The Uses of Differential Geometry in Finance

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Overview

Joint with P. Hagan and D. Woodward

- Motivation: Varadhan's theorem
- Differential geometry
- SABR model
- Geometry of no arbitrage

Varadhan's theorem

This work has been motivated by the classical result of Varadhan.

- It relates the short time asymptotic of the *Green's function* of the backward Kolmogorov equation to the differential geometry of the state space.
- From the probabilistic point of view, the Green's function represents the transition probability of the diffusion, and it thus carries all the information about the process.
- Consequently, the geometry of the diffusion provides a natural book keeping device for calculations.

Varadhan's theorem

Let $W_t^1, \ldots, W_t^d$ be an N-dimensional Borwnian motion with

$$\mathbf{E}[dW_t^a dW_t^b] = \rho^{ab} dt,$$

where ρ is a constant correlation matrix. We consider a driftless, time homogeneous diffusion:

$$dX_t^j = \sum_a \sigma_a^j(X_t) dW_t^a,$$

defined in an open subset of $\mathbb{R}^N$ ("state space"). Define the following positive definite matrix:

$$g^{ij}\left(x\right) = \sum_{ab} \rho^{ab} \sigma_a^i\left(x\right) \sigma_b^j\left(x\right).$$

Varadhan's theorem

Let $G_{T,X}(t,x)$ denote the transition probability (or Green's function). It satisfies the following terminal value problem for the corresponding backward Kolmogorov equation:

$$\frac{\partial G_{T,X}}{\partial t}(t,x) + \frac{1}{2}\sum_{i,j} g^{ij}(x) \; \frac{\partial G_{T,X}}{\partial x^i \partial x^j}(t,x) = 0,$$

$$G_{T,X}(T,x) = \delta(x - X).$$

Substitution $t \to T - t$ transforms this into the initial value problem for the heat equation:

$$\frac{\partial G_X}{\partial t}(t,x) = \frac{1}{2}\sum_{i,j} g^{ij}(x) \; \frac{\partial G_X}{\partial x^i \partial x^j}(t,x),$$

$$G_X(0,x) = \delta(x - X).$$

Varadhan's theorem

Let $\mathcal{M}$ denote the state space of the diffusion. Varadhan's theorem states that

$$\lim_{t \to 0} t \log G_X(t, x) = -\frac{d(x, X)^2}{2} .$$

Here $d(x, X)$ is the geodesic distance on $\mathcal{M}$ with respect to a Riemannian metric given by the coefficients $g^{ij}(x)$ of the Kolmogorov equation. This gives us the leading order behavior of the Green's function:

$$G_X(t, x) \sim \exp\left(-\frac{d(x, X)^2}{2t}\right) .$$

Varadhan's theorem

To extract usable asymptotic information about the transition probability, more accurate analysis is necessary, but the choice of the Riemannian structure on $\mathcal{M}$ dictated by Varadhan's theorem turns out to be key. Indeed, that Riemannian geometry becomes an important tool in carrying out the calculations. Technically speaking, we are led to studying the asymptotic properties of the perturbed Laplace - Beltrami operator on a Riemannian manifold. The relevant techniques go by the names of the *geometric optics* or the *WKB method*.

Manifolds

A smooth manifold is a set $\mathcal{M}$ along with an open covering $\{U_\alpha\}$ and maps $h_\alpha : U_\alpha \longrightarrow \mathbb{R}^N$ such that all functions $h_\beta \circ h_\alpha^{-1} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ are infinitely differentiable. The maps h_α are called local coordinate systems. Thus a manifold locally looks like the flat space. In the following, all our manifolds will admit one global system of coordinates.

Tangent bundle

A tangent vector to a manifold $\mathcal{M}$ at a point x is a first order differential operator

$$V(x) = \sum_i V_i(x) \, \frac{\partial}{\partial x^i} \; .$$

The vector space $T_x\mathcal{M}$ of all tangent vectors at x is called the tangent space at x, the union $T\mathcal{M} = \bigcup_x T_x\mathcal{M}$ is called the tangent bundle.

Riemannian manifolds

A Riemannian metric on a manifold is a symmetric, positive definite form ("inner product") on the tangent bundle. The corresponding line element is

$$ds^2 = \sum_{ij} g_{ij}(x)\, dx^i dx^j .$$

A manifold equipped with a Riemannian metric is called a *Riemannian manifold*.

The Laplace-Beltrami operator Δ_g on a Riemannian manifold $\mathcal{M}$ is

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \sum_{ij} \frac{\partial}{\partial x^i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x^j} \right),$$

where $f \in C^\infty(\mathcal{M})$.

Example: Poincare geometry

The *Poincare plane* is the upper half plane $\mathbb{H}^2 = \{(x, y) : \; y > 0\}$ equipped with the line element

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

This line element comes from the metric tensor given by

$$h = \frac{1}{y^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

For convenience, we introduce complex coordinates on $\mathbb{H}^2$, $z = x + iy$; the defining condition then reads $\mathrm{Imz} > 0$.

Example: Poincare geometry

By $d(z, Z)$ we denote the *geodesic distance* between two points $z, Z \in \mathbb{H}^2$, $z = x + iy$, $Z = X + iY$, i.e. the length of the shortest path connecting z and Z. There is an explicit expression for $d(z, Z)$:

$$\cosh d(z, Z) = 1 + \frac{|z - Z|^2}{2yY} ,$$

where $|z - Z|$ denotes the Euclidean distance between z and Z.
The Laplace-Beltrami operator on $\mathbb{H}^2$ is given by:

$$\Delta_h = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) .$$

Heat equation on the Poincare plane

The heat kernel on the Poincare plane satisfies:

$$\frac{\partial}{\partial t} G_Z(t,z) = \Delta_h G_Z(t,z),$$
$$G_Z(0,z) = Y^2 \delta(z - Z).$$

McKean's formula gives an explicit representation for $G_Z(s,z)$:

$$G_Z(t,z) = \frac{e^{-t/4}\sqrt{2}}{(4\pi t)^{3/2}} \int_{d(z,Z)}^{\infty} \frac{u e^{-u^2/4t}}{\sqrt{\cosh u - \cosh d(z,Z)}} \, du.$$

Asymptotic of McKean's formula

Key for us is the following asymptotic expansion as $t \longrightarrow 0$:

$$G_Z(t, z) = \frac{1}{4\pi s}\, \exp\left(-\frac{d^2}{4t}\right) \times$$
$$\sqrt{\frac{d}{\sinh d}}\left[1 - \frac{1}{4}\left(\frac{d\coth d - 1}{d^2} + 1\right)t + O\left(t^2\right)\right].$$

Instantaneous covariance structure

The instantaneous covariance structure of a diffusion

$$dX_t^i = A^i(X_t, t)dt + \sum_a \sigma_a^i(X_t, t)dW_t^a$$

Quadratic variation of X

$$< dX^i, dX^j >_t = g^{ij}(X_t, t)dt,$$

with

$$\mathbf{E}[dW_t^a\, dW_t^b] = \rho^{ab}dt,$$

defines a time dependent Riemannian metric on $\mathcal{M}$.:

$$g^{ij}(X_t, t) = \sum_{ab} \rho^{ab}\sigma_a^i(X_t, t)\sigma_b^i(X_t, t).$$

SABR model

The dynamics of the forward rate F_t is

$$dF_t = \Sigma_t C(F_t) dW_t,$$
$$d\Sigma_t = \alpha \Sigma_t dZ_t.$$

Here Σ_t is the stochastic volatility parameter,

$$C(F) = F^\beta,$$

and W_t and Z_t are Brownian motions with

$$\mathbf{E}\left[dW_t dZ_t\right] = \rho dt.$$

We supplement the dynamics with the initial condition $F_0 = f,\ \Sigma_0 = \sigma$.

Special case: normal SABR model

The normal SABR model is a special case in which $\beta = 0$, and $\rho = 0$:

$$
\begin{aligned}
dF_t &= \Sigma_t dW_t, \\
d\Sigma_t &= \alpha \Sigma_t dZ_t .
\end{aligned}
$$

and

$$
\mathbf{E}\left[dW_t dZ_t\right] = 0.
$$

Backward Kolmogorov's equation reads:

$$
\begin{gathered}
\frac{\partial G_{T,F,\Sigma}}{\partial t} + \frac{1}{2}\,\sigma^2 \left(\frac{\partial^2 G_{T,F,\Sigma}}{\partial f^2} + \alpha^2\, \frac{\partial^2 G_{T,F,\Sigma}}{\partial \sigma^2} \right) = 0, \\
G_{T,F,\Sigma}\left(t, f, \sigma\right) = \delta\left(f - F,\, \sigma - \Sigma\right),\ \text{at } t = T.
\end{gathered}
$$

Special case: normal SABR model

After the transformation $t \to T - t$:

$$\frac{\partial G_{F,\Sigma}}{\partial t} = \frac{1}{2}\,\sigma^2 \left(\frac{\partial^2 G_{T,F,\Sigma}}{\partial f^2} + \alpha^2\,\frac{\partial^2 G_{F,\Sigma}}{\partial \sigma^2} \right),$$

$$G_{F,\Sigma}\left(0, f, \sigma\right) = \delta\left(f - F,\, \sigma - \Sigma\right).$$

This resembles the heat equation on the Poincare plane!

Normal SABR model and Poincare geometry

Brownian motion on the Poincare plane is described by:

$$dX_t = Y_t dW_t,$$
$$dY_t = Y_t dZ_t,$$

with

$$\mathbf{E}\left[dW_t dZ_t\right] = 0.$$

This is the normal SABR model if we make the following identifications:

$$X_t = F_{\alpha^2 t},$$
$$Y_t = \frac{1}{\alpha}\,\Sigma_{\alpha^2 t}\ .$$

Geometry of the full SABR model

The state space associated with the general SABR model has a somewhat more complicated geometry. Let $\mathbb{S}^2$ denote the upper half plane $\{(x, y) : \; y > 0\}$, equipped with the following metric g:

$$g = \frac{1}{\sqrt{1-\rho^2}\, y^2 C\left(x\right)^2} \begin{pmatrix} 1 & -\rho C\left(x\right) \\ -\rho C\left(x\right) & C\left(x\right)^2 \end{pmatrix}.$$

This metric is a generalization of the Poincare metric: the case of $\rho = 0$ and $C\left(x\right) = 1$ reduces to the Poincare metric.

Geometry of the full SABR model

The metric g is the pullback of the Poincare metric under the following diffeomorphism. We choose $p \geq 0$, and define a map $\phi_p : \mathbb{S}^2 \rightarrow \mathbb{H}^2$ by

$$\phi_p(z) = \left(\frac{1}{\sqrt{1-\rho^2}} \left(\int_p^x \frac{du}{C(u)} - \rho y \right), \quad y \right).$$

A consequence of this fact is that we have an explicit formula for the geodesic distance $\delta(z, Z)$ on $\mathbb{S}^2$:

$$\begin{aligned} \cosh \delta(z, Z) &= \cosh d\left(\phi_p(z), \phi_p(Z)\right) \\ &= 1 + \frac{\left(\int_X^x \frac{du}{C(u)} \right)^2 - 2\rho (y - Y) \int_X^x \frac{du}{C(u)} + (y-Y)^2}{2(1-\rho^2)\, yY}. \end{aligned}$$

Geometry of the full SABR model

We use invariant notation. Let $z^1 = x,\ z^2 = y$, and let $\partial_i = \partial / \partial z^i$, $i = 1,\ 2$, denote the corresponding partial derivatives. We denote the components of g^{-1} by g^{ij}, and use g^{-1} and g to raise and lower the indices: $z_i = \sum_j g_{ij} z^j$, $\partial^i = \sum_j g^{ij} \partial_j$. Explicitly,

$$\partial^1 = y^2 \left(C\left(x\right)^2 \partial_1 + \rho C\left(x\right) \partial_2 \right),$$
$$\partial^2 = y^2 \left(\rho C\left(x\right) \partial_1 + \partial_2 \right).$$

Consequently, the initial value problem can be written in the form:

$$\frac{\partial}{\partial s} K_Z\left(s, z\right) = \frac{1}{2} \varepsilon \sum_i \partial^i \partial_i K_Z\left(s, z\right),$$
$$K_Z\left(0, z\right) = \delta\left(z - Z\right) .$$

Geometry of the full SABR model

Except for the normal case, the operator $\sum_i \partial^i \partial_i$ does not coincide with the Laplace-Beltrami operator Δ_g on $\mathbb{S}^2$. One verifies that

$$\begin{aligned}\sum_i \partial^i \partial_i f &= \Delta_g f - \frac{1}{\sqrt{\det g}} \sum_{ij} \frac{\partial}{\partial x^j} \left(\sqrt{\det g}\, g^{ij} \right) \frac{\partial f}{\partial x^i} \\ &= \Delta_g f - \frac{1}{\sqrt{1-\rho^2}}\, y^2 C C' \,\frac{\partial f}{\partial x}.\end{aligned}$$

We treat the Laplace-Beltrami part by mapping it to the Laplace-Beltrami operator in the Poincare plane. The first order operator is treated as a regular perturbation of the Laplace-Beltrami operator.

Asymptotics of the Green's function

Using the asymptotic expansion of McKean's formula we derive the following asymptotic expansion for the Green's function:

$$K_Z(t,z) = \frac{1}{2\pi\lambda\sqrt{1-\rho^2}\,Y^2 C(X)} \exp\left(-\frac{\delta^2}{2\lambda}\right)\sqrt{\frac{\delta}{\sinh\delta}}$$
$$\left[1 - \frac{\delta}{\sinh\delta}\,q - \left(\frac{1}{8} + \frac{\delta\coth\delta - 1}{8\delta^2} - \frac{3(1-\delta\coth\delta)+\delta^2}{8\delta\sinh\delta}\,q\right)\lambda + \dots\right].$$

Here $\lambda = t\alpha^2$, and

$$q(z,Z) = -\frac{yC'(x)}{2(1-\rho^2)^{3/2}\,Y}\left(\int_X^x \frac{du}{C(u)} - \rho(y-Y)\right).$$

Probability distribution in the SABR model

STEP 1. We integrate the asymptotic joint density over the terminal volatility variable Y to find the marginal density for the forward x:

$$
\begin{aligned}
P_X(t,x,y) &= \int_0^\infty K_Z(t,z)\,dY \\
&= \frac{1}{2\pi\lambda\sqrt{1-\rho^2}\,C(X)}\int_0^\infty e^{-\delta^2/2\lambda}\sqrt{\frac{\delta}{\sinh\delta}}\,\left[1-\frac{\delta}{\sinh\delta}\,q\right. \\
&\left.-\frac{1}{8}\,\lambda\left(1+\frac{\delta\coth\delta-1}{\delta^2}-\frac{3\left(1-\delta\coth\delta\right)+\delta^2}{\delta\sinh\delta}\,q\right)\right]\frac{dY}{Y^2}\,.
\end{aligned}
$$

Here again, $\delta(z,Z)$ denotes the geodesic distance on $\mathbb{S}^2$.

Probability distribution in the SABR model

STEP 2. We evaluate this integral asymptotically by using the steepest descent method. Assume that $\phi(u)$ is positive and has a unique minimum u_0 in $(0, \infty)$ with $\phi''(u_0) > 0$. Then, as $\epsilon \longrightarrow 0$,

$$\int_0^\infty f(u) e^{-\phi(u)/\epsilon}\, du = \sqrt{\frac{2\pi\epsilon}{\phi''(u_0)}}\, e^{-\phi(u_0)/\epsilon} \times$$

$$\left\{ f(u_0) + \epsilon \left[\frac{f''(u_0)}{2\phi''(u_0)} - \frac{\phi^{(4)}(u_0) f(u_0)}{8\phi''(u_0)^2} \right.\right.$$

$$\left.\left. - \frac{f'(u_0)\,\phi^{(3)}(u_0)}{2\phi''(u_0)^2} + \frac{5\phi^{(3)}(u_0)^2 f(u_0)}{24\phi''(u_0)^3} \right] + O\left(\epsilon^2\right) \right\}.$$

Probability distribution in the SABR model

STEP 3. The exponent:

$$\phi(Y) = \frac{1}{2}\,\delta(z, Z)^2 \ ,$$

has a unique minimum at Y_0 given by

$$Y_0 = y\sqrt{\zeta^2 - 2\rho\zeta + 1}\ ,$$

where

$$\zeta = \frac{1}{y}\int_X^x \frac{du}{C(u)}\ .$$

Y_0 is the “most likely value” of Y, and thus $Y_0 C(X)$ (when expressed in the original units) is the leading contribution to the observed implied volatility.

Probability distribution in the SABR model

STEP 4. Let $D(\zeta)$ denote the value of $\delta(z, Z)$ with $Y = Y_0$. Explicitly,

$$D(\zeta) = \log \frac{\sqrt{\zeta^2 - 2\rho\zeta + 1} + \zeta - \rho}{1 - \rho} \ .$$

We also introduce the notation:

$$\begin{aligned} I(\zeta) &= \sqrt{\zeta^2 - 2\rho\zeta + 1} \\ &= \cosh D(\zeta) - \rho \sinh D(\zeta) \ . \end{aligned}$$

Probability distribution in the SABR model

Then, to within $O\left(\lambda^2\right)$, we find that

$$
\begin{aligned}
P_X\left(t,x,y\right) = \frac{1}{\sqrt{2\pi\lambda}}\,\frac{1}{yC\left(X\right)I^{3/2}}\,\exp\left\{-\frac{D^2}{2\lambda}\right\}\Bigg\{1+\frac{yC'\left(x\right)D}{2\sqrt{1-\rho^2}\,I} \\
-\frac{1}{8}\,\lambda\left[1+\frac{yC'\left(x\right)D}{2\sqrt{1-\rho^2}\,I}+\frac{6\rho yC'\left(x\right)}{\sqrt{1-\rho^2}\,I^2}\,\cosh\left(D\right)\right. \\
\left.-\left(\frac{3\left(1-\rho^2\right)}{I}+\frac{3yC'\left(x\right)\left(5-\rho^2\right)D}{2\sqrt{1-\rho^2}\,I^2}\right)\frac{\sinh\left(D\right)}{D}\right]+\ldots\Bigg\}\,.
\end{aligned}
$$

Probability distribution in the SABR model

In terms of the original variables:

$$P_F\left(t,f,\sigma\right)=\frac{1}{\sqrt{2\pi\tau}}\,\frac{1}{\sigma C\left(F\right)I^{3/2}}\,\exp\left\{-\frac{D^2}{2tv^2}\right\}\left\{1+\frac{\sigma C'\left(f\right)D}{2v\sqrt{1-\rho^2}\,I}\right.$$

$$-\frac{1}{8}\,\tau v^2\left[1+\frac{\sigma C'\left(f\right)D}{2v\sqrt{1-\rho^2}\,I}+\frac{6\rho\sigma C'\left(f\right)}{v\sqrt{1-\rho^2}\,I^2}\,\cosh\left(D\right)\right.$$

$$\left.\left.-\left(\frac{3\left(1-\rho^2\right)}{I}+\frac{3\sigma C'\left(f\right)\left(5-\rho^2\right)D}{2v\sqrt{1-\rho^2}\,I^2}\right)\frac{\sinh\left(D\right)}{D}\right]+\dots\right\}.$$

Line bundles over manifolds

A line bundle over a manifold $\mathcal{M}$ is a manifold $\mathcal{L}$ together with a map $\pi : \mathcal{L} \to \mathcal{M}$ such that:

- Each *fiber* $\pi^{-1}(x)$ is isomorphic to $\mathbb{R}$.
- Each $x \in \mathcal{M}$ has a neighborhood $U \subset \mathcal{M}$ and a diffeomorphism $\pi^{-1}(U) \simeq U \times \mathbb{R}$.

The simplest example of a line bundle is the trivial line bundle, $\mathcal{L} = \mathcal{M} \times \mathbb{R}$. A general line bundle looks locally like a trivial line bundle.

A smooth map $\phi : \mathcal{M} \to \mathcal{L}$ is called a *section* of $\mathcal{L}$ if $\pi \circ \phi = \mathrm{Identity}$. The set of all sections is denoted by $\Gamma(\mathcal{L})$.

Connections on line bundles

A connection ∇ on a line bundle $\mathcal{L} \to \mathcal{M}$ is a way to calculate derivatives of sections of $\mathcal{L}$ along tangent vectors.

- For $a, b \in \mathbb{R}$, and $\phi, \psi \in \Gamma(\mathcal{L})$,

$$\nabla(a\phi + b\psi) = a\nabla\phi + b\nabla\psi.$$

- For $f \in C^\infty(\mathcal{M})$, and $\phi \in \Gamma(\mathcal{L})$,

$$\nabla(f\phi) = df \otimes \phi + f\nabla\phi.$$

For example, on a trivial bundle $\mathcal{L} = \mathcal{M} \times \mathbb{R}$, all connections are of the form $\nabla = d + \omega$, where ω is a 1-form on $\mathcal{M}$.

Numeraire line bundle

Each nonzero $x \in \mathcal{M} \subset \mathbb{R}^N$ determines a direction in $\mathbb{R}^N$. Define a line bundle $\mathcal{L}$ by

$$\mathcal{L} = \{(x,\ \lambda x) :\ x \in \mathcal{M}, \lambda \in \mathbb{R}\}\ ,$$

and

$$\pi\,(x,\ \lambda x) = x.$$

Action of the group $\mathbb{R}_+$ on $\mathcal{L}$:

$$\Lambda\,(x)\,\phi\,(x)\ ,$$

where $\Lambda\,(x)$ is the ratio of two numeraires.

Holonomy and arbitrage

For $\phi \in \Gamma(\mathcal{L})$ there is a 1-form ω such that $\nabla(\phi) = \omega\phi$. If $\phi'(x) = \Lambda(x)\phi(x)$, then

$$\nabla(\phi') = \nabla(\Lambda\phi) = (d\Lambda + \Lambda\omega)\phi,$$

and so

$$\begin{aligned} \omega' &= \Lambda^{-1} d\Lambda + \omega \\ &= d\log\Lambda + \omega. \end{aligned}$$

The absence of arbitrage is related to the flatness of ω, $d\omega = 0$.

LIBOR market model

We are given N LIBOR forwards $L_t^1, \ldots, L_t^N$, expiring at T_j, with the dynamics

$$dL_t^j = \Delta^j(L_t, t)dt + C^j(L_t^j, t)dW_t^j,$$

with

$$\mathbf{E}[dL_t^i\, dL_t^j] = \rho^{ij}dt,$$

Choose the drifts so that this dynamics is arbitrage free!

The instantaneous Riemannian metric is given by

$$g^{ij}(L) = \rho^{ij}C^i(L^i)C^j(L^j).$$

LIBOR market model

We let $P_j(L)$ denote the price of the zero coupon bond expiring at T_j,

$$P_j(L) = \prod_{1 \leq i \leq j} \frac{1}{1 + \delta_i L^i} \,,$$

where δ_i is the day count fraction. Therefore, if

$$\Lambda(L) = \frac{P_k(L)}{P_j(L)} \,,$$

the corresponding connection forms differ by

$$d \log \left(\prod_{j+1 \leq i \leq k} \frac{1}{1 + \delta_i L^i} \right) .$$

LIBOR market model

Therefore,

$$\omega_i = \begin{cases} -\dfrac{\delta_i}{1+\delta_i L_t^i}\,, & \text{if } j+1 \le i \le k, \\ \dfrac{\delta_i}{1+\delta_i L_t^i}\,, & \text{if } k+1 \le i \le j, \\ 0, & \text{otherwise.} \end{cases}$$

and thus

$$\begin{aligned} \Delta^j &= \sum_i g^{ji}\omega_i \\ &= C^j \sum_i \rho^{ji} C^i \omega_i . \end{aligned}$$

LIBOR market model

As a consequence, the dynamics of the LMM has the familiar form:

Under Q^k,

$$dL_t^j = C^j(L_t^j, t) \times \begin{cases} -\sum_{j+1 \le i \le k} \dfrac{\rho_{ji}\delta_i C^i(L_t^i, t)}{1 + \delta_i L_t^i}\, dt + dW_t^j, & \text{if } j < k, \\ dW_t^j, & \text{if } j = k, \\ \sum_{k+1 \le i \le j} \dfrac{\rho_{ji}\delta_i C^i(L_t^i, t)}{1 + \delta_i L_t^i}\, dt + dW_t^j, & \text{if } j > k\,. \end{cases}$$

Extended LIBOR market model

We consider an extension of the LIBOR market model with stochastic volatility parameters denoted by $\sigma_t^1, \ldots, \sigma_t^N$:

$$dL_t^j = \Delta^j(L_t, \sigma_t, t)dt + C^j(L_t^j, \sigma_t^j, t)dW_t^j,$$
$$d\sigma_t^j = \Gamma^j(L_t, \sigma_t, t)dt + D^j(L_t^j, \sigma_t^j, t)dZ_t^j,$$

with

$$\mathbf{E}[dW_t^i\, dW_t^j] = \rho^{ij}\, dt,$$
$$\mathbf{E}[dW_t^i\, dZ_t^j] = r^{ij}\, dt,$$
$$\mathbf{E}[dZ_t^i\, dZ_t^j] = \pi^{ij}\, dt.$$

Choose the drifts so that this dynamics is arbitrage free!

Extended LIBOR market model

The state space of this model has dimension $2N$ and the instantaneous Riemannian metric is given by:

$$g^{ij}(L,\sigma) = \begin{cases} \rho^{ij} C^i(L^i,\sigma^i) C^j(L^j,\sigma^j), & \text{if } 1 \le i,j \le N, \\ \pi^{ij} C^i(L^i,\sigma^i) D^j(L^j,\sigma^j), & \text{if } 1 \le i \le N,\ N+1 \le j \le 2N, \\ \pi^{ij} D^i(L^i,\sigma^i) C^j(L^j,\sigma^j), & \text{if } N+1 \le i \le 2N,\ 1 \le j \le N, \\ r^{ij} D^i(L^i,\sigma^i) D^j(L^j,\sigma^j), & \text{if } N+1 \le i,j \le 2N. \end{cases}$$

Calculation of the connection form ω is identical to that of the LIBOR model.

Extended LIBOR market model

Under Q^k,

$$dL_t^j = C^j(L_t^j, \sigma_t^j, t) \times \begin{cases} -\sum_{j+1 \le i \le k} \dfrac{\rho_{ji}\delta_i C^i(L_t^i, \sigma_t^i, t)}{1+\delta_i L_t^i}\, dt + dW_t^j, & \text{if } j < k, \\ dW_t^j, & \text{if } j = k, \\ \sum_{k+1 \le i \le j} \dfrac{\rho_{ji}\delta_i C^i(L_t^i, \sigma_t^i, t)}{1+\delta_i L_t^i}\, dt + dW_t^j, & \text{if } j > k\,, \end{cases}$$

$$d\sigma_t^j = D^j(L_t^j, \sigma_t^j, t) \times \begin{cases} -\sum_{j+1 \le i \le k} \dfrac{r_{ji}\delta_i C^i(L_t^i, \sigma_t^i, t)}{1+\delta_i L_t^i}\, dt + dZ_t^j, & \text{if } j < k, \\ dZ_t^j, & \text{if } j = k, \\ \sum_{k+1 \le i \le j} \dfrac{r_{ji}\delta_i C^i(L_t^i, \sigma_t^i, t)}{1+\delta_i L_t^i}\, dt + dZ_t^j, & \text{if } j > k\,. \end{cases}$$