WKB Method for Swaption Smile*

Andrew Lesniewski
BNP Paribas
New York

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Abstract

We study a three-parameter stochastic volatility model, originally proposed by P. Hagan, for the forward swap rate. The model is essentially a stochasticied version of the CEV model, where the volatility parameter is itself a stochastic process. We construct a computationally efficient, asymptotic solution to this model. This solution allows one to fit a variety of shapes of the volatility smiles in the swaption markets. The technique used to obtain this solution is a WKB expansion around geodesic motion on a suitable hyperbolic manifold.

Contents

*Transparencies for the talk at the Mathematical Finance Seminar at the Courant Institute of Mathematical Sciences, New York

References

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1 Introduction

A *swaption* is an option to enter into a swap. A *receiver* (*payer*) swaption gives the owner the right to receive (pay) fixed rate on the swap.

Market lingo: a T into M swaption is a swaption expiring T years from now on an M year swap. Thus a 7% 5 into 10 receiver is an option to receive 7% on a 10 year swap starting 5 years from now.

The market practice is to quote prices and calculate risk parameters of European swaptions in terms of Black's model:

$$dF_t = \sigma F_t dW_t,$$

where F_t is the forward swap rate, and σ is the implied volatility or "Black's volatility". For simplicity, rather than pricing calls and puts, we consider the Arrow-Debreu security whose payoff at time T is given by $\delta\left(F_T - F\right)$, where δ denotes Dirac's delta function. The time t price $G\left(t, f; T, F\right)$ of this security (or, the transition probability) is the solution to the following parabolic problem:

$$\frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 f^2 \frac{\partial^2 G}{\partial f^2} = 0,$$

$$G(t, f; T, F) = \delta(f - F), \text{ at } t = T.$$

Thus the price V of a payer struck at K is

$$V = \mathcal{N} \int G(t, f; T, F) \max(F - K, 0)\, dF,$$

where $\mathcal{N}$ is depends on the notional principal of the transaction and today's term structure of rates but not σ.

The solution is

$$G(t, f; T, F) = \sqrt{\frac{f}{2\pi\tau F^3 \sigma^2}} \exp\left(-\frac{(\log(f/F))^2}{2\tau\sigma^2} - \frac{\tau\sigma^2}{8}\right),$$

where $\tau = T - t$. This leads to the well known Black's formula for pricing swaptions.

The reality of the market is that implied volatility is a function $\sigma = \sigma(K, T, f)$ of

- strike K,
- time to expiry T,
- today's value of the forward f.

The dependence of σ on K reflects economic realities, and is referred to as the option *smile*.

Why modeling smile is important?

- Transaction pricing;
- Portfolio mark to market;
- Portfolio risk: smile adjusted delta is

$$\begin{aligned} \Delta &= \frac{\partial V}{\partial f} + \frac{\partial V}{\partial \sigma}\frac{\partial \sigma}{\partial f} \\ &= \Delta_0 + \Lambda \frac{\partial \sigma}{\partial f}, \end{aligned}$$

  where Δ_0 is the B-S delta, and Λ is the B-S vega. Similarly for the other greeks;

- Calibrating term structure models for pricing exotic structures, transactions with embedded options, etc.

How to model smile?

- Interpolate brokers' quotes;
- Shifted lognormal model:

$$dF_t = (\sigma_1 F_t + \sigma_0)\, dW_t;$$

- CEV model

$$dF_t = \sigma F_t^{\beta} dW_t,\ 0 \leq \beta \leq 1;$$

- Stochastic volatility models;
- ...

Implementation issues:

- Exact solutions;
- Numerical implementations (tree, MC, PDE);
- Approximate analytic solutions.

2 Stochastic CEV model

Replace Black's model by the system

$$\begin{aligned} dF_t &= \sigma_t b\,(F_t)\, dW_t, \\ d\sigma_t &= v \sigma_t dZ_t, \end{aligned}$$

where the two Brownian motions are correlated:

$$E\,[dW_t dZ_t] = \rho dt.$$

For the CEV model

$$b\,(F) = F^{\beta}.$$

Hagan's formula

Implied normal volatility σ_n has the following asymptotic expansion

$$\begin{aligned}\sigma_n &= v\frac{f-K}{\delta_0\left(f,K\right)}\times \\ &\quad \left(1+\frac{1}{24}\left[\left(2\gamma_2-\gamma_1^2\right)\sigma^2 b\left(f\right)^2+6\rho v\sigma\gamma_1 b\left(f\right)+\left(2-3\rho^2\right)v^2\right]\tau+\dots\right),\end{aligned}$$

where

$$\begin{aligned}\delta_0\left(\zeta\right) &= \log\frac{\sqrt{\zeta^2-2\rho\zeta+1}+\zeta-\rho}{1-\rho}, \\ \zeta &= \frac{v}{\sigma}\int_K^f \frac{du}{b\left(u\right)},\end{aligned}$$

and $\tau=T-t, \gamma_1=b'\left(f\right), \gamma_2=b''\left(f\right)$.

We consider the Arrow-Debreu security whose payoff at time T is given by $\delta\left(F_T-F,\sigma_T-\Sigma\right)$, where δ denotes Dirac's delta function. The time t price $G\left(t,f,\sigma;T,F,\Sigma\right)$ of this security is the solution to the following parabolic problem:

$$\begin{gathered}\frac{\partial G}{\partial t}+\frac{1}{2}\sigma^2\left(b\left(f\right)^2\frac{\partial^2 G}{\partial f^2}+2v\rho b\left(f\right)\frac{\partial^2 G}{\partial f\partial\sigma}+v^2\frac{\partial^2 G}{\partial\sigma^2}\right)=0, \\ G\left(t,f,\sigma;T,F,\Sigma\right)=\delta\left(f-F,\sigma-\Sigma\right),\ \text{at } t=T.\end{gathered}$$

The price of a payer struck at K is now

$$\begin{aligned}V &= \mathcal{N}\int G\left(t,f,\sigma;T,F,\Sigma\right)\max\left(F-K,0\right)dFd\Sigma \\ &= \mathcal{N}\int G_\sigma\left(t,f;T,F\right)\max\left(F-K,0\right)dF,\end{aligned}$$

where

$$G_\sigma\left(t,f;T,F\right)=\int G\left(t,f,\sigma;T,F,\Sigma\right)d\Sigma$$

is the marginal transition probability.

The coefficients in this problem are time independent, and so G is a function of $\tau=T-t$. Denote this function by $G\left(f,\sigma,F,\Sigma;\tau\right)$. Introduce the following variables:

$$\begin{aligned}
s &= \tau/T, \\
x &= f, \\
X &= F, \\
y &= \sigma/v, \\
Y &= \Sigma/v, \\
K\left(x,y,X,Y;s\right) &= vTG\left(x,vy,X,vY;sT\right).
\end{aligned}$$

In terms of these variables, the initial value problem can be recast as:

$$\begin{aligned}
\frac{\partial K}{\partial s} &= \frac{1}{2}\varepsilon y^2\left(b\left(x\right)^2\frac{\partial^2 K}{\partial x^2}+2\rho b\left(x\right)\frac{\partial^2 K}{\partial x\partial y}+\frac{\partial^2 K}{\partial y^2}\right), \\
K\left(x,y,X,Y;0\right) &= \delta\left(x-X,y-Y\right), \text{ at } s=0,
\end{aligned}$$

where

$$\varepsilon = v^2 T.$$

It will be assumed that ε is small and it will serve as the parameter of our expansion. The heuristic picture behind this idea is that the volatility varies slower than the forward, and the rates of variability of f and σ/v are similar. The time T defines the time scale of the problem, and thus s is a natural dimensionless time variable. Expressed in terms of the new variables, our problem has a natural differential geometric content which is key to its solution.

3 Exactly solvable case

Let $b\left(F\right)=1$, and $\rho=0$:

$$\begin{aligned}
dF_t &= \sigma_t dW_t, \\
d\sigma_t &= v\sigma_t dZ_t, \\
E\left[dW_t dZ_t\right] &= 0.
\end{aligned}$$

Also define $x=f, y=\sigma/v$. Then the problem becomes:

$$\begin{aligned}
\frac{\partial K}{\partial \tau} &= \frac{1}{2}y^2\left(\frac{\partial^2 K}{\partial x^2}+\frac{\partial^2 K}{\partial y^2}\right), \\
K\left(x,y,X,Y;\tau\right) &= \delta\left(x-X,y-Y\right), \text{ at } \tau=0.
\end{aligned}$$

This is the heat equation on the Poincare plane!

Recall that the Poincare plane is the upper half plane $\mathbb{H}^2 = \{(x, y) : \ y > 0\}$ with the line element

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

This comes from the metric tensor

$$h = \frac{1}{y^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The geodesic distance $d(z, Z)$ between two points $z, Z \in \mathbb{H}^2$, $z = x + iy$, $Z = X + iY$ is

$$\cosh d(z, Z) = 1 + \frac{|z - Z|^2}{2yY},$$

where $|z - Z|$ is the Euclidean distance between z and Z.

The heat equation on the Poincare plane can be solved in closed form:

$$K(z, Z; \tau) =$$

$$\frac{e^{-d^2/2\tau}\sqrt{2}}{(4\pi\tau)^{3/2}} \int_{d(z,Z)}^{\infty} \frac{ue^{-u^2/4\tau}}{\sqrt{\cosh u - \cosh d(z, Z)}} du$$

Asymptotic expansion as $\tau \to 0$:

$$\begin{aligned} K(z, Z; \tau) &= \frac{1}{2\pi\tau} e^{-d^2/2\tau} \sqrt{\frac{d}{\sinh d}} \times \\ & \left(1 - \frac{1}{8}\left(\frac{d \coth d - 1}{d^2} + 1\right) \tau + O\left(\tau^2\right)\right). \end{aligned}$$

4 Geometry of the full model

Let $\mathbb{M}^2$ denote the first quadrant $\{(x, y) : x > 0, \ y > 0\}$, and let g denote the metric:

$$g = \left(1 - \varrho^2\right)^{-1} \begin{pmatrix} \frac{1}{y^2 b(x)^2} & -\frac{\rho}{y^2 b(x)} \\ -\frac{\rho}{y^2 b(x)} & \frac{1}{y^2} \end{pmatrix}.$$

The case of $\rho = 0$ and $b(x) = 1$ reduces to the Poincare metric. The metric g is the pullback of the Poincare metric under a diffeomorphism: choose $p > 0$, and define $\phi_p : \mathbb{M}^2 \to \mathbb{H}^2$ by

$$\phi_p(z) = \left(\frac{1}{\sqrt{1-\varrho^2}} \left(\int_p^x \frac{du}{b(u)} - \rho y \right), \; y \right),$$

where $z = (x, y)$. The Jacobian $\nabla\phi_p$ of ϕ_p is

$$\nabla\phi_p(z) = \begin{pmatrix} \frac{1}{\sqrt{1-\varrho^2} b(x)} & -\frac{\rho}{\sqrt{1-\varrho^2}} \\ 0 & 1 \end{pmatrix},$$

and so $\phi_p^* h = g$, where ϕ_p^* denotes the pullback of ϕ_p.

The manifold $\mathbb{M}^2$ is thus isometrically diffeomorphic with a submanifold of the Poincare plane. Consequently, we have an explicit formula for the geodesic distance $\delta(z, Z)$ on $\mathbb{M}^2$:

$$\begin{aligned} \cosh\delta(z,Z) &= \cosh d(\phi_p(z), \phi_p(Z)) \\ &= 1 + \frac{\left(\int_X^x \frac{du}{b(u)} \right)^2 - 2\rho(y-Y) \int_X^x \frac{du}{b(u)} + (y-Y)^2}{2(1-\rho^2) yY}, \end{aligned}$$

where $z = (x, y)$ and $Z = (X, Y)$ are points on $\mathbb{M}^2$. The volume element on $\mathbb{M}^2$ is given by

$$\frac{1}{\sqrt{1-\varrho^2}} \frac{dxdy}{b(x) y^2}.$$

Let $z^1 = x$, $z^2 = y$, and let $\partial_\mu = \partial/\partial z^\mu$, $\mu = 1, 2$, denote the corresponding partial derivatives. We denote the components of g^{-1} by $g^{\mu\nu}$, and use g^{-1} and g to raise and lower the indices: $z_\mu = g_{\mu\nu} z^\nu$, $\partial^\mu = g^{\mu\nu}\partial_\nu = \partial/\partial z_\mu$, where we sum over the repeated indices. Explicitly,

$$\begin{aligned} \partial^1 &= y^2 \left(b(x)^2 \partial_1 + \rho b(x) \partial_2 \right), \\ \partial^2 &= y^2 \left(\partial_2 + \rho b(x) \partial_1 \right). \end{aligned}$$

Consequently, the initial value problem can be written as:

$$\begin{aligned} \frac{\partial}{\partial s} K(z, Z; s) &= \frac{1}{2} \varepsilon \partial^\mu \partial_\mu K(z, Z; s), \\ K(z, Z; 0) &= \delta(z - Z). \end{aligned}$$

Except for $b(x) = 1$, $\partial^\mu\partial_\mu$ *is not* the Laplace-Beltrami operator on $\mathbb{M}^2$.

5 WKB method

We seek $K(z, Z; s)$ in the form

$$K(z, Z; s) = \frac{1}{2\pi\varepsilon} R(z, Z; s) \exp\left(-\frac{1}{\varepsilon} S(z, Z; s)\right),$$

$S(z, Z; s)$ is assumed independent of ε. $R(z, Z; s)$ depends on ε and is assumed smooth at $\varepsilon = 0$. Substituting we obtain the following two PDEs:

$$S_s + \frac{1}{2}\partial^\mu S\ \partial_\mu S = 0, \tag{1}$$

where the subscript s denotes the derivative with respect to s, and

$$\left(R^2\right)_s + \partial^\mu\left(R^2 \partial_\mu S\right) = \varepsilon R \partial^\mu \partial_\mu R.$$

Equation (1) is the *Hamilton-Jacobi equation* for a free motion of a particle on $\mathbb{M}^2$, $S(z, Z; s)$ is the *action function*.

Now factor out the ε-independent part of $R(z, Z; s)$:

$$R(z, Z; s) = q(z, Z)^{1/2} Q(z, Z; s)$$

and make the following asymptotic expansion in ε:

$$Q(z, Z; s) = \sum_{k\geq 0} \varepsilon^k Q^{(k)}(z, Z; s),$$

with

$$Q^{(0)}(z, Z; s) = 1.$$

Then the function $q(z, Z; s)$ satisfies the *transport equation*,

$$q_s + \partial^\mu\left(q \partial_\mu S\right) = 0,$$

and for $Q(z, Z; s)$ we find the equation:

$$Q_s + \partial^\mu Q\ \partial_\mu S = \varepsilon \frac{1}{2q^{1/2}} \partial^\mu \partial_\mu \left(q^{1/2} Q\right).$$

This last equation is equivalent to an infinite system:

$$Q_s^{(k+1)} + \partial^\mu Q^{\ (k+1)} \partial_\mu S = \frac{1}{2\sqrt{q}} \partial^\mu \partial_\mu \left(\sqrt{q} Q^{(k)}\right),$$

which we call the *WKB hierarchy*. The three equations:

- the Hamilton-Jacobi equation
- the transport equation
- the WKB hierarchy

form a basis for solution of our problem.

Solve the Hamilton-Jacobi equation

We set

$$S(z, Z; s) = S^0(\phi(z), \phi(Z); s),$$

where $\phi = \phi_X$. Then S^0 satisfies the Hamilton-Jacobi equation on the Poincare plane:

$$S_s^0 + \frac{1}{2}y^2\left(\left(S_x^0\right)^2 + \left(S_y^0\right)^2\right) = 0.$$

Seek a solution $S^0(w, W; s) = f(r; s)$, where $r = d(w, W)$. Then f satisfies

$$f_s + \frac{1}{2}f_r^2 = 0,$$

and so

$$f(r; s) = \frac{1}{2s}r^2 + const.$$

We will choose $const = 0$. Consequently,

$$\begin{aligned} S(z, Z; s) &= \frac{1}{2s}d(\phi(z), \phi(Z))^2 \\ &= \frac{1}{2s}\delta(z, Z)^2 \end{aligned}$$

It is (after reinstating the constant $const$) the complete integral of the Hamilton-Jacobi equation.

Solve the transport equation

We set

$$\begin{aligned} q(z, Z; s) &= p(\phi(z), \phi(Z); s)\det\nabla\phi(z) \\ &= \frac{1}{\sqrt{1-\varrho^2 b(x)}}p(\phi(z), \phi(Z); s) \end{aligned}$$

to find that

$$p_s + \partial^\mu\left(p\partial_\mu S^0\right) = y^2\partial_1(\log B)\,\partial_1 S^0 p,$$

where $B(\phi(z)) = b(x)$. Substituting $p = q^0 B$ we find

$$q_s^0 + y^2 \left(\left(q^0 S_x^0\right)_x + \left(q^0 S_y^0\right)_y \right) = 0.$$

This is the transport equation on the Poincare plane. We seek a radial solution $q^0 = f(r,s)$. Then

$$s\,(f r \sinh r)_s + r\,(f r \sinh r)_r = 0.$$

This means that $f(r,s)\, r \sinh r$ is a function of r/s:

$$f(r,s) = \frac{\chi(r/s)}{r \sinh r},$$

and so

$$\begin{aligned} q(z,Z;s) &= \frac{b\left(\phi^{-1}(x)\right)}{\sqrt{1-\varrho^2 b(x)}} \times \\ &\quad \frac{\chi(\delta(z,Z)/s)}{\delta(z,Z)\sinh\delta(z,Z)}. \end{aligned}$$

Solve the WKB hierarchy
Well, ok, let's stick with $Q^{(0)}(z,Z;s) = 1$.

6 Probability distribution

We put everything together and verify that in order for the initial condition to be satisfied we need to choose $\chi(u) = u^2$. Hence finally we obtain the asymptotic formula

$$\begin{aligned} K(z,Z;s) &= \frac{1}{2\pi\varepsilon s\sqrt{1-\rho^2}}\sqrt{\frac{B(x,X)}{b(x)}} \\ &\quad \times \sqrt{\frac{\delta(z,Z)}{\sinh\delta(z,Z)}} \exp\left\{-\frac{\delta(z,Z)^2}{2\varepsilon s}\right\}(1+O(s\varepsilon)). \end{aligned}$$

First we calculate the approximate marginal probability distribution from the WKB Green's function:

$$K_y(x,X;s) = \int_0^\infty K(z,Z;s)\,dY$$

$$
\begin{aligned}
&= \frac{1}{2\pi\varepsilon s\sqrt{1-\rho^2}}\sqrt{\frac{B(x,X)}{b(x)}}\times \\
&\int_0^\infty \sqrt{\frac{\delta(z,Z)}{\sinh\delta(z,Z)}}\exp\left(-\frac{\delta(z,Z)^2}{2\varepsilon s}\right)\frac{dY}{Y^2},
\end{aligned}
$$

To evaluate this integral we use the Laplace method: as $\varepsilon \to 0$,

$$
\int_0^\infty f(u)\, e^{-\phi(u)/\varepsilon} du = \sqrt{\frac{2\pi\varepsilon}{\phi''(u_0)}} e^{-\phi(u_0)/\varepsilon} f(u_0)\,(1+O(\varepsilon)),
$$

if u_0 is the unique minimum of ϕ with $\phi''(u_0) > 0$.

Let us introduce the notation:

$$
\zeta = \frac{1}{y}\int_X^x \frac{du}{b(u)}.
$$

Given x, X, and y, let Y_0 be the value of Y which minimizes the distance $\delta(z,Z)$, and let $\delta_0(\zeta)$ be the corresponding value of $\delta(z,Z)$. Explicitly,

$$
\begin{aligned}
Y_0(\zeta,y) &= y\sqrt{\zeta^2 - 2\rho\zeta + 1}, \\
\delta_0(\zeta) &= \log\frac{\sqrt{\zeta^2-2\rho\zeta+1}+\zeta-\rho}{1-\rho}.
\end{aligned}
$$

Introduce the notation

$$
I(\zeta) = \sqrt{\zeta^2 - 2\rho\zeta + 1}.
$$

This yields

$$
K_y(x,X;s) = \frac{1}{\sqrt{2\pi\varepsilon s y^2 I(\zeta)^3}}\sqrt{\frac{B(x,X)}{b(x)}}\exp\left\{-\frac{\delta_0^2}{2\varepsilon s}\right\}(1+O(s\varepsilon))
$$

This is the desired asymptotic form of the marginal probability distribution.

Let us now compare this result with the normal distribution function:

$$
P(x,X;s) = \frac{1}{\sqrt{2\pi s\varepsilon y_n^2}}\exp\left(-\frac{(x-X)^2}{2s\varepsilon y_n^2}\right),
$$

where y_n is related to the normal volatility σ_n by $y_n = \sigma_n/\varepsilon^{1/2}$. We shall relate the cumulative distribution function

$$\begin{aligned}\int_K^\infty P(x, X; s)\, dX &= \frac{1}{\sqrt{2\pi\varepsilon s}} \int_K^\infty \exp\left\{-\frac{h^2}{2\varepsilon s}\right\} dh \\ &= \frac{1}{2}\operatorname{erfc}\left(\frac{K - x}{\sqrt{2s\varepsilon y_n}}\right)\end{aligned}$$

to the cumulative distribution function of $K_y(x, X; s)$. Neglecting the terms of order $O(\varepsilon^2)$, we have:

$$\int_K^\infty K_y(x, X; s)\, dX =$$

$$\frac{1}{\sqrt{2\pi\varepsilon s y^2}} \int_K^\infty \sqrt{\frac{b(B(x, X))}{I(\zeta)^3\, b(x)}} \exp\left\{-\frac{\delta_0^2}{2\varepsilon s}\right\} dX.$$

We substitute a new variable in the integral above,

$$h = h(X) = \delta_0 \left(\frac{1}{y} \int_X^x \frac{du}{b(u)}\right),$$

which yields

$$\int_K^\infty K_y(x, X; s)\, dX =$$

$$\frac{1}{\sqrt{2\pi\varepsilon s}} \int_{h(K)}^\infty \sqrt{\frac{B(x, X(h))}{I(\zeta)\, b(x)}} \exp\left\{-\frac{h^2}{2\varepsilon s}\right\} dh$$

Expanding as $\varepsilon \to 0$, and comparing the terms we obtain Hagan's formula.

7 Further developments

- Mean reversion in the volatility dynamics

$$\begin{aligned}dF_t &= \sigma_t b(F_t)\, dW_t, \\ d\sigma_t &= \lambda\sigma_t \left(\log \frac{\overline{\sigma}}{\sigma_t}\right) dt + v\sigma_t dZ_t, \\ E[dW_t dZ_t] &= \rho dt\end{aligned}$$

- Time dependent parameters

$$\begin{aligned} dF_t &= a_t b\left(F_t\right)\sigma_t dW_t, \\ d\sigma_t &= v_t \sigma_t dZ_t, \\ E\left[dW_t dZ_t\right] &= \rho_t dt. \end{aligned}$$

- Term structure model with stochastic volatility consistent with the vanilla model.