Commun. Math. Phys. 118, 1-14 (1988)

Communications in
Mathematical
Physics

Quantum K-Theory

I. The Chern Character*

Arthur Jaffe, Andrzej Lesniewski, and Konrad Osterwalder**
Harvard University, Cambridge, MA 02138, USA

Abstract. We construct a cocycle on an infinite dimensional generalization of a p-summable Fredholm module. Our framework is related to Connes' cyclic cohomology and is motivated by our work on index theory on infinite dimensional manifolds. The p-summability condition is characteristic of dimension $O(p)$. We replace this assumption by the requirement that there exists an underlying heat kernel which is trace class. Then we use the heat kernel to regularize states in dimension-independent fashion. Our cocycle may be interpreted as an infinite dimensional Chern character.

I. Introduction

This paper continues our work on index theory on infinite dimensional manifolds [8-12] and connects it with Connes' theory of cyclic cohomology [2-4]. Our aim is to formulate non-commutative differential geometry in an infinite dimensional setting. We define the notion of a quantum algebra and formulate a cohomology theory for such algebras. Our framework is similar to Connes' theory of Θ-summable Fredholm modules [4].

Non-trivial, infinite-dimensional examples of these structures arise from existence theorems for two-dimensional, supersymmetric, i.e. $\mathbb{Z}_2$-graded quantum fields. In this case, the underlying, infinite-dimensional manifold is the loop space of a finite-dimensional manifold M, namely the space ΛM of smooth maps from the circle to M,

$$\Lambda M = \{\varphi : S^1 \to M\} . \tag{I.1}$$

The field theory examples constructed so far [8–12] arise from the choices $M = \mathbb{C}$ and $M = \mathbb{R}$. They yield a Hilbert space

$$\mathscr{H} = L_2(\Lambda M) \oplus L_2(\Lambda M) ,$$

* Supported in part by the National Foundation under Grant DMS/PHY 86-45122
** Permanent address: Mathematics Department, E.T.H.-Zentrum, CH-8092 Zürich, Switzerland

defined in terms of a measure $d\mu$ on the space of functions on ΛM. In these examples, there is a Dirac operator Q on $\mathscr{H}$, whose Atiyah-Singer index we have computed in [8–12]. Another example of these structures arising from representation theory is given in [3].

In order to find further invariants, we follow the methods of Connes. However, his main analytic tool, the p-summability assumption, cannot be used in the infinite dimensional context. Motivated by our field theory models, we propose to replace p-summability by *heat kernel regularization.* That means that we set

$$H = Q^2,$$

and use the trace class operator $\exp(-\beta H)$ to regularize the states we study. This allows us to construct a coboundary operator and a character form which is a cocycle, even though our algebras are not p-summable Fredholm modules. An alternative proposal along these lines was given in [4].

We define a quantum algebra $\mathscr{A}$ as a $\mathbb{Z}_2$-graded algebra of operators on a Hilbert space, with certain additional structure, including a graded derivation d.

Our cocycle τ^β has an $n = 2k^{\text{th}}$ component $\tau_n^\beta(a_0, \ldots, a_n)$ of the general form,

$$(-\beta)^{-n/2} \int_{0 \leqq t_1 \leqq \ldots \leqq t_n \leqq \beta} \operatorname{Str}(a_0 da_1(t_1) da_2(t_2) \ldots da_n(t_n) e^{-\beta H}) dt; \tag{I.2}$$

see Sect. 5 for a complete definition. The cohomology classes defined by τ^β are independent of β.

We plan to use this cocycle to investigate other invariants of the theory. Heat kernel regularization allows us to work with functional integral representations of the cocycle, as we already did in the discussion of the index of Q [8–12].

II. Quantum Algebras

Quantum algebras arise as a mathematical abstraction of graded (supersymmetric) quantum field theory.

Definition II.1. A quantum algebra is a quadruple $(\mathscr{A}, \mathscr{H}, \Gamma, Q)$ with the following structures (i)–(v):

(i) *The algebra $\mathscr{A}$ acts on $\mathscr{H}$.* The space $\mathscr{H}$ is a separable Hilbert space over $\mathbb{C}$. The algebra $\mathscr{A}$ acts as a unital algebra of bounded linear operators on $\mathscr{H}$.

(ii) *Γ is a $\mathbb{Z}_2$ grading of $\mathscr{A}$, of $\mathscr{L}(\mathscr{H})$, and of $\mathscr{H}$.* The operator Γ is a selfadjoint, unitary operator on $\mathscr{H}$. Thus Γ is a $\mathbb{Z}_2$ grading operator for $\mathscr{H}$, and

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-, \tag{II.1}$$

where $\mathscr{H}_\pm$ are the positive and negative eigenspaces of Γ. Furthermore Γ defines a $\mathbb{Z}_2$ grading of $\mathscr{L}(\mathscr{H})$, the algebra of bounded operators on $\mathscr{H}$, namely

$$\mathscr{L}(\mathscr{H}) = \mathscr{L}(\mathscr{H})_+ \oplus \mathscr{L}(\mathscr{H})_-, \tag{II.2}$$

where $\mathscr{L}(\mathscr{H})_+$ are the diagonal entries of $\mathscr{L}(\mathscr{H})$ with respect to the decomposition (II.1). This is equivalent to writing $A = A_+ + A_-$, where

$$A_\pm = \tfrac{1}{2}(A \pm A^\Gamma), \quad \text{and} \quad A^\Gamma \equiv \Gamma A \Gamma. \tag{II.3}$$

Thus $\mathscr{L}(\mathscr{H})_\pm$ are respectively even or odd under conjugation by Γ. We define $\deg(A_+)=0$, $\deg(A_-)=1$, so

$$A_\pm^\Gamma=(-1)^{\deg(A_\pm)}A_\pm\,.$$

We require that $\mathscr{A}\subset\mathscr{L}(\mathscr{H})$ is invariant under conjugation by Γ, namely $\mathscr{A}^\Gamma=\Gamma\mathscr{A}\Gamma\subset\mathscr{A}$. Thus

$$\mathscr{A}=\mathscr{A}_+\oplus\mathscr{A}_-\,.$$

(iii) *Q is selfadjoint and odd.* The operator Q is a selfadjoint transformation on $\mathscr{H}$ and

$$Q^\Gamma=-Q\,. \tag{II.4}$$

Thus

$$Q=\begin{pmatrix}0 & Q_-\\ Q_+ & 0\end{pmatrix}, \tag{II.5}$$

where $Q_\pm:\mathscr{H}_\pm\to\mathscr{H}_\mp$ and $Q_-=Q_+^*$. Furthermore we define

$$H=Q^2=\begin{pmatrix}Q_+^*Q_+ & 0\\ 0 & Q_-^*Q_-\end{pmatrix}. \tag{II.6}$$

In our examples, Q is a Dirac operator and H is a Hamiltonian (or a Laplacian).

(iv) *The heat kernel is trace class.* We assume that for all $\beta>0$, $\exp(-\beta H)$ is trace class.

(v) *$\mathscr{A}$ has a graded derivation d.* We assume that each operator in $Q\mathscr{A}$ is densely defined on $\mathscr{H}$. Thus we can define

$$da=[iQ,a_+]+\{iQ,a_-\}\,, \tag{II.7}$$

where $[a,b]=ab-ba$ and $\{a,b\}=ab+ba$. We assume that

$$d:\mathscr{A}\to\mathscr{A}\,, \tag{II.8}$$

or in other words

$$d:\mathscr{A}_\pm\to\mathscr{A}_\mp\,. \tag{II.9}$$

It follows that

$$da=i\Gamma[\Gamma Q,a]\,. \tag{II.10}$$

We define the norm

$$\|a\|_*=\|a\|+\|da\|\,,\qquad a\in\mathscr{A}\,, \tag{II.11}$$

where $\|\cdot\|$ is the operator norm. $\mathscr{A}$ is a normed algebra with respect to $\|\cdot\|_*$,

$$\|a_1a_2\|_*\leqq\|a_1\|_*\|a_2\|_*\,. \tag{II.12}$$

For $a\in\mathscr{A}$ note that

$$a=(a^\Gamma)^\Gamma\,, \tag{II.13}$$

$$(da)^\Gamma=-d(a^\Gamma)\,, \tag{II.14}$$

and for $a, b \in \mathscr{A}$,

$$d(ab) = (da)b + a^{\Gamma}(db). \tag{II.15}$$

Thus we also obtain

$$d(a_1 a_2 \dots a_n) = \sum_{j=1}^{n} (a_1 \dots a_{j-1})^{\Gamma} da_j (a_{j+1} \dots a_n). \tag{II.16}$$

As a special case,

$$d(a_1 da_2 \dots da_n) = da_1 da_2 \dots da_n + \sum_{j=2}^{n} (-1)^j (a_1^{\Gamma} \dots da_{j-1}^{\Gamma}) d^2 a_j (da_{j+1} \dots da_n). \tag{II.17}$$

For $a \in \mathscr{A}$ and $0 \leqq t$, define

$$a(t) \equiv e^{-tH} a e^{tH}. \tag{II.18}$$

Clearly $a(t)$ is an (unbounded) operator with domain Range($\exp(-tH)$). Hence $Qa(t)$ and $a(t)Q$ are densely defined and we set

$$d(a(t)) = e^{-tH} da e^{tH} = (da)(t).$$

Proposition II.2. *For $a \in \mathscr{A}$ and $t \geqq 0$,*

$$d^2 a(t) = \frac{d}{dt} a(t). \tag{II.19}$$

Proof. By assumption $d: \mathscr{A} \to \mathscr{A}$, so $d^2 a(t)$ is well defined. Furthermore from (II.9) and the identities

$$\{iQ, [iQ, a]\} = -[Q^2, a] = [iQ, \{iQ, a\}],$$

we obtain (II.19).

Let I_p denote the p-th Schatten class of bounded operators on $\mathscr{H}$ with the norm

$$\|A\|_p = (\mathrm{Tr}(A^* A)^{p/2})^{1/p}. \tag{II.20}$$

For $A \in I_1$, we define the supertrace by

$$\mathrm{Str}(A) = \mathrm{Tr}(\Gamma A). \tag{II.21}$$

Definition II.3. Let $A \in \mathscr{L}(\mathscr{H})$ and $\varepsilon > 0$, then $A^{(\varepsilon)} \equiv e^{-\varepsilon H} A e^{-\varepsilon H}$ is the heat kernel regularization of A.

Proposition II.4. *The heat kernel regularization of $A \in \mathscr{L}(\mathscr{H})$ is trace class: $A^{(\varepsilon)} \in I_1$, for $\varepsilon > 0$.*

Proof. Since $e^{-\varepsilon H}$ is assumed to be trace class for all $\varepsilon > 0$, the desired fact follows from Hölder's inequality on I_1.

Proposition II.5. *Let $A = B^{(\varepsilon)}$ be the heat kernel regularization of $B \in \mathscr{L}(\mathscr{H})$. Then*

$$\mathrm{Str}(dA) = 0. \tag{II.22}$$

Proof. With our assumptions, both QA and AQ are bounded. Also by (II.10),

$$\mathrm{Str}(dA) = i\,\mathrm{Tr}([\Gamma Q, A]) = i\,\mathrm{Tr}(\Gamma Q A - A \Gamma Q). \tag{II.23}$$

Let $R=e^{-\varepsilon H/2}$. Then using the fact that $\mathrm{Tr}(CD)=\mathrm{Tr}(DC)$ for C bounded and $D\in I_1$, we have

$$\mathrm{Tr}(\Gamma QA)=\mathrm{Tr}((\Gamma QR)RBR^2)=\mathrm{Tr}(RBR^2\Gamma QR)=\mathrm{Tr}(R^2BR^2\Gamma Q)=\mathrm{Tr}(A\Gamma Q).$$

Thus (II.23) vanishes and the proof is complete.

III. Cohomologies of Quantum Algebras

In this section we summarize cohomology theory of quantum algebras. Our framework is based on the work of Connes [2–4] and also Loday and Quillen [13].

Let $\mathscr{A}$ denote a quantum algebra, and let $\mathscr{C}^n(\mathscr{A})$ denote the space of $(n+1)$-linear functionals on A, continuous with respect to the norm $\|\cdot\|_*$ of (II.11). We define a norm $\|\cdot\|_*$ on $\mathscr{C}^n(\mathscr{A})$ by

$$\|f_n\|_*=\sup_{\|a_j\|_*=1}|f_n(a_0,a_1,\dots,a_n)|. \tag{III.1}$$

Let

$$f=(f_0,f_1,f_2,\dots) \tag{III.2}$$

denote a sequence of $f_n\in\mathscr{C}^n(\mathscr{A})$. We use the following definition of Connes [4]:

Definition III.1. The space $\mathscr{C}(\mathscr{A})$ of entire cochains consists of sequences (III.2) such that

$$\|f\|_z=\sum_{n\geqq 0}(n!)^{1/2}\|f_n\|_*z^n \tag{III.3}$$

is an entire function of z. For z positive, $\|f\|_z$ defines a norm on $\mathscr{C}(\mathscr{A})$.

The grading Γ of $\mathscr{A}$ lifts to a grading Γ of $\mathscr{C}(\mathscr{A})$. The action of Γ on $\mathscr{C}^n(\mathscr{A})$ is defined by

$$(\Gamma f_n)(a_0,\dots,a_n)=f_n(a_0^\Gamma,\dots,a_n^\Gamma). \tag{III.4}$$

Also $\Gamma f\equiv(\Gamma f_0,\Gamma f_1,\dots)$. Thus

$$\mathscr{C}(\mathscr{A})=\mathscr{C}_+(\mathscr{A})\oplus\mathscr{C}_-(\mathscr{A}),$$

where

$$\mathscr{C}_\pm(\mathscr{A})=\tfrac{1}{2}(\mathbf{1}\pm\Gamma)\mathscr{C}(\mathscr{A}) \tag{III.5}$$

are the even and odd cochains under Γ.

We also decompose $\mathscr{C}(\mathscr{A})$ according to whether n is odd or even. Write

$$\mathscr{C}(\mathscr{A})=\mathscr{C}^e(\mathscr{A})\oplus\mathscr{C}^o(\mathscr{A}), \tag{III.6}$$

where

$$(f_0,f_2,f_4,\dots)\in\mathscr{C}^e(\mathscr{A}),\qquad(f_1,f_3,f_5,\dots)\in\mathscr{C}^o(\mathscr{A}). \tag{III.7}$$

Clearly,

$$\mathscr{C}^e(\mathscr{A})=\mathscr{C}^e_+(\mathscr{A})\oplus\mathscr{C}^e_-(\mathscr{A}), \tag{III.8}$$

$$\mathscr{C}^o(\mathscr{A})=\mathscr{C}^o_+(\mathscr{A})\oplus\mathscr{C}^o_-(\mathscr{A}). \tag{III.9}$$

In this paper we are concerned with

$$\mathscr{C}_+(\mathscr{A}) = \mathscr{C}^e_+(\mathscr{A}) \oplus \mathscr{C}^o_+(\mathscr{A}).$$

An operator $S: \mathscr{C}_+(\mathscr{A}) \to \mathscr{C}_+(\mathscr{A})$ of the form

$$(Sf)_n = S_n f_n,$$

where S_n acts on $\mathscr{C}^n(\mathscr{A})$ is called diagonal on $\mathscr{C}_+(\mathscr{A})$. Let T denote the diagonal operator on $\mathscr{C}_+(\mathscr{A})$ defined by

$$(T_n f_n)(a_0, a_1, \ldots, a_n) = (-1)^n f(a_n^\Gamma, a_0, \ldots, a_{n-1}). \tag{III.10}$$

Note that $T_n^{n+1} = \mathbf{1}$ on $\mathscr{C}^n_+(\mathscr{A})$, so (III.10) defines an action of the cyclic group on $\mathscr{C}^n_+(\mathscr{A})$. Also define the diagonal operator N by

$$N_n = \sum_{j=0}^{n} T_n^j. \tag{III.11}$$

Note that

$$(T_n - \mathbf{1}) N_n = 0. \tag{III.12}$$

We now define two coboundary operators

$$B_n : \mathscr{C}^n_+(\mathscr{A}) \to \mathscr{C}^{n-1}_+(\mathscr{A}),$$

$$b_n : \mathscr{C}^n_+(\mathscr{A}) \to \mathscr{C}^{n+1}_+(\mathscr{A}),$$

which anticommute.

To simplify later calculations we first introduce two operators U and V on $\mathscr{C}_+(\mathscr{A})$ by setting

$$(Uf_n)(a_0, \ldots, a_{n-1}) = (-1)^n f_n(a_0, a_1, \ldots, a_{n-1}, \mathbf{1}), \tag{III.13}$$

$$(Vf_n)(a_0, \ldots, a_{n+1}) = (-1)^{n+1} f_n(a_{n+1}^\Gamma a_0, a_1, \ldots, a_n). \tag{III.14}$$

Then we set

$$B_n = N_{n-1} U (T_n - \mathbf{1}), \tag{III.15}$$

$$b_n = \sum_{j=0}^{n+1} T_{n+1}^{-(j+1)} V T_n^j. \tag{III.16}$$

Notice that $\{B_n\}$ and $\{b_n\}$ induce continuous homomorphisms B and b on $\mathscr{C}_+(\mathscr{A})$ which map $\mathscr{C}^e_+(\mathscr{A})$ into $\mathscr{C}^0_+(\mathscr{A})$ and *vice versa.*

More explicitly B_n may be defined by

$$B_n = N_{n-1} B_n^0,$$

where

$$(B_n^0 f_n)(a_0, a_1, \ldots, a_{n-1}) = f_n(\mathbf{1}, a_0, \ldots, a_{n-1}) + (-1)^{n-1} f_n(a_0, \ldots, a_{n-1}, \mathbf{1}).$$

Alternatively

$$\begin{aligned}(B_n f_n)(a_0, a_1, \ldots, a_{n-1}) = \sum_{j=0}^{n-1} (-1)^{(n-1)j} (&f_n(\mathbf{1}, a_{n-j}^\Gamma, \ldots, a_{n-1}^\Gamma, a_0, \ldots, a_{n-j-1}) \\ &+ (-1)^{n-1} f_n(a_{n-j}^\Gamma, \ldots, a_{n-j-1}, \mathbf{1})).\end{aligned} \tag{III.17}$$

Similarly b_n is given by

$$(b_n f_n)(a_0, a_1, \ldots, a_{n+1}) = \sum_{j=0}^{n} (-1)^j f_n(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1})$$
$$+(-1)^{n+1} f_n(a_{n+1}^{\Gamma} a_0, a_1, \ldots, a_n). \tag{III.18}$$

Proposition III.1. *The operators B and b, as defined by* (III.15–16) *are coboundary operators, i.e.*

$$B^2 = 0, \tag{III.19}$$

$$b^2 = 0, \tag{III.20}$$

which anticommute

$$\{B, b\} = 0. \tag{III.21}$$

Proof. a) Equation (III.19) follows from

$$B_n B_{n+1} = N_{n-1} U(T_n - \mathbf{1}) N_n U(T_{n+1} - \mathbf{1}),$$

and from Eq. (III.12): $(T_n - \mathbf{1})N_n = 0$.

b) To verify (III.20) we use

Lemma 1. *For* $0 \leqq k \leqq n+1$,

$$V T_{n+1}^{-(k+1)} V T_n^k = -T_{n+2}^{-(k+1)} V T_{n+1}^k V.$$

This follows readily from (III.10, 13–14).

Now we get

$$b_{n+1} b_n = \sum_{r=0}^{n+2} \sum_{s=0}^{n+1} T^{-r-1} V T^{r-s-1} V T^s$$
$$= \left(\sum_{r=1}^{n+2} \sum_{s=0}^{r-1} + \sum_{r=0}^{n+1} \sum_{s=r}^{n+1} \right) T^{-r-1} V T^{r-s-1} V T^s. \tag{III.22}$$

Using Lemma 1, we write the first double sum on the right-hand side of this equation as

$$-\sum_{r=1}^{n+2} \sum_{s=0}^{r-1} T^{-s-1} V T^{s-r} V T^{r-1} = -\sum_{s=0}^{n+1} \sum_{r=0}^{s} T^{-r-1} V T^{r-s-1} V T^s$$
$$= -\sum_{r=0}^{n+1} \sum_{s=r}^{n+1} T^{-r-1} V T^{r-s-1} V T^s,$$

which is up to the sign equal to the second double sum in (III.22). Therefore, $b_{n+1} b_n = 0$, which proves (III.20).

c) To prove (III.21) we use

Lemma 2. (i) $UV = \mathbf{1}$.

(ii) $T_n^{-1} U T_{n+1} V = -\mathbf{1}$.

(iii) $T_n^r U T_{n+1}^{-r} V = -V T_{n-1}^{r-1} U T_n^{-r+1}$, *for* $1 \leqq r \leqq n$.

From (III.15–16) we get

$$B_{n+1}b_n+b_{n-1}B_n=\sum_{j=0}^{n}\sum_{k=0}^{n+1}(T_n^jUT_{n+1}^{-k}VT_n^k-T_n^jUT_{n+1}^{-k-1}VT_n^k)$$
$$+\sum_{j=0}^{n}\sum_{k=0}^{n-1}(T_n^{-j-1}VT_{n-1}^{j+k}UT_n-T_n^{-j-1}VT_{n-1}^{j+k}U). \quad \text{(III.23)}$$

In the second double sum we change the order of V and U, using (iii) of Lemma 2. Reordering terms and renaming the summation variables we then show that all the terms in (III.23) cancel, if we use the fact that $T_n^{n+1}=\mathbf{1}$. This proves (III.21) and the proof of Proposition III.1 is complete.

Now we set

$$\partial=b+B. \quad \text{(III.24)}$$

Note that

$$(\partial f)_n(a_0,a_1,\dots,a_n)=(b_{n-1}f_{n-1})(a_0,a_1,\dots,a_n)+(B_{n+1}f_{n+1})(a_0,a_1,\dots,a_n). \quad \text{(III.25)}$$

Relations (III.19)–(III.21) imply that $\partial^2=0$, i.e., ∂ is a coboundary operator.

Definition III.2. We call the cohomology of the complex

$$\dots\to\mathscr{C}_+^e(\mathscr{A})\xrightarrow{\partial}\mathscr{C}_+^0(\mathscr{A})\xrightarrow{\partial}\mathscr{C}_+^e(\mathscr{A})\to\dots \quad \text{(III.26)}$$

the entire cyclic cohomology of the quantum algebra $\mathscr{A}$. The corresponding cohomology groups are denoted by $H_+^e(\mathscr{A})$ and $H_+^0(\mathscr{A})$.

IV. The Regularized Trace Form

In this section we study a regularized trace form which is multilinear on the algebra $\mathscr{L}(\mathscr{H})^{n+1}$. The regularization of bounded operators is provided by heat kernel regularization as described in Chap. II.

Definition IV.1. Given $\beta>0$ and $n\in\mathbb{Z}_+$, we denote by F_n^β or $\langle\cdot\rangle=\langle\cdot\rangle_{\beta,n}$ the regularized trace form on $\mathscr{L}(\mathscr{H})^{n+1}$,

$$F_n^\beta(A_0,\dots,A_n)=\langle A_0,\dots,A_n\rangle_{\beta,n}=\int_{\sigma_n}\mathrm{Str}(A_0A_1(t_1)\dots A_n(t_n)e^{-\beta H})dt. \quad \text{(IV.1)}$$

Here $\sigma_n=\sigma_n^\beta$ denotes the n-simplex

$$\sigma_n=\{(t_1,\dots,t_n):0\leqq t_1\leqq t_2\leqq\dots\leqq\beta\} \quad \text{(IV.2)}$$

and

$$A(t)=e^{-tH}Ae^{tH}. \quad \text{(IV.3)}$$

Proposition IV.2. *The regularized trace form is a continuous multilinear functional and satisfies the bound*

$$|F_n^\beta(A_0,\dots,A_n)|=|\langle A_0,\dots,A_n\rangle_{\beta,n}|\leqq\frac{1}{n!}\beta^n\,\mathrm{Tr}(e^{-\beta H})\prod_{j=0}^{n}\|A_j\|. \quad \text{(IV.3)}$$

Proof. Define $t_0 \equiv 0$, $t_{n+1} \equiv \beta$, and $\delta_j = (t_j - t_{j-1})/\beta$ for $j = 1, 2, \ldots, n+1$. For $t \in \sigma_n$, $\sum \delta_j = 1$. Thus we can use Hölder's inequality on the integrand in (IV.1), namely on the $2n+3$ factors in the I_1 Schatten class

$$\Gamma A_0 A_1(t_1) \ldots A_n(t_n) e^{-\beta H} = \Gamma A_0 e^{-\beta\delta_1 H} A_1 e^{-\beta\delta_2 H} \ldots A_n e^{-\beta\delta_{n+1} H}. \tag{IV.4}$$

Choose the I_∞ Schatten class norm [the operator norm on $\mathscr{L}(\mathscr{H})$] for Γ and A_j, and choose the $I_{\delta_j^{-1}}$ norm for $\exp(-\beta\delta_j H)$, or the L_∞ norm if $\delta_j = 0$. Since $\|\Gamma\| = 1$ and

$$\|e^{-\beta\delta H}\|_{\delta^{-1}} = (\mathrm{Tr}(e^{-\beta H}))^\delta,$$

it follows that for $t \in \sigma_n$,

$$|\mathrm{Str}(A_0 A_1(t_1) \ldots A_n(t_n))| \leqq \mathrm{Tr}(e^{-\beta H}) \prod_{j=0}^{n+1} \|A_j\|. \tag{IV.5}$$

The volume of σ_n is $\beta^n/n!$, yielding the desired bound.

Proposition IV.3. *The regularized trace form satisfies*

$$F_n^\beta(A_0, \ldots, A_n) = F_n^\beta(A_n^\Gamma, A_0, \ldots, A_{n-1}) = F_n^\beta(A_0^\Gamma, \ldots, A_n^\Gamma). \tag{IV.6}$$

Thus with T defined in (III.10), $(TF^\beta)_n = (-1)^n F_n^\beta$.

Proof. Use the fact that σ_n is parameterized by $\delta_j \geqq 0$, $\sum \delta_j = 1$; hence σ_n is invariant under cyclic permutation of the δ_j's. The desired identity follows from

$$\begin{aligned}\mathrm{Str}(A_0 A_1(t_1) \ldots A_n(t_n)) &= \mathrm{Tr}(\Gamma A_0 e^{-\beta\delta_1 H} A_1 \ldots A_n e^{-\beta\delta_{n+1} H}) \\ &= \mathrm{Tr}(\Gamma A_n^\Gamma e^{-\beta\delta_{n+1} H} A_0 e^{-\beta\delta_1 H} A_1 \ldots A_{n-1} e^{-\beta\delta_n H}).\end{aligned} \tag{IV.7}$$

Proposition IV.4 (Integration by parts formula). *Assume that $A_i \in \mathscr{L}(\mathscr{H})$ for $i = 0, \ldots, n+1$. Then*

(i) *If $\dot{A}_j \in \mathscr{L}(\mathscr{H})$, then*

$$\begin{aligned}F_{n+1}^\beta(A_0, \ldots, A_{j-1}, \dot{A}_j, A_{j+1}, \ldots, A_{n+1}) &= F_n^\beta(A_0, \ldots, A_{j-1}, A_j A_{j+1}, \ldots, A_{n+1}) \\ &\quad - F_n^\beta(A_0, \ldots, A_{j-1} A_j, A_{j+1}, \ldots, A_{n+1})\end{aligned} \tag{IV.8}$$

for $j = 1, \ldots, n$.

(ii) *If $\dot{A}_0 \in \mathscr{L}(\mathscr{H})$, then*

$$F_{n+1}^\beta(\dot{A}_0, A_1, A_2, \ldots, A_{n+1}) = F_n^\beta(A_0 A_1, A_2, \ldots, A_{n+1}) - F_n^\beta(A_1, \ldots, A_{n+1} A_0^\Gamma). \tag{IV.9}$$

(iii) *If $\dot{A}_{n+1} \in \mathscr{L}(\mathscr{H})$, then*

$$F_{n+1}^\beta(A_0, \ldots, A_n, \dot{A}_{n+1}) = F_n^\beta(A_{n+1}^\Gamma A_0, A_1, \ldots, A_n) - F_n^\beta(A_0, A_1, \ldots, A_n A_{n+1}). \tag{IV.10}$$

Proof. We prove (IV.8) for $j = 1$; then the other desired identities follow from the cyclicity property (IV.6) of the trace form F_n^β.

Consider the integration over t_1 in

$$\begin{aligned}\int_0^{t_2} &\operatorname{Str}(A_0\dot{A}_1(t_1)A_2(t_2)\dots A_{n+1}(t_{n+1})e^{-\beta H})dt_1\\ &=\operatorname{Str}(A_0A_1(t_2)A_2(t_2)\dots)-\operatorname{Str}(A_0A_1(0)A_2(t_2)\dots)\\ &=\operatorname{Str}(A_0(A_1A_2)(t_2)\dots)-\operatorname{Str}((A_0A_1)A_2(t_2)\dots).\end{aligned}$$

Here we have used the fact that $A_1(0)=A_1$. If we now integrate over $t_2,\dots,t_n\in\sigma_{n-1}$, we obtain (IV.8) with $j=1$.

Proposition IV.5. *For $A_0, A_1, \dots, A_n$, and $dA_0, \dots, dA_n \in \mathscr{L}(\mathscr{H})$,*

$$\int_{\sigma_n} \operatorname{Str}(d(A_0A_1(t_1)\dots A_n(t_n))e^{-\beta H})dt=0, \tag{IV.11}$$

and

$$F_n^\beta(dA_0, dA_1, \dots, dA_n)=-\sum_{j=1}^{n} F_n^\beta(A_0^\Gamma, (dA_1)^\Gamma, \dots, (dA_{j-1})^\Gamma, \dot{A}_j, dA_{j+1}, \dots, dA_n). \tag{IV.12}$$

Proof. Let $A=A_0A_1(t_1)\dots A_n(t_n)e^{-\beta H}$. We first assume that $t_n<\beta$. Thus by Proposition II.5, $\operatorname{Str}(dA)=0$. Since Q and Γ commute with $\exp(-\beta H)$, it follows that $dA=d(A_0A_1(t_1)\dots A_n(t_n))e^{-\beta H}$. The integrand dA, as in the proof of Proposition IV.2, is continuous in t also at the boundary. Hence the proof of (IV.11) is complete. Equation (IV.12) follows from (IV.11) and (II.16, 19).

Proposition IV.6. *For $A_0, \dots, A_{n-1} \in \mathscr{L}(\mathscr{H})$,*

$$\sum_{j=0}^{n-1} F_n^\beta(A_0, \dots, A_j, \mathbf{1}, A_{j+1}, \dots, A_{n-1})=\beta F_{n-1}^\beta(A_0, \dots, A_{n-1}). \tag{IV.13}$$

Proof. Notice that $F_n^\beta(A_0, \dots, A_n)$ may be written as

$$\int_{\xi_i\geqq 0} \operatorname{Str}(A_0e^{-\xi_1 H}A_1e^{-\xi_2 H}\dots A_ne^{-\xi_{n+1}H})\delta\left(\sum_{i=1}^{n+1}\xi_i-\beta\right)d\xi.$$

Hence

$$\begin{aligned}F_n^\beta(A_0, \dots, A_j, \mathbf{1}, A_{j+1}, \dots, A_{n-1})=\int_{\xi_i\geqq 0} &\operatorname{Str}(A_0e^{-\xi_1 H}A_1\dots A_je^{-(\xi_j+\xi_{j+1})H}\\ &\times A_{j+1}\dots A_{n-1}e^{-\xi_{n+1}H})\delta\left(\sum_{i=1}^{n+1}\xi_i-\beta\right)d\xi.\end{aligned}$$

Now we change variables:

$$\begin{aligned}\xi_i&=\xi_i' \qquad \text{for} \quad 1\leqq i\leqq j-1,\\ \xi_j+\xi_{j+1}&=\xi_j',\\ \xi_i&=\xi_{i-1}', \quad \text{for} \quad j+2\leqq i\leqq n+1,\\ \xi_j&=\eta.\end{aligned}$$

The range of integration of η is the interval $[0, \xi'_j]$, while all the ξ'_i are only restricted by $\xi'_i \geqq 0$. Nothing in the integral depends on η, thus integration over η yields a factor ξ'_j. We thus find that

$$F_n^\beta(A_0, \ldots, A_j, \mathbf{1}, A_{j+1}, \ldots, A_{n-1})$$
$$= \int_{\xi'_i \geqq 0} \mathrm{Str}(A_0 e^{-\xi'_1 H} A_1 e^{-\xi'_2 H} \ldots A_{n-1} e^{-\xi'_n H}) \xi'_j \delta\left(\sum_{i=1}^n \xi'_1 - \beta\right) d\xi .$$

Summing over j and using $\sum_{j=1}^n \xi'_j = \beta$ in the integral, we obtain the desired result.

V. The Character Form

In this section we construct a cocycle $\tau = \tau^\beta \in \mathscr{C}_+^e(\mathscr{A})$. We show that τ is closed with respect to the coboundary operator ∂, and therefore it defines a cohomology class in $H_+^e(\mathscr{A})$. We call τ the character form.

Definition V.1. For $a_0, \ldots, a_{2k} \in \mathscr{A}$, let

$$\tau_{2k}^\beta(a_0, \ldots, a_{2k}) = (-\beta)^{-k} F_{2k}^\beta(a_0, da_1^\Gamma, da_2, da_3^\Gamma, \ldots, da_{2k}), \tag{V.1}$$

where F_n^β is the regularized trace form (IV.1).

Equivalently,

$$\tau_{2k}^\beta(a_0, \ldots, a_{2k}) = (-\beta)^{-k} \int_{\sigma_{2k}} \mathrm{Tr}(\Gamma a_0 \Gamma da_1(t_1) \Gamma da_2(t_2) \ldots \Gamma da_{2k}(t_{2k}) e^{-\beta H}) dt . \tag{V.2}$$

Also let

$$\tau^\beta = (\tau_0^\beta, \tau_2^\beta, \tau_4^\beta, \ldots) . \tag{V.3}$$

Proposition V.2. *For* $r \geqq 0$, *the norm* (III.3) *satisfies*

$$\|\tau^\beta\|_r \leqq \exp(r^2 \beta) \,\mathrm{Tr}(e^{-\beta H}) . \tag{V.4}$$

Proof. We use Proposition IV.2. Thus the norm (III.1) can be estimated for τ_{2k}^β from

$$|\tau_{2k}^\beta(a_0, \ldots, a_{2k})| \leqq \frac{1}{(2k)!} \beta^k \,\mathrm{Tr}(e^{-\beta H}) \,\|a_0\| \prod_{j=1}^{2k} \|da_j\|$$
$$\leqq \frac{1}{(2k)!} \beta^k \,\mathrm{Tr}(e^{-\beta H}) \prod_{j=0}^{2k} \|a_j\|_* .$$

Thus

$$\|\tau_{2k}^\beta\|_* \leqq \frac{1}{(2k)!} \beta^k \,\mathrm{Tr}(e^{-\beta H}) ,$$

from which (V.4) follows.

Proposition V.3.

$$\tau^\beta \in \mathscr{C}_+^e(\mathscr{A}) .$$

Proof. Since we have verified the entire growth condition, we need only check that τ_{2k}^{β} is even under Γ, namely $\Gamma\tau_{2k}^{\beta}=\tau_{2k}^{\beta}$. However, this follows from the representation (V.2) and cyclicity of the trace.

Theorem V.4 (The cocycle condition).

$$\partial\tau^{\beta}=(b+B)\tau^{\beta}=0\,.$$

Proof. We claim that

$$\left.\begin{matrix}-(B_{2k+2}\tau_{2k+2}^{\beta})(a_0,\ldots,a_{2k+1})\\ (b_{2k}\tau_{2k}^{\beta})(a_0,\ldots,a_{2k+1})\end{matrix}\right\}=(-\beta)^{-k}F_{2k+1}^{\beta}(da_0,da_1^{\Gamma},\ldots,da_i^{\Gamma^i},\ldots,da_{2k+1}^{\Gamma^{2k+1}})\,. \tag{V.5}$$

Verification of (V.5) *for B:* Using formula (III.17) we get for $n=2k+2$,

$$\begin{aligned}(B_n\tau_n^{\beta})(a_0,\ldots,a_{n-1})=\sum_{j=0}^{n-1}(-1)^j(\tau_n^{\beta}(\mathbf{1},a_{n-j}^{\Gamma},\ldots,a_{n-1}^{\Gamma},a_0,\ldots,a_{n-j-1})\\ -\tau_n^{\beta}(a_{n-j}^{\Gamma},\ldots,a_{n-1}^{\Gamma},a_0,\ldots,a_{n-j-1},\mathbf{1}))\,.\end{aligned}$$

Substituting (V.1) and using $d\mathbf{1}=0$ as well as Proposition IV.3, we obtain

$$\begin{aligned}&(B_n\tau_n^{\beta})(a_0,\ldots,a_{n-1})\\ &=(-\beta)^{-(k+1)}\sum_{j=0}^{n-1}F_n^{\beta}(da_0,da_1^{\Gamma},\ldots,da_{n-j-1}^{\Gamma^{n-j-1}},\mathbf{1},da_{n-j}^{\Gamma^{n-j}},\ldots,da_{n-1}^{\Gamma^{n-1}})\,.\end{aligned}$$

Notice that the factors of (-1) have been used to bring $da_0,\ldots,da_{n-j-1}$ in front, as $(da)^{\Gamma}=-da^{\Gamma}$. Now it follows from Proposition IV.6 that

$$(B_n\tau_n^{\beta})(a_0,\ldots,a_{n-1})=-(-\beta)^{-k}F_{n-1}^{\beta}(da_0,da_1^{\Gamma},\ldots,da_{n-1}^{\Gamma^{n-1}})\,,$$

as claimed in (V.5), $n=2k+2$.

Verification of (V.5) *for b:* We set $n=2k$ and calculate the terms in the sum of (III.18).

i) For $j=0$,

$$\tau_n^{\beta}(a_0a_1,a_2,\ldots,a_{n+1})=(-\beta)^{-k}F_n^{\beta}(a_0a_1,da_2^{\Gamma},da_3^{\Gamma^2},\ldots,da_{n+1}^{\Gamma^n})\,.$$

ii) For $j=1$,

$$\begin{aligned}\tau_n^{\beta}(a_0,a_1a_2,a_3,\ldots)&=(-\beta)^{-k}F_n^{\beta}(a_0,d(a_1^{\Gamma}a_2^{\Gamma}),da_3^{\Gamma^2},\ldots)\\ &=(-\beta)^{-k}F_n^{\beta}(a_0,da_1^{\Gamma}\cdot a_2^{\Gamma},da_3^{\Gamma^2},\ldots)+(-\beta)^{-k}F_n^{\beta}(a_0,a_1da_2^{\Gamma},da_3^{\Gamma^2},\ldots)\\ &=(-\beta)^{-k}F_n^{\beta}(a_0,da_1^{\Gamma}\cdot a_2^{\Gamma},da_3^{\Gamma^2},\ldots)+(-\beta)^{-k}F_n^{\beta}(a_0a_1,da_2^{\Gamma},da_3^{\Gamma^2},\ldots)\\ &\quad+(-\beta)^{-k}F_{n+1}^{\beta}(a_0,\dot{a}_1,da_2^{\Gamma},da_3^{\Gamma^2},\ldots)\,,\end{aligned}$$

where we have used (IV.8).

iii) For $1<j\leqq n$,

$$\begin{aligned}
&\tau_n^\beta(a_0,\ldots,a_ja_{j+1},\ldots,a_{n+1})\\
&\quad=(-\beta)^{-k}F_n^\beta(a_0,da_1^\Gamma,\ldots,d(a_ja_{j+1})^{\Gamma^{j+1}},\ldots,da_{n+1}^{\Gamma^n})\\
&\quad=(-\beta)^{-k}F_n^\beta(a_0,da_1^\Gamma,\ldots,da_j^{\Gamma^j}\cdot a_{j+1}^{\Gamma^j},\ldots,da_{n+1}^{\Gamma^n})\\
&\qquad+(-\beta)^{-k}F_n^\beta(a_0,da_1^\Gamma,\ldots,a_j^{\Gamma^{j+1}}da_{j+1}^{\Gamma^j},\ldots,da_{n+1}^{\Gamma^n})\\
&\quad=(-\beta)^{-k}F_n^\beta(a_0,da_1^\Gamma,\ldots,da_j^{\Gamma^j}\cdot a_{j+1}^{\Gamma^j},da_{j+2}^{\Gamma^{j+1}},\ldots,da_{n+1}^{\Gamma^n})\\
&\qquad+(-\beta)^{-k}F_n^\beta(a_0,da_1^\Gamma,\ldots,da_{j-1}^{\Gamma^{j-1}}\cdot a_j^{\Gamma^{j-1}},da_{j+1}^{\Gamma^j},\ldots,da_{n+1}^{\Gamma^n})\\
&\qquad+(-\beta)^{-k}F_{n+1}^\beta(a_0,da_1^\Gamma,\ldots,da_{j-1}^{\Gamma^{j-1}},\dot a_j^{\Gamma^{j-1}},da_{j+1}^{\Gamma^j},\ldots,da_{n+1}^{\Gamma^n}).
\end{aligned}$$

iv) For $j=n+1$,

$$\tau_n^\beta(a_{n+1}^\Gamma a_0,a_1,\ldots,a_{n+1})=(-\beta)^{-k}F_n^\beta(a_{n+1}^\Gamma a_0,da_1^\Gamma,\ldots,da_n^{\Gamma^n}).$$

Multiplying the terms in (i)–(iv) by $(-1)^j$, using (III.18), and summing yields

$$\begin{aligned}
(b_n\tau_n^\beta)(a_0,a_1,\ldots,a_{n+1})&=(-\beta)^{-k}\sum_{j=1}^{n}(-1)^j\\
&\quad\times F_{n+1}^\beta(a_0,da_1^\Gamma,\ldots,da_{j-1}^{\Gamma^{j-1}},\dot a_j^{\Gamma^{j-1}},da_{j+1}^{\Gamma^j},\ldots,da_{n+1}^{\Gamma^n})\\
&\quad+(-\beta)^{-k}(F_n^\beta(a_0,da_1^\Gamma,\ldots,da_n^{\Gamma^n}\cdot a_{n+1}^{\Gamma_n})\\
&\quad-F_n^\beta(a_{n+1}^\Gamma a_0,da_1^\Gamma,\ldots,da_n^{\Gamma^n}))\\
&=(-\beta)^{-k}\sum_{j=1}^{n+1}(-1)^j\\
&\quad\times F_{n+1}^\beta(a_0,da_1^\Gamma,\ldots,da_{j-1}^{\Gamma^{j-1}},\dot a_j^{\Gamma^{j-1}},da_{j+1}^{\Gamma^j},\ldots,da_{n+1}^{\Gamma^n}).
\end{aligned}$$

Using (IV.12), (IV.6), and (II.14), we infer

$$(b_n\tau_n^\beta)(a_0,a_1,\ldots,a_{n+1})=(-\beta)^{-k}F_{n+1}^\beta(da_0,da_1^\Gamma,\ldots,da_j^{\Gamma^j},\ldots,da_{n+1}^{\Gamma^{n+1}}),$$

as claimed in (V.5). This completes the proof of Theorem V.4.

Remark 1. As explained in Sects. 2–3 and Sect. 7 of Connes [4], if a cocycle τ^β exists, then it defines a pairing between $H_+^e(\mathscr{A})$ and $K_0(\mathscr{A})$. See also [5] for elaboration of this point.

Remark 2. In fact, if $Q'=\beta^{1/2}Q$ and $d'a=i\Gamma[\Gamma Q',a]$, then for $n=2k$,

$$\tau_n^\beta(a_0,\ldots,a_n)=(-1)^k\int_{\sigma_n^1}\mathrm{Str}(a_0d'a_1^\Gamma(t_1)\ldots d'a_n^{\Gamma^n}(t_n)e^{-H'})dt. \tag{V.6}$$

Thus using $H'=\frac{1}{2i}d'Q'$, we infer that

$$\begin{aligned}
\beta\frac{d}{d\beta}\tau_n^\beta(a_0,\ldots,a_n)&=\frac{n}{2}\tau_n^\beta(a_0,\ldots,a_n)\\
&\quad+(-1)^k\frac{i}{2}\sum_{j=0}^{n}F_{n+1}^1(a_0,d'a_1^\Gamma,\ldots,d'a_j^{\Gamma^j},d'Q',d'a_{j+1}^{\Gamma^{j+1}},\ldots,d'a_n^{\Gamma^n}),
\end{aligned}\tag{V.7}$$

where the first term arises from differentiating $d'a_j$, while the second one arises from differentiating $e^{-H'}$. Define the cochain

$$G^{\beta}_{2k-1}(a_0, \dots, a_{2k-1}) = -\frac{i}{2} \sum_{j=0}^{2k-1} (-1)^{j+k} F^{1}_{2k}(a_0, d'a_1^{\Gamma}, \dots, d'a_j^{\Gamma^j}, Q', d'a_{j+1}^{\Gamma^{j+2}}, \dots, d'a_{2k-1}^{\Gamma^{2k}}) . \qquad (V.8)$$

We compute ∂G^{β}_n as above and obtain from (V.7) that

$$\beta \left(\frac{d\tau^{\beta}}{d\beta} \right) = \partial G^{\beta} . \qquad (V.9)$$

Hence the cohomology classes defined by τ^{β}_n are independent of β.

Acknowledgements. We wish to thank Alain Connes and Ping Feng for comments and helpful discussions.

References

1. Blackadar, B.: K-Theory for operator algebras. Berlin, Heidelberg, New York: Springer 1986
2. Connes, A.: Noncommutative differential geometry. Publ. Math. IHES **62**, 257–360 (1986)
3. Connes, A.: Compact metric spaces, Fredholm modules and hyperfiniteness (preprint)
4. Connes, A.: Entire cyclic cohomology of Banach algebras and characters of θ-summable Fredholm modules (preprint)
5. Getzler, E., Szenes, A.: Preprint
6. Helton, J.W., Howe, R.: Traces of commutators of integral operators. Acta Math. **135**, 271–305 (1975)
7. Hochschild, G., Kostant, B., Rosenberg, A.: Differential forms on regular affine algebras. Trans. Am. Math. Soc. **102**, 383–408 (1962)
8. Jaffe, A., Lesniewski, A.: A priori estimates for $N=2$ Wess-Zumino models on a cylinder. Commun. Math. Phys. **114**, 553–576 (1988)
9. Jaffe, A., Lesniewski, A., Lewenstein, M.: Ground state structure in supersymmetric quantum mechanics. Ann. Phys. **178**, 313–329 (1987)
10. Jaffe, A., Lesniewski, A., Weitsman, J.: Index of a family of Dirac operators on loop space. Commun. Math. Phys. **112**, 75–88 (1987)
11. Jaffe, A., Lesniewski, A., Weitsman, J.: The two-dimensional, $N=2$ Wess Zumino model on a cylinder. Commun. Math. Phys. **114**, 147–166 (1988)
12. Jaffe, A., Lesniewski, A., Weitsman, J.: The $S_1 \to \mathbb{R}$ loop space and supersymmetric quantum fields. Ann. Phys. (to appear)
13. Loday, J.L., Quillen, D.: Cyclic homology and the Lie algebra homology of matrices. Comment. Math. Helvetici **59**, 565–591 (1984)
14. Quillen, D.: Superconnections and the Chern character. Topology **24**, 89–95 (1985)
15. Taylor, J.: Homology and cohomology for topological algebras. Adv. Math. **9**, 137–182 (1972)
16. Witten, E.: Constraints on supersymmetry breaking. Nucl. Phys. B **202**, 253–316 (1982)

Communicated by A. Jaffe

Received March 28, 1988; in revised form April 19, 1988