Dynamic Mortgage Rate Replication and Risk Management of MBSs

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Outline

Recap of Mortgage Mathematics

Mortgage backed securities (MBS) are fixed income instruments collateralized by mortgage loans. A (fixed coupon) mortgage carries an annual coupon $\widehat{C}$ and matures in N months. Denote:

$$c = \frac{\widehat{C}}{12},$$
$$d = \frac{1}{1+c}.$$

On a principal of \$1:

- Scheduled monthly payment is

$$m = \frac{c}{1-d^N}.$$

Recap of Mortgage Mathematics

- The principal repayment portion p_j of m for month j is
$$p_j = \frac{cd^{N-j+1}}{1-d^N}\,.$$
- The interest portion i_j of m for month j is
$$i_j = \frac{c\left(1-d^{N-j+1}\right)}{1-d^N}\,.$$
- Balance B_j outstanding at the end of month j is
$$B_j = \frac{1-d^{N-j}}{1-d^N}\,.$$
- The amortization schedule is defined so that
$$i_j = cB_{j-1}.$$

Types of MBSs

- Agency pass-throughs are the basic MBSs: all principal repayments and interest (less servicing and credit spread) of the underlying pool of collateral are paid to the holder.
- TBA (to be announced) pass-throughs, or simply TBAs.
- Mortgage options = options on TBAs.
- Structured MBSs such as IOs, POs, CMOs: cash flows are carved out from the cash flows of the underlying pool of collateral.
- Constant maturity mortgage (CMM) products.
- ...

TBAs

- A TBA is a futures contract on a pool of conventional, fixed coupon mortgage loans.
- It carries a coupon C reflective of the coupons on the deliverable loans. The values of C are spaced in 50 bp increments: 3.5%, 4.0%, 4.5%, etc.
- There is a standard delivery date, the PSA date, in each month. A vast majority of the trading takes place in either the nearest or the once-deferred month, but the market quotes prices for three or four TBAs settling on the next PSA dates.
- A party long a contract at settlement takes the delivery of a pool of mortgage loans satisfying the good delivery guidelines.

TBAs

For example, here is a snapshot of the TBA market on June 2, 2010:

Cpn \ PSA	Jun 2010	Jul 2010	Aug 2010
3.5	95.594	95.211	94.773
4.0	99.125	98.789	98.484
4.5	102.086	101.695	101.359
5.0	104.852	104.430	104.047
5.5	106.906	106.500	106.141
6.0	107.930	107.609	107.328
6.5	108.953	108.656	108.375

We let $P_C(T)$ denote the price of the TBA with coupon C and settlement date T.

Mortgage options

Mortgage options are European calls and puts on TBAs.

- They expire one calendar week before the PSA day on the underlying TBA.
- The strikes on the TBA options are standardized: $\mathrm{ATM} \equiv P_C(T)$, $\mathrm{ATM} \pm 1/2$ pt, $\mathrm{ATM} \pm 1$ pt.

Here is a market snapshot taken on June 2, 2000:

June 2010 expirations

Cpn \ Strike	-1.0	-0.5	ATM	+0.5	+1.0
4.0	0.0326	0.1120	0.2904	0.1016	0.0247
4.5	0.0143	0.0651	0.2279	0.0547	0.0052
5.0	0.0059	0.0299	0.1667	0.0195	N/A
5.5	N/A	0.0104	0.1185	0.0039	N/A

Mortgage options

July 2010 expirations

Cpn \ Strike	-1.0	-0.5	ATM	+0.5	+1.0
4.0	0.4141	0.5755	0.7839	0.5443	0.3555
4.5	0.2852	0.4245	0.6237	0.3828	0.2096
5.0	0.1641	0.2760	0.4570	0.2227	0.0833
5.5	0.0833	0.1628	0.3216	0.1081	0.0195

August 2010 expirations

Cpn \ Strike	-1.0	-0.5	ATM	+0.5	+1.0
4.0	0.6992	0.8737	1.0820	0.8307	0.6133
4.5	0.5078	0.6628	0.8607	0.6029	0.3958
5.0	0.3255	0.4531	0.6341	0.3789	0.1914
5.5	0.1940	0.2930	0.4518	0.2083	0.0677

CMM rate and instruments

A CMM rate is an index representing the yield on a hypothetical TBA pricing at par. Spot CMM rate is calculated by interpolation:

- For each coupon C, synthesize a TBA which settles in $T = 30$ days. Its price $P_C(T)$ is defined as the linear interpolation of the two bracketing instruments.
- Synthesize a par TBA with settlement 30 calendar days from today by linearly interpolating the prices $P_C(T)$. Its coupon, expressed in terms of the bracketing coupons C_1 and C_2 (with $C_1 < C_2$),

  $$M = w_1 C_1 + w_2 C_2,$$

  is the spot CMM rate.

CMM rates and instruments

Forward CMM rates are quoted in the CMM markets (forward rate agreements and swaps).

- A CMM FRA is structured as follows:
  - The counterparties agree on the contract rate K.
  - The spot CMM rate M is fixed two business days before the start date T. The net rate $M - K$ is applied to the notional over the accrual period $[T, T_{\mathrm{pay}}]$.
  - The payment of the net amount is made on T_{pay}.
- A CMM swap is a multi-period version of a CMM FRA.

CMM rates and instruments

- The break even value of K is called the CMM rate $M_0(T)$.
- We let $\mathcal{M}_0$ denote the curve $[0, \infty) \ni T \to M_0(T)$, the CMM curve (or mortgage rate curve).

Generally, there is a good deal of liquidity in the CMM markets for T out to about a year, and it is possible to get quotes for settlements further out.

Interest Rate Process

The primary source of risk of MBSs is the interest rates risk modeled by a term structure model.

Assumption 1. *The underlying interest rates dynamics is modeled by a term structure model whose state variables are denoted by* $X_1(t), \ldots, X_k(t)$. *Underlying this dynamics is a probability space* $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathrm{P})$ *with a (multi-dimensional) Wiener process driving the interest rates dynamics.*

The model describes the evolution of the forward curve $\mathcal{C}(t)$. Its detailed specification will play no role in the following, it can be any model such as the Hull-White model, LMM, SABR/LMM, ... It is of practical importance that the model is properly specified and capable of accurate calibration to the market data.

Interest Rate Process

We assume the following about the model.

(i) The initial value $X_1(0), \ldots, X_k(0)$ of the dynamics is given by the current LIBOR forward curve $\mathcal{C}_0$. Here, $\mathcal{C}_0$ is defined in terms of a smooth (say, twice continuously differentiable) function $f_0 : \mathbb{R}_+ \to \mathbb{R}$, representing the instantaneous forward rate. The rates dynamics depends smoothly on $\mathcal{C}_0$.

(ii) The current interest rates volatility $\mathcal{V}_0$, represented as a smooth surface $\sigma_0 : \mathbb{R}_+^2 \to \mathbb{R}$, is mapped smoothly on the dynamics of the interest rates process. Ideally, in order to accurately take into account the volatility smile, one should represent $\mathcal{V}_0$ as a three dimensional object $\sigma_0 : \mathbb{R}_+^3 \to \mathbb{R}$. This mapping is such that the model prices correctly relevant benchmark interest rates options.

Prepayment Intensity Process

A key feature of MBSs is a variety of event risks embedded in the collateral. In particular, a borrower has the right to prepay the loan. This prepayment risk is modeled by a random time $\mathcal{T}$, the time to event. $\mathcal{T}$ is not necessarily a stopping time with respect to $(\mathscr{F}_t)_{t\geq 0}$. A borrower's decision to prepay is driven by factors which are partly exogenous to the interest rates.

Assumption 2. *The probability of prepayment depends only on the information up to time t,*

$$P\left(\mathcal{T} > t \,|\, \mathscr{F}_\infty\right) = P\left(\mathcal{T} > t \,|\, \mathscr{F}_t\right).$$

There exists a doubly stochastic Cox process $\lambda(t)$ such that

$$P\left(\mathcal{T} > t \,|\, \mathscr{F}_t\right) = \exp\left(-\int_0^t \lambda(s)\, ds\right).$$

Mortgage Rate Process

The intensity process $\lambda(t)$ represents the conditional prepayment probability density. The quantity $S(0,t) = \exp\left(-\int_0^t \lambda(s)\,ds\right)$ is the prepayment survival probability. Note that $S(0,t)$ itself is random.

Key among the factors affecting $\lambda(t)$ is the mortgage rate curve $\mathcal{M}(t)$.

Assumption 3. *Mortgage rate process is modeled as a diffusion defined on a suitable probability space $(\Omega_1, (\mathscr{G}_t)_{t\geq 0}, \mathsf{P}_1)$. The prepayment intensity $\lambda(t)$ depends exogenously on the mortgage rate curve at time t:*

$$\lambda(t) = \lambda(t, \mathcal{M}(t)).$$

Empirical Model of the Prepayment Intensity

Prepayment behavior shows the following characteristics:

- The intensity of prepayments λ increases, as the mortgage rate M goes down, and decreases, as it goes up.
- If C is the coupon on the existing loan, λ approaches finite saturation levels λ_0 and λ_1, as $C - M \to \infty$ and $C - M \to -\infty$, respectively.

This pattern of prepayments is fairly well captured by the logistic function (the “S-curve”) model of the prepayment intensity:

$$\lambda(t) = \lambda_0 + (\lambda_1 - \lambda_0) \frac{1}{1 + \gamma e^{\kappa(M(t) - C)}} .$$

Here, $0 < \lambda_0 < \lambda_1$, and $\gamma, \kappa > 0$.

TBA Valuation

The holder of a TBA is short an American call: the cash flows on a mortgage are uncertain because of the borrower's right to prepay. For calculation of the TBA's PV, the cash flows should be discounted by the survival probability S.

Let T denote the TBA's settlement date, and let $T_1, \ldots, T_N$ denote the payment dates. The scheduled cash flow on the date T_j is denoted by C_j, and B_j denotes the outstanding balance. The price of a TBA is

$$\begin{aligned} P(T) = \mathrm{E}\Big[& \sum\nolimits_j Z(T, T_j) \\ & \times \Big(S(T, T_{j-1})\, C_j + \big(S(T, T_{j-1}) - S(T, T_j) \big) B_j \Big) \Big]. \end{aligned}$$

Here E is the expected value under the risk neutral measure, and $Z(T, T_j)$ denotes the discount factor.

TBA Valuation

Alternatively, we can write this as

$$P(T) = \mathsf{E}\left[\sum_j Z^\pi(T, T_j)\left(C_j + \overline{\lambda}(T_{j-1}, T_j)\, B_j\right)\right],$$

where $Z^\pi(T, T_j) = Z(T, T_j)\, S(T, T_{j-1})$ is the prepaying discount factor, and

$$\begin{aligned}\overline{\lambda}(T_{j-1}, T_j) &= \frac{S(T, T_{j-1}) - S(T, T_j)}{S(T, T_{j-1})}\\ &\simeq \int_{T_{j-1}}^{T_j} \lambda(t)\, dt\end{aligned}$$

is the conditional probability of prepayment in month j, also known as the single month mortality (SMM).

TBA Valuation

The scheduled cash flow on a TBA is $C_j = p_j + i_j^{\mathrm{net}}$, where p_j is the scheduled principal repayment, and i_j^{net} is the interest less the servicing and credit spread F,

$$i_j^{\mathrm{net}} = \frac{\widehat{C} - F}{12} B_{j-1}.$$

The rate $C = \widehat{C} - F$ is the net coupon. We can thus recast the valuation formula as:

$$P(T) = \mathrm{PO}(T) + \mathrm{IO}(T).$$

TBA Valuation

- The first term on the RHS is the *principal only* (PO),

$$\mathrm{PO}\left(T\right)=\mathsf{E}\left[\sum_j Z^{\pi}\left(T,T_j\right)\left(p_j+\overline{\lambda}\left(T_{j-1},T_j\right)B_j\right)\right].$$

  Note that the cash flows consist of the scheduled and prepayed principal repayments.

- The second term is the *interest only* (IO),

$$\begin{aligned}\mathrm{IO}\left(T\right)&=\mathsf{E}\left[\sum_j Z^{\pi}\left(T_0,T_j\right)i_j^{\mathrm{net}}\right]\\&=\frac{C}{12}\,\mathsf{E}\left[\sum_j Z^{\pi}\left(T,T_j\right)B_{j-1}\right].\end{aligned}$$

CMM rate

In order to define the forward CMM rate we will make the following completeness assumption about the TBA market.

Assumption 4. *For each coupon $C > 0$ and each settlement date $T > 0$, there exists a traded TBA of that coupon and maturity.*

Since this assumption is in practice violated, TBAs for arbitrary settlements and coupons have to be created synthetically:

- For settlements not exceeding the longest traded PSA date, interpolate the coupons and settlement dates.
- For settlement dates past the longest traded PSA date, we model the TBA prices based on the currently calibrated term structure and prepayment models.

CMM rate

From the TBA valuation formula,

$$C = \frac{P_C(T) - \mathrm{PO}_C(T)}{L^{\mathrm{IO}}(T)} ,$$

where

$$L_C^{\mathrm{IO}}(T) = \frac{1}{12} \sum_j Z^{\pi}(T, T_j) B_{j-1}$$

is the IO annuity associated with the specified TBA. Let $Y(T)$ denote the coupon on the par TBA. Then

$$Y(T) = \frac{1 - \mathrm{PO}(T)}{L^{\mathrm{IO}}(T)} ,$$

where $L^{\mathrm{IO}}(T) = L^{\mathrm{IO}}_{Y(T)}(T)$ is the IO annuity associated with the par TBA.

CMM rate

As a consequence, $Y(T)$ is a tradable asset, if the IO annuity is used as a numeraire, and its dynamics is given by a martingale. The corresponding martingale measure Q^{IO} is called the IO measure.

As far as we know, no well developed market for CMM swaptions currently exists. Should such a market ever come to existence, the IO measure introduced above would be the natural martingale measure for CMM swaption valuation, very much like the swap measure is the natural martingale measure for interest rate swaption valuation. For example, the price of a CMM receiver swaption struck at K would be

$$\mathrm{RecSwpt}_T = L^{\mathrm{IO}}(T)\,\mathsf{E}^{\mathrm{Q}^{\mathrm{IO}}}\left[\max\left(S(T) - K, 0\right)\right].$$

CMM rate

The CMM rate is the contract rate K in the definition of a CMM FRA. This leads to the following definition:

$$M(T \mid T_{\text{pay}}) = \mathsf{E}^{\mathsf{Q}_{T_{\text{pay}}}}[Y(T_{\text{fix}} + \vartheta)],$$

where ϑ is the 30 day settlement delay on the underlying TBA.

- We will be assuming that the payment on the FRA is made on the start date T, and, for simplicity, we will neglect the small convexity correction coming from the fact that the rate fixes on T_{fix} rather than on T.
- We thus define the forward CMM rate for the date T as

$$M(T) = \mathsf{E}^{\mathsf{Q}_T}[Y(T + \vartheta)].$$

Continuously Compounded Mortgage

We now describe a simple empirical model of a TBA. To motivate, consider a mortgage which pays a continuous stream of cash. By $\widehat{C}$ we denote the annual coupon on the mortgage, and assume that its term is T_m years.

- The (constant) payment $dm(t)$ in $[t, t + dt]$ is

$$dm(t) = \frac{\widehat{C}\, dt}{1 - e^{-\widehat{C} T_m}} .$$

- The outstanding balance at time t is

$$B(t) = \frac{1 - e^{-\widehat{C}(T_m - t)}}{1 - e^{-\widehat{C} T_m}} .$$

- The interest portion $di(t)$ of the payment $dm(t)$ is

$$di(t) = \widehat{C} B(t)\, dt .$$

Perpetual TBA

Consider now a TBA collateralized by a pool of such mortgages, and assume that :

- All cash flows are discounted on a constant interest rate r.
- The prepayment intensity is time independent, $\lambda = \lambda(r, \widehat{C})$.

The price of the TBA is given by the integral

$$P_C(T) = \int_T^{T_m} e^{-(r+\lambda)t} \left[dm(t) + (\lambda - F) B(t) \, dt \right],$$

where F denotes the servicing and credit spread. This integral is closed (albeit lengthy) form. Assuming that $T = 0$, and $T_m \to \infty$,

$$P = \frac{C + \lambda}{r + \lambda} .$$

Perpetual TBA

Note that $P_C = 1$, if and only if $r = \widehat{C} - F \equiv C$. We expand $P = P(r)$ around $r = C$,

$$P(r) \simeq 1 - D(r - C),$$

where

$$D = -\frac{d}{dr} \log P(r)\big|_{r=C}$$

is the duration. This yields:

$$D \simeq \frac{1}{r + \lambda}.$$

Explicitly, we can write the duration in the form:

$$D = D_0 + (D_1 - D_0) \frac{1}{1 + \Gamma e^{-\kappa(r-C)}}.$$

Empirical Model of a TBA

This leads us to the following model of a TBA. Let $M = M(T)$ denote the CMM rate on the TBA settlement date T. The duration of a TBA is assumed to be a function of

$$X(T) = M(T) - C.$$

We parameterize the duration of a TBA by means of a logistic function of X:

$$\mathbf{D}(X) = L + (U - L)\,\frac{1}{1 + e^{-\kappa(X-\Delta)}}\,.$$

Empirical Model of a TBA

Integrating $\mathbf{D}(X)$ and exponentiating the result yields the following shape of the TBA model price function:

$$\mathbf{P}(X) = \mu \exp\left(-\frac{(L+U)X}{2}\right)\left\{\frac{\cosh\frac{\kappa\Delta}{2}}{\cosh\frac{\kappa(X-\Delta)}{2}}\right\}^{\frac{U-L}{\kappa}} .$$

The convexity of a TBA is given by the following function:

$$\mathbf{K}(X) = \mathbf{D}(X)^2 - \frac{\kappa(U-L)}{4}\,\frac{1}{\cosh^2\frac{\kappa(X-\Delta)}{2}} .$$

Empirical Model of a TBA

The price of a TBA in the empirical model is given by

$$P_C(T) = \mathrm{P}(M(T) - C).$$

The model parameters L, U, κ and Δ and $\mu = \mathbf{P}(0)$ are determined to fit to the market.

Note:

- The key assumption behind this model is that the price of a TBA depends on the forward mortgage rate for the same settlement as that of the TBA (rather than on the entire CMM curve).
- This is somewhat unsatisfying, as one might believe that prepayment behavior depends on the entire rates outlook (i.e. forward rates) rate than the spot rate only.

Valuation under a Term Structure Model

Calls and puts struck at K and expiring at T are valued according to

$$\mathrm{Call}\left(P_C, K\right) = Z\left(T\right) \mathsf{E}^{\mathsf{Q}_T}\left[\max\left(P_C\left(T+\delta\right) - K,\ 0\right)\right],$$
$$\mathrm{Put}\left(P_C, K\right) = Z\left(T\right) \mathsf{E}^{\mathsf{Q}_T}\left[\max\left(K - P_C\left(T+\delta\right),\ 0\right)\right],$$

where δ is the time lag between the option expiration and the settlement date of the underlying TBA. Note that:

- These valuation formulas do not lead to closed form expressions.
- The expectations are evaluated by means of Monte Carlo simulations based on a term structure model, and they require substantial computational resources.

Empirical Model of Mortgage Options

Since mortgage options have short expirations, it is a common practice to value them within a single rate framework. We assume that the CMM rate follows a normal SABR process,

$$\begin{aligned} dM(t) &= \sigma(t)\, dW(t), \\ d\sigma(t) &= \alpha\sigma(t)\, dB(t), \end{aligned}$$

where $\mathsf{E}[dW(t)\, dB(t)] = \rho dt$, and $M(0) = M_0, \sigma(0) = \sigma_0$. The system has the following strong solution

$$M(t) = M_0 + \sigma_0 \int_0^t \exp\left(-\frac{\alpha^2 s}{2} + \alpha B(s)\right) dW(s),$$

which can easily and efficiently be implemented by a Monte Carlo simulation.

Empirical Model of Mortgage Options

Consider a coupon C TBA whose price process P_C is given by

$$P_C(t) = \mathbf{P}(M(t) - C),$$

where $\mathbf{P}(x)$ is the model price function. Let T denote the option expiration. Shifting from the IO measure to the forward measure Q_T yields

$$\frac{dP_C(t)}{P_C(t)} = \sigma(t) D_C(t) dW(t).$$

Under Q_T,

$$\mathsf{E}^{\mathsf{Q}_T}[P_C(t)] = P_{C,0},$$

where $P_{C,0}$ denotes here the current market price.

Empirical Model of Mortgage Options

Since

- $P_C(t)$ is given by an explicit expression,
- the Monte Carlo paths are generated from a simple CMM dynamics rather than a term structure model,

the computation is rapid and accurate. The model can easily be calibrated:

- Choose the parameter μ so that the mean of the price distribution is the market price of the underlying TBA.
- Choose the parameters σ_0, α and ρ of the SABR model to match the option premia. These parameters may show a dependence on the coupon C.

Replication Formula

The CMM rate can be replicated in terms of mortgage options. Since

$$\begin{aligned} P_C(T) < 1, &\quad \text{if } Y(T) > C, \\ P_C(T) = 1, &\quad \text{if } Y(T) = C, \\ P_C(T) > 1, &\quad \text{if } Y(T) < C, \end{aligned}$$

we can write

$$\begin{aligned} Y(T) &= \int_0^{Y(T)} dC \\ &= \int_0^{\infty} \Theta(1 - P_C(T))\, dC, \end{aligned}$$

where $\Theta(x)$ denotes the step function.

Replication Formula

From the definition of the CMM rate,

$$M_0(T) = \mathrm{E}^{\mathrm{Q}_T}\left[Y(T+\vartheta)\right],$$

we infer that

$$M_0(T) = \int_0^\infty \mathrm{E}^{\mathrm{Q}_T}\Big[\Theta\Big(1 - P_C(T+\vartheta)\Big)\Big]dC.$$

Finally, we note that the integrand in the integral above equals, up to the discount factor, to the price of a digital put on a TBA struck at par.

Replication Formula

This proves the following CMM replication formula:

Let $\mathrm{DigiPut}_T\left(P_C\left(t\right), K\right)$ denote the price of a digital put option on a TBA which expires on T and is struck at K. Then

$$M_0\left(T\right) = \frac{1}{Z\left(T\right)} \int_0^{\infty} \mathrm{DigiPut}_T\left(P_C\left(T+\vartheta\right), 1\right) dC.$$

In practice, the prices of digital puts are approximated as put spreads:

$$\begin{aligned}
&\mathrm{DigiPut}_T\left(P_C\left(T+\vartheta\right), 1\right) \\
&\qquad \approx \frac{\mathrm{Put}_T\left(P_C\left(T+\vartheta\right), 1+\epsilon\right) - \mathrm{Put}_T\left(P_C\left(T+\vartheta\right), 1-\epsilon\right)}{2\epsilon},
\end{aligned}$$

where ϵ is a strike spread.

Replication Formula

Let now $\Delta_{\text{DigiPut}_T}(P_C(t), K)$ denote the price delta of the digital put above, and let

$$w_C(T) = -\frac{1}{Z(T)}\,\Delta_{\text{DigiPut}_T}(P_C(T+\vartheta), 1)\, P_C(T+\vartheta)\, D_C(T+\vartheta).$$

Since the delta of a digital put is non-negative, $w_C(T) \geq 1$. Furthermore, as a consequence of the replication formula,

$$\int_0^\infty w_C(T)\, dC = 1.$$

The number $w_C(T)$ represents thus the *dynamic weight* of the coupon C TBA in the replication of the CMM rate.

Risk of a portfolio of MBSs

We consider a portfolio of CMOs whose current value is equal to Π. The key risks are:

- Mortgage rate risk
- Interest rates risk
- Volatility risk

The mortgage rate risk of the portfolio is its sensitivity to the CMM curve $\mathcal{M}_0$:

$$\delta_M \Pi(T) = \frac{\delta \Pi}{\delta M_0(T)} .$$

Mortgage Rate Risk

In practice, this risk is measured either by

- shifting the entire CMM curve in parallel by ε and computing the finite difference

$$\delta_M^{\|}\Pi = \frac{\Pi\left(\mathcal{M}_0 + \varepsilon\right) - \Pi\left(\mathcal{M}_0 - \varepsilon\right)}{2\varepsilon},$$

- or dividing up the curve $\mathcal{M}_0$ into segments (“buckets”) $\mathcal{M}_0^i$, and computing the corresponding partial finite difference $\delta_M^i\Pi\left(t\right)$.

Note that the latter approach amounts to:

$$\delta_M^i\Pi \approx \frac{1}{b-a}\int_a^b \delta_M\Pi\left(T\right)dT,$$

where a and b indicate the start and end points of the bucket.

Risk of a TBA

The risk profile of a TBA is particularly easy to understand within the empirical model:

$$\frac{\partial P_C(T)}{\partial M_0(T)} = D_C(T)\,P_C(T)\,,$$
$$\frac{\delta P_C(T)}{\delta f_0(t)} = D_C(T)\,P_C(T)\,\frac{\delta M_0(T)}{\delta f_0(t)}\,,$$
$$\frac{\delta P_C(T)}{\delta \sigma_0(t,\tau)} = D_C(T)\,P_C(T)\,\frac{\delta M_0(T)}{\delta \sigma_0(t,\tau)}\,.$$

In other words, within the empirical model:

- TBA's sensitivity to the mortgage rate is given by its duration.
- The sensitivities to the interest rates and their volatility are given by the TBA's duration multiplied by the relevant sensitivity of the mortgage rate.

Replicating Portfolio of TBAs

Let us compute the hedge ratios of the portfolio to each of the TBAs:

$$\begin{aligned}\frac{\delta\Pi}{\delta P_C(T)} &= \frac{\delta M_0(t)}{\delta P_C(T)}\,\frac{\delta\Pi}{\delta M_0(t)}\\ &= w_C(T)\,\delta_M\Pi(t)\,,\end{aligned}$$

where $w_C(T)$ are the weights from the replication formula. In practice, we face two issues:

(i) Only few coupons trade.

(ii) TBAs with settlements beyond few months do not trade.

Replicating Portfolio of TBAs

(i) Let $C_1, \ldots, C_m$ denote the available coupons, and calculate the hedge ratios:

$$
\begin{aligned}
w_1(T) &= \int_0^{(C_1+C_2)/2} w_C(T)\,dC, \\
w_i(T) &= \int_{(C_{i-1}+C_i)/2}^{(C_i+C_{i+1})/2} w_C(T)\,dC, \text{ for } i = 2, \ldots, m-1, \\
w_m(T) &= \int_{(C_{m-1}+C_m)/2}^{\infty} w_C(T)\,dC.
\end{aligned}
$$

(ii) The risk to CMM rates with settlement longer than the longest traded PSA is projected onto the short end of the CMM curve.

Interest Rate and Volatility Risk

Other risks are computed analogously, as sensitivities to:

- Current forward curve:

$$\delta_f \Pi\,(T) = \frac{\delta \Pi}{\delta f_0\,(T)}\,.$$

- Current volatilities:

$$\delta_\sigma \Pi\,(T, \tau) = \frac{\delta \Pi}{\delta \sigma_0\,(T, \tau)}\,.$$

Portfolio Risk Management

We hedge a portfolio of MBS in the following steps.

- Calculate the portfolio's sensitivity to the CMM rate. Using the dynamic replication methodology, compute the face values of the TBA's required to offset that risk.
- Calculate the vega risk of the combined portfolio of the MBSs and TBAs, and offset that risk by a position in swaptions and caps / floors.
- Calculate the delta risk of the combined portfolio of MBSs, TBAs, and interest rate options, and offset that risk by a position in LIBOR swaps, FRAs, and Eurodollar futures.

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