colon \mathscr{L}^2(D, d\mu) \to \mathscr{H}(D, d\mu)$ is an integral operator whose kernel $K(\zeta, \eta)$ is called the Berman kernel. For $f \in C(\bar{D})$ we define $T(f) := PM(f)$, where $M(f)$ is the operator of multiplication by f. Then

$\|T(f)\| \leqslant \|f\|$ and $T(f)$ maps $\mathscr{H}(D)$ into itself. We call $T(f)$ a Toeplitz operator with symbol f. Explicitly, for $\phi \in \mathscr{H}(D)$,

$$(T(f)\,\phi)(\zeta) = \int_D K(\zeta, \eta)\, f(\eta)\, \phi(\eta)\, d\mu(\eta). \tag{VI.1}$$

Case I. $\mu < 1 - q$. Let $D = U$ and let

$$d\mu^{\mathrm{I}}(\zeta) := \frac{1 - R^{-1}}{\pi} \frac{(|\zeta|^2 q; q)_\infty}{(|\zeta|^2 R^{-1}; q)_\infty} \sum_{n \geqslant 0} q^n\, \delta(|\zeta|^2 - q^n)\, d^2\zeta, \tag{VI.2}$$

where $(a; q)_\infty := \prod_{n \geqslant 0} (1 - aq^n)$, where $\delta(|\zeta|^2 - \rho^2)\, d^2\zeta$ is the Dirac measure concentrated on the circle $|\zeta|^2 = \rho^2$, and where $R^{-1} < 0$ is defined by (IV.9). Then, for $n \in \mathbb{Z}_+$,

$$\begin{aligned}
\int_U |\zeta|^{2n}\, d\mu^{\mathrm{I}}(\zeta) &= (1 - R^{-1}) \sum_{m \geqslant 0} q^{m + nm} \frac{(q^{m+1}; q)_\infty}{(q^m R^{-1}; q)_\infty} \\
&= [r]_q \int_0^1 t^n \frac{(tq; q)_\infty}{(tq^r; q)_\infty}\, d_q t,
\end{aligned}$$

where $r \in \mathbb{C}$ is defined by $q^r = R^{-1}$, where $[r]_q := (1 - q^r)/(1 - q)$, and where $\int_0^1 f(t)\, d_q t$ is the Jackson integral [GR]. By the familiar formulas of the q-calculus [GR], the last expression can be rewritten in terms of the q-gamma function $\Gamma_q(z) := ((q; q)_\infty/(q^z; q)_\infty)(1 - q)^{1-z}$ as

$$[r]_q \frac{\Gamma_q(n+1)\, \Gamma_q(r)}{\Gamma_q(n + r + 1)} = \frac{\Gamma_q(n+1)\, \Gamma_q(r+1)}{\Gamma_q(n + r + 1)}$$

(in particular, $d\mu^{\mathrm{I}}$ is a probability measure). As a consequence, the functions

$$\phi_n(\zeta) := \left\{ \frac{\Gamma_q(n+1)\, \Gamma_q(r+1)}{\Gamma_q(n + r + 1)} \right\}^{-1/2} \zeta^n, \qquad n \in \mathbb{Z}_+, \tag{VI.3}$$

form an orthonormal basis for $\mathscr{H}(D)$. The corresponding Bergman kernel is

$$\begin{aligned}
K^{\mathrm{I}}(\zeta, \eta) &= \Gamma_q(r+1)^{-1} \sum_{n \geqslant 0} \frac{\Gamma_q(n + r + 1)}{\Gamma_q(n+1)} (\zeta \bar{\eta})^n \\
&= \frac{(q^{r+1}; q)_\infty}{(q; q)_\infty} \sum_{n \geqslant \infty} \frac{(q^{n+1}; q)_\infty}{(q^{n+r+1}; q)_\infty} (\zeta \bar{\eta})^n \\
&= \sum_{n \geqslant 0} \frac{(q^{r+1}; q)_n}{(q; q)_n} (\zeta \bar{\eta})^n,
\end{aligned}$$

where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. Using the q-binomial formula [GR] we finally obtain

$$K^{\mathrm{I}}(\zeta,\eta)=\frac{(q\zeta\bar{\eta}/R;q)_\infty}{(\zeta\bar{\eta};q)_\infty}. \tag{VI.4}$$

Consider now the Toeplitz operator $T(\zeta)$. We find

$$T(\zeta)\,\phi_n=\left\{\frac{1-q^{n+1}}{1-q^{n+1+r}}\right\}^{1/2}\phi_{n+1}, \tag{VI.5}$$

and so we have proven the following proposition.

PROPOSITION VI.1. *With the above definitions, representation π^{I} is equivalent to the following representation on $\mathscr{H}(U, d\mu^{\mathrm{I}})$:*

$$z\to T(\zeta),\qquad \bar{z}\to T(\bar{\zeta}).$$

Case II. $\mu=1-q$, $q<1$. Let $D=U^* := U\setminus\{0\}$ and let

$$d\mu^{\mathrm{II}}(\zeta):=\frac{1}{\pi(-tq;q)_\infty}\sum_{n\geqslant 0}\frac{q^{n(n+1)/2}}{(q;q)_n}\,t^n\,\delta(|\zeta|^2-q^n)\,d^2\zeta. \tag{VI.6}$$

Then, for $n\in\mathbb{Z}$,

$$\begin{aligned}\int_{U^*}|\zeta|^{2n}\,d\mu^{\mathrm{II}}(\zeta)&=\frac{1}{(-tq;q)_\infty}\sum_{n\geqslant 0}\frac{q^{m(m+1)/2}}{(q;q)_m}(tq^n)^m\\&=\frac{(-tq^{n+1};q)_\infty}{(-tq;q)_\infty}.\end{aligned}$$

As a consequence, the functions

$$\phi_n(\zeta):=\left\{\frac{(-tq;q)_\infty}{(-tq^{n+1};q)_\infty}\right\}^{1/2}\zeta^n,\qquad n\in\mathbb{Z}, \tag{VI.7}$$

from an orthonormal basis for $\mathscr{H}(U^*, d\mu^{\mathrm{II}})$. The corresponding Bergman kernel is

$$\begin{aligned}K_t^{\mathrm{II}}(\zeta,\eta)&=(-tq;q)_\infty\sum_{n\in\mathbb{Z}}\frac{1}{(-tq^{n+1};q)_\infty}(\zeta\bar{\eta})^n\\&=\sum_{n\in\mathbb{Z}}(-tq;q)_n\,(\zeta\bar{\eta})^n.\end{aligned}$$

Using Ramanujan's summation formula [GR] we thus obtain

$$K_t^{\mathrm{II}}(\zeta, \eta) = (q; q)_\infty \frac{(-qt\zeta\bar{\eta}; q)_\infty (-1/(t\zeta\bar{\eta}); q)_\infty}{(\zeta\bar{\eta}; q)_\infty (-1/t; q)_\infty}. \tag{VI.8}$$

Consider now the Toeplitz operator $T(\zeta)$. From (VI.7),

$$T(\zeta)\phi_n = (1 + tq^{n+1})^{-1/2} \phi_{n+1}, \tag{VI.9}$$

and so we have proven the following proposition,

PROPOSITION VI.2. *With the above definitions, representation π_t^{II} is equivalent to the following representation on $\mathscr{H}(U^*, d\mu_t^{\mathrm{II}})$:*

$$z \to T(\zeta), \qquad \bar{z} \to T(\bar{\zeta}).$$

Case III. $\mu > 1 - q$, $q < 1$. Let us first consider the case of π_R^{III}. We set $D = U_R := \{\zeta : |\zeta| < R\}$, and define the measure

$$d\mu_R^{\mathrm{III}}(\zeta) := \frac{1}{\pi}(1 - R)\frac{(|\zeta|^2 qR^{-1}; q)_\infty}{(|\zeta|^2; q)_\infty} \sum_{n \geqslant 0} q^n \delta(|\zeta|^2 - Rq^n)\, d^2\zeta. \tag{VI.10}$$

Then, for $n \in \mathbb{Z}_+$,

$$\begin{aligned}\int_{U_R} |\zeta|^{2n}\, d\mu_R^{\mathrm{III}}(\zeta) &= R^n [s]_q \int_0^1 t^n \frac{(tq; q)_\infty}{(tq^s; q)_\infty}\, d_q t \\ &= R^n \frac{\Gamma_q(n+1)\, \Gamma_q(s+1)}{\Gamma_q(s+n+1)},\end{aligned}$$

where $q^s = R$. Consequently, the functions

$$\phi_n(\zeta) := \left\{R^{-n} \frac{\Gamma_q(s+n+1)}{\Gamma_q(n+1)\, \Gamma_q(s+1)}\right\}^{1/2} \zeta^n \tag{VI.12}$$

form an orthonormal basis for $\mathscr{H}(U_R, d\mu_R^{\mathrm{III}})$. The corresponding Bergman kernel is easily found to be

$$K_R^{\mathrm{III}}(\zeta, \eta) = \frac{(q\zeta\bar{\eta}; q)_\infty}{(\zeta\bar{\eta}/R; q)_\infty}. \tag{VI.13}$$

The Toeplitz operator $T(\zeta)$ satisfies

$$T(\zeta)\, \phi_n = \left\{R \frac{1 - q^{n+1}}{1 - R^{n+1}}\right\}^{1/2} \phi_{n+1},$$

and so we have the following result.

PROPOSITION VI.3. *With the above definitions, representation* π_R^{III} *is equivalent to the following representation on* $\mathcal{H}(U_R, d\mu_R^{\mathrm{III}})$:

$$\bar{z} \to T(\zeta), \qquad z \to T(\bar{\zeta}).$$

We were not able to find a Bergman space representation for $\pi_{1,t}^{\mathrm{III}}$, except for $s = \log_q R \in \mathbb{N}$. In this case, we set $D = U_R^* := \{\zeta \in \mathbb{C} : R < |\zeta| < 1\}$ and define

$$d\mu_{1,t}^{\mathrm{III}}(\zeta) = \frac{1}{\pi(-tq;q)_s} \sum_{0 \leqslant n \leqslant s} \frac{(q;q)_s}{(q;q)_n (q;q)_{s-n}} t^n q^{n(n+1)/2} \, \delta(|\zeta|^2 - q^n) \, d^2\zeta. \tag{VI.14}$$

Then, using the q-binomial theorem, we find that

$$\int_{U_R^*} |\zeta|^{2n} \, d\mu_{1,t}^{\mathrm{III}}(\zeta) = \frac{(-tq^{n+1};q)_s}{(-tq;q)_s}, \qquad n \in \mathbb{Z}, \tag{VI.15}$$

and so the functions

$$\phi_n(\zeta) := \left\{ \frac{(-tq;q)_s}{(-tq^{n+1};q)_s} \right\}^{1/2} \zeta^n \tag{VI.16}$$

form an orthonormal basis for $\mathcal{H}(U_R^*, d\mu_{1,t}^{\mathrm{III}})$. The Toeplitz operator $T(\zeta)$ satisfies

$$T(\zeta) \, \phi_n = \left\{ \frac{1 + tq^{n+1}R}{1 + tq^{n+1}} \right\}^{1/2} \phi_{n+1},$$

and so we have the following proposition.

PROPOSITION VI.4. *Let* $s = \log_q R \in \mathbb{N}$. *The* $\pi_{1,t}^{\mathrm{III}}$ *is unitarily equivalent to the following representation on* $\mathcal{H}(U_R^*, d\mu_{1,t}^{\mathrm{III}})$:

$$z \to T(\zeta), \qquad \bar{z} \to T(\bar{\zeta}).$$

VII. $U_q(sl(2))$-ACTION ON $C_{\mu,q}(\bar{U})$

In this section we construct an action of the quantized universal enveloping algebra $U_q(sl(2))$ [Dr] on the $\mathbb{C}^*$-algebra $C_{\mu,q}(\bar{U})$. This action is implemented by certain unbounded linear operators on $C_{\mu,q}(\bar{U})$ and hence it is defined only on a dense subalgebra. More generally, if $\mathscr{C}$ is a (Hopf) algebra and $\mathscr{A}$ a $\mathbb{C}^*$-algebra, we say that $\mathscr{C}$ acts densely on $\mathscr{A}$ if it acts on a dense subalgebra $\mathscr{A}_0$ of $\mathscr{A}$.

Recall that $U_q(sl(2))$ is a unital Hopf algebra with generators E, F, K, and K^{-1} and relations

$$\begin{gathered} KK^{-1} = K^{-1}K = I, \\ KEK^{-1} = q^{1/2}E, \qquad KFK^{-1} = q^{-1/2}F, \\ [E, F] = \frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}}. \end{gathered} \tag{VII.1}$$

Let $C_{\mu,q}(\bar{U})_0 \subset C_{\mu,q}(\bar{U})$ be the dense $*$-subalgebra of all polynomials in z and $\bar{z}$. We set

$$\begin{array}{lll} E(I) = 0, & E(z) = I, & E(\bar{z}) = -\bar{z}^2, \\ F(I) = 0, & F(z) = -z^2, & F(\bar{z}) = I, \\ K(I) = I, & K(z) = q^{-1/2}z, & K(\bar{z}) = q^{1/2}\bar{z}, \end{array} \tag{VII.2}$$

and require that for all $a, b \in C_{\mu,q}(\bar{U})_0$,

$$\begin{aligned} E(ab) &= E(a)\, K(b) + K^{-1}(a)\, E(b), \\ F(ab) &= F(a)\, K(b) + K^{-1}(a)\, F(b), \\ K(ab) &= K(a)\, K(b). \end{aligned} \tag{VII.3}$$

In other words, K is an automorphism of $C_{\mu,q}(\bar{U})_0$ and E and F are K-twisted derivations. Conditions (VII.3) are natural in the sense that they are consistent with the coproduct structure on $U_q(sl(2))$.

PROPOSITION VII.1. *Formulas* (VII.2) *and* (VII.3) *define linear maps of* $C_{\mu,q}(\bar{U})_0$ *into itself.*

Proof. We have to verify that E, F, and K are consistent with (IV.1). Observe first that

$$K(z\bar{z}) = z\bar{z}, \qquad K(\bar{z}z) = \bar{z}z, \tag{VII.4}$$

and so K preserves (IV.1). Apply now E to the left-hand side of (IV.1):

$$\begin{aligned} E(qz\bar{z} - \bar{z}z) &= qE(z)\, K(\bar{z}) + qK^{-1}(z)\, E(\bar{z}) - E(\bar{z})\, K(z) - K^{-1}(\bar{z})\, E(z) \\ &= q^{3/2}(I - z\bar{z})\, \bar{z} - q^{-1/2}\bar{z}(I - \bar{z}z). \end{aligned} \tag{VII.5}$$

On the other hand, applying E to the right-hand side of (IV.1) and using (VII.4) we obtain

$$\begin{aligned} &-\mu E(\bar{z}z)(I - z\bar{z}) - \mu(I - \bar{z}z)\, E(z\bar{z}) \\ &\quad = -\mu q^{-1/2}\bar{z}(I - \bar{z}z)(I - z\bar{z}) - \mu q^{1/2}(I - \bar{z}z)(I - z\bar{z})\bar{z}, \end{aligned}$$

which, upon using (IV.1), is equal to the right-hand side of (VII.5). As a consequence, E is consistent with (IV.1). In the same fashion we verify that F is consistent with (IV.1). ∎

PROPOSITION VII.2. *E, F, and K define a dense action of $U_q(sl(2))$ on $C_{\mu,q}(\bar{U})$.*

Proof. We verify that E, F, and K satisfy the algebra (VII.1). The first pair of relations holds trivially. To verify the second pair of relations we note first that $E' := KEK^{-1}$ and $F' := KFK^{-1}$ are K-twisted derivations in the sense of (VII.3). Indeed,

$$E'(ab) = KE(K^{-1}(a)\, K^{-1}(b)) = K(EK^{-1}(a)\, b + K^{-2}(a)\, EK^{-1}(b))$$
$$= E'(a)\, K(b) + K^{-1}(a)\, E'(b).$$

Consequently, it is sufficient to verify these relations when applied to the generators of $C_{\mu,q}(\bar{U})$. Clearly, they hold when applied to I. Furthermore,

$$KEK^{-1}(z) = q^{1/2} KE(z) = q^{1/2} K(I) = q^{1/2} I = q^{1/2} I = q^{1/2} E(z),$$

and

$$KEK^{-1}(\bar{z}) = q^{-1/2} KE(\bar{z}) = q^{-1/2} K(\bar{z}^2) = q^{1/2} \bar{z}^2 = q^{1/2} E(\bar{z}).$$

In the same manner we verify that $KFK^{-1}(z) = q^{-1/2} F(z)$ and $KFK^{-1}(\bar{z}) = q^{-1/2} F(\bar{z})$.

To verify the last relation in (VII.1) we set $V := [E, F]$ and $W := K^2 - K^{-2}$. We claim that V and W are K^2-twisted derivations. Indeed,

$$\begin{aligned} V(ab) &= EF(ab) - FE(ab) \\ &= E(F(a)\, K(b) + K^{-1}(a)\, F(b)) - F(E(a)\, K(b) + K^{-1}(a)\, E(b)) \\ &= (EF(a) - FE(a))\, K^2(b) + K^{-2}(a)(EF(b) - FE(b)) \\ &= V(a)\, K^2(b) + K^{-2}(b)\, V(a), \end{aligned}$$

and

$$\begin{aligned} W(ab) &= K^2(a)\, K^2(b) - K^{-2}(a)\, K^{-2}(b) \\ &= (K^2(a) - K^{-2}(a))\, K^2(b) + K^{-2}(a)(K^2(b) - K^{-2}(b)) \\ &= W(a)\, K^2(b) + K^{-2}(a)\, W(b). \end{aligned}$$

As a consequence, it is sufficient to verify the relation when applied to the generators. It clearly holds when applied to I. Moreover,

$$[E, F](z) = -E(z^2) = -(q^{-1/2} + q^{1/2})\, z = \frac{K^2(z) - K^{-2}(z)}{q^{1/2}\zeta - q^{-1/2}},$$

and

$$[E, F](\bar{z}) = F(\bar{z}^2) = -(q^{-1/2} + q^{1/2})\, \bar{z} = \frac{K^2(\bar{z}) - K^{-2}(\bar{z})}{q^{1/2} - q^{-1/2}},$$

and the proposition is proven. ∎

VIII. Concluding Remarks

In conclusion, we would like to discuss briefly two points which seem to be interesting and which were not touched upon in this paper.

First, it would be interesting to study the norm limits (II.2). Given our experience from [KL], we believe that the Bergman space representations of Section VI should provide a convenient framework for studying this problem. Formally, the structure of the limits is as follows. Let γ: $[0, 1] \to S$ be given by

$$\begin{aligned} \mu(t) &= \mu t, \\ q(t) &= 1 - (1 - q)t. \end{aligned} \tag{VIII.1}$$

Then

$$[z, \bar{z}] = t(1 - \bar{z}z)(\mu - (1 - q) - \mu\bar{z}z) + O(t^2), \tag{VIII.2}$$

and so the classical limit along γ yields a Poisson structure of the form (III.6). Observe that the three ranges of the parameters (μ, q) discussed in previous sections give rise to Poisson structures of entirely different characters: For $\mu < 1 - q$, the Poisson structure is symplectic, for $\mu = 1 - q$, it has two symplectic leaves, $\{0\}$ and $U \setminus \{0\}$, while for $\mu > 1 - q$, it has the following symplectic leaves: the points on the circle $|\zeta| = R$, the disc $|\zeta| < R$, and the annulus $R < |\zeta| < 1$, where R is given by (IV.9).

Second, it would be interesting to describe the quantized unit disc as a quantum homogeneous space. Formally, the algebra $C_{\mu, q}(\bar{U})$ admits an action of the quantum group $SU_q(1, 1)$. However, due to domain problems of the unbounded generators of $SU_q(1, 1)$ a quantum space description of $SU_q(1, 1)$ is unknown [Wo]. The dense $U_q(sl(2))$-action constructed in Section VII seems to be a residue of this putative $SU_q(1, 1)$-action.

APPENDIX. $\mathbb{C}^*$-ALGEBRAS BY WEIGHTED SHIFTS

In this Appendix we summarize some results on the structure of $\mathbb{C}^*$-algebras generated by hyponormal weighted shifts. More details and proofs can be found in [Co].

Let A be an irreducible operator on a Hilbert space $\mathscr{H}$ such that $[A^*, A]$ is compact. Let $\mathbb{C}^*(A)$ denote the unital $\mathbb{C}^*$-algebra generated by A and let I be its commutator ideal. Then $I = \mathscr{K}(\mathscr{H})$, the $\mathbb{C}^*$-algebra of compact operators on $\mathscr{H}$, and so the $\mathbb{C}^*$-algebra $\mathbb{C}^*(A)/\mathscr{K}$ is Abelian. Therefore, we have the short exact sequence of $\mathbb{C}^*$-algebras

$$0 \to \mathscr{K} \to \mathbb{C}^*(A) \to C(\sigma) \to 0, \tag{A.1}$$

where σ is a compact space (the spectrum of $\mathbb{C}^*(A)$). σ can be identified with the essential spectrum of A. Another useful characterization of σ is $\sigma = \{\lambda \in \mathbb{C}$: there is a multiplicative functional ϕ on $\mathbb{C}^*(A)$ such that $\phi(A) = \lambda\}$.

In this paper we are concerned with A, a weighted shift. Let $\mathscr{H} = l^2(\mathbb{Z}_+)$ and let $\{\phi_n\}_{n \in \mathbb{Z}_+}$ be the canonical basis for $\mathscr{H}$. Then A is a unilateral weighted shift on $\mathscr{H}$ if $A\phi_n = \alpha_n \phi_{n+1}$, $n \in \mathbb{Z}_+$, $\alpha_n \in \mathbb{C}$.

THEOREM A.1. *Let A be a unilateral weighted shift satisfying*

$$0 < \alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} < \cdots. \tag{A.2}$$

Then A is irreducible and $[A^*, A]$ is compact. Furthermore,

$$\sigma = \{\lambda \in \mathbb{C} : |\lambda| = \alpha_\infty\}, \tag{A.3}$$

where $\alpha_\infty := \lim_{n \to \infty} \alpha_n = \|A\|$.

Now let $\mathscr{H} = l^2(\mathbb{Z})$ and let $\{\phi_n\}_{n \in \mathbb{Z}}$ be the canonical basis for $\mathscr{H}$. Then A is a bilateral weighted shift on $\mathscr{H}$ if $A\phi_n = \alpha\phi_{n+1}$, $n \in \mathbb{Z}$, $\alpha_n \in \mathbb{C}$.

THEOREM A.2. *Let A be a bilateral weighted shift satisfying*

$$0 \leqslant \cdots \leqslant \alpha_n < \alpha_{n+1} < \cdots. \tag{A4}$$

Then A is irreducible and $[A^*, A]$ is compact. Furthermore,

$$\sigma = \{\lambda \in \mathbb{C} : |\lambda| = \alpha_{-\infty}\} \cup \{\lambda \in \mathbb{C} : |\lambda| = \alpha_\infty\}, \tag{A.5}$$

where $\alpha_{\pm\infty} := \lim_{n \to \pm\infty} \alpha_n$.

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