Notes on the CEV model

Andrew Lesniewski
Ellington Management Group
53 Forest Avenue
Old Greenwich, CT 06870

First draft of March 12, 2004
This draft of November 6, 2009

1 Statement of the problem

We construct a family of Green's functions $G_X(\tau, x)$ for a forward following the CEV dynamics. Specifically, we consider the initial value problem:

$$\begin{aligned} \frac{\partial}{\partial \tau} G_X(\tau, x) &= \frac{1}{2} b(x)^2 \frac{\partial^2}{\partial x^2} G_X(\tau, x), \\ G_X(0, x) &= \delta_X(x), \end{aligned} \tag{1}$$

where $\delta_X(x) = \delta(x - X)$ denotes Dirac's delta supported at X. This is actually the terminal value problem for the backward Kolmogorov equation written in terms of the time variable $\tau = T - t$. The function $b(x)$ has the form[1]:

$$b(x) = \sigma x^\beta, \qquad \beta < 1. \tag{2}$$

We impose the natural boundary condition:

$$G_X(\tau, x) \to 0, \quad \text{as } x \to \infty. \tag{3}$$

In addition, at $x = 0$, we impose the following family of boundary conditions (the Robin problem [2]):

$$\frac{\partial}{\partial x} G_X(\tau, x) + \mu G_X(\tau, x) \Big|_{x=0} = 0. \tag{4}$$

This reduces to the reflecting (Neumann) problem for $\mu = 0$, and the absorbing (Dirichlet) problem for $\mu \to \infty$. It is these two boundary value problems that we consider in this manuscript.

Taking the Laplace transform of $G_X(\tau, x)$,

$$G_X(\tau, x) = \int_0^\infty e^{-\lambda\tau} g(\lambda, x)\, d\lambda, \tag{5}$$

[1]Peter Carr informed me that the restriction $\beta \geq 0$ is not necessary

we find that

$$\frac{1}{2}\,\sigma^2 x^{2\beta} g'' + \lambda g = 0. \tag{6}$$

Simple algebra shows that g can be expressed as

$$g\left(\lambda, x\right) = \sqrt{x}\; u\left(\frac{2\nu\sqrt{2\lambda}}{\sigma}\; x^{\frac{1}{2\nu}}\right), \tag{7}$$

where

$$\nu = \frac{1}{2\left(1-\beta\right)}\;, \quad \text{i.e. } \nu > 0, \tag{8}$$

and where $u\left(z\right)$ satisfies Bessel's equation [4]:

$$u'' + \frac{1}{z}\; u' + \left(1 - \frac{\nu^2}{z^2}\right) u = 0. \tag{9}$$

Now (see e.g. [5]), any solution to Bessel's equation is a linear combination of Bessel's functions $J_\nu\left(z\right)$ and $Y_\nu\left(z\right)$ of the first and second kind, respectively:

$$u\left(z\right) = A_\nu J_\nu\left(z\right) + B_\nu Y_\nu\left(z\right). \tag{10}$$

Recall (we refer to [4] for details) that $J_\nu\left(z\right)$ is defined by the series

$$J_\nu\left(z\right) = \sum_{k\geq 0} \frac{\left(-1\right)^k}{k!\,\Gamma\left(\nu+k+1\right)} \left(\frac{z}{2}\right)^{2k+\nu}\;, \tag{11}$$

and

$$Y_\nu\left(z\right) = \frac{J_\nu\left(z\right)\cos\left(\nu\pi\right) - J_{-\nu}\left(z\right)}{\sin\left(\nu\pi\right)}\;. \tag{12}$$

We shall determine the constants A_ν and B_ν so that all the conditions imposed on the solution are satisfied. Using (11) and (12) as well as the well known properties of the gamma function we observe that, as $x \to 0$,

$$\begin{aligned} g\left(\lambda, x\right) \sim\; & \frac{1}{\Gamma\left(\nu+1\right)} \left(\frac{\nu\sqrt{2\lambda}}{\sigma}\right)^{\nu} \big(A_\nu\left(\lambda\right) + \cot\left(\nu\pi\right) B_\nu\left(\lambda\right)\big) x \\ & - B_\nu\left(\lambda\right)\; \frac{\Gamma\left(\nu\right)}{\pi} \left(\frac{\sigma}{\nu\sqrt{2\lambda}}\right)^{\nu} \left(1 + \frac{2\nu\lambda}{\sigma^2}\; x^{\frac{1}{\nu}}\right). \end{aligned} \tag{13}$$

2 Dirichlet boundary condition

Let us start with the Dirichlet problem. Since from (13),

$$g\left(\lambda, 0\right) = -B_\nu\left(\lambda\right)\; \frac{\Gamma\left(\nu\right)}{\pi} \left(\frac{\sigma}{\nu\sqrt{2\lambda}}\right)^{\nu}\;, \tag{14}$$

we infer that *the Dirichlet problem has a solution for all values of* ν. Furthermore, we have $B_\nu(\lambda) = 0$, and thus

$$g(\lambda, x) = A_\nu(\lambda)\sqrt{x}\, J_\nu\left(\frac{2\nu\sqrt{2\lambda}}{\sigma}\, x^{\frac{1}{2\nu}}\right), \tag{15}$$

and we need to determine $A_\nu(\lambda)$.

In order to do this recall first that the Hankel transform (see e.g. [1]) $\mathcal{H}_\nu f$ of a function f is given by

$$(\mathcal{H}_\nu f)(p) = \int_0^\infty f(x)\, J_\nu(px)\, x\, dx. \tag{16}$$

Its key property is that

$$f(x) = \int_0^\infty (\mathcal{H}_\nu f)(p)\, J_\nu(px)\, p\, dp. \tag{17}$$

In particular, note that

$$(\mathcal{H}_\nu \delta_X)(p) = X J_\nu(pX). \tag{18}$$

As a consequence, we get the following expression for $A_\nu(\lambda)$:

$$A_\nu(\lambda) = \frac{2\nu}{\sigma^2}\, X^{\frac{1}{\nu}-\frac{3}{2}}\, J_\nu\left(\frac{2\nu\sqrt{2\lambda}}{\sigma}\, X^{\frac{1}{2\nu}}\right). \tag{19}$$

Using the identity [4]:

$$\int_0^\infty e^{-\tau p^2} J_\nu(ap)\, J_\nu(bp)\, p\, dp = \frac{1}{2\tau}\, \exp\left(-\frac{a^2+b^2}{4\tau}\right) I_\nu\left(\frac{ab}{2\tau}\right), \tag{20}$$

we finally obtain the following explicit representation for the Dirichlet Green's function $G_X^D(\tau, x)$:

$$\begin{aligned} G_X^D(\tau, x) &= \frac{\left(xX^{1-4\beta}\right)^{1/2}}{(1-\beta)\,\sigma^2\tau} \\ &\quad \times \exp\left(-\frac{x^{2(1-\beta)} + X^{2(1-\beta)}}{2(1-\beta)^2\,\sigma^2\tau}\right) I_\nu\left(\frac{(xX)^{1-\beta}}{(1-\beta)^2\,\sigma^2\tau}\right). \end{aligned} \tag{21}$$

Recall that the non-central χ^2 distribution with r degrees of freedom and the non-centrality parameter λ is given by the following probability density distribution:

$$p(x; r, \lambda) = \frac{1}{2}\left(\frac{x}{\lambda}\right)^{(r-2)/4} \exp\left(-\frac{x+\lambda}{2}\right) I_{(r-2)/2}\left(\sqrt{\lambda x}\right), \tag{22}$$

and thus the Dirichlet Green's function can be written as

$$G_X^D(\tau, x) = \frac{4\nu x X^{1/\nu-2}}{\sigma^2\tau}\, p\left(\frac{4\nu^2 X^{1/\nu}}{\sigma^2\tau};\ 2\nu+2,\ \frac{4\nu^2 x^{1/\nu}}{\sigma^2\tau}\right). \tag{23}$$

Note that the total mass of $G_X^D(\tau, x)$ is indeed less than one, meaning that there is a nonzero probability of absorption at zero. Using the series expansion [4]:

$$I_\nu(z) = \sum_{k\geq 0} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k+\nu}, \tag{24}$$

we readily find that

$$\int_0^\infty G_X^D(\tau, x)\, dX = 1 - \frac{1}{\Gamma(\nu)}\, \Gamma\left(\nu,\, \frac{2\nu^2 x^{1/\nu}}{\sigma^2\tau}\right), \tag{25}$$

where

$$\Gamma(\nu,\, x) = \int_x^\infty t^{\nu-1} e^{-t} dt \tag{26}$$

is the complementary incomplete gamma function [5]. The quantity

$$\frac{1}{\Gamma(\nu)}\, \Gamma\left(\nu,\, \frac{2\nu^2 x^{1/\nu}}{\sigma^2\tau}\right) \tag{27}$$

is the probability of absorption at zero. For example, in the square root process case, i.e. $\nu = 1$, that probability equals $\exp\left(-\frac{2x}{\sigma^2\tau}\right)$.

We shall now follow [3] in order to express the option pricing function in terms of the cumulative noncentral χ^2 distribution function:

$$\chi^2(x; r, \lambda) = \int_0^x p(y; r, \lambda)\, dy. \tag{28}$$

The pricing function of a call struck at c is given by the integral

$$\begin{aligned} &\int_0^\infty \max(X-c, 0)\, G_X^D(\tau, x)\, dX \\ &\qquad = \int_c^\infty X G_X^D(\tau, x)\, dX - c\int_c^\infty G_X^D(\tau, x)\, dX. \end{aligned} \tag{29}$$

The first term on the right hand side of (29) is easy to calculate: after substituting

$$z = \frac{4\nu^2 X^{1/\nu}}{\sigma^2\tau} \tag{30}$$

we find that it evaluates to

$$x\left(1 - \chi^2\left(\frac{4\nu^2 c^{1/\nu}}{\sigma^2\tau};\ 2\nu+2, \frac{4\nu^2 x^{1/\nu}}{\sigma^2\tau}\right)\right). \tag{31}$$

In order to calculate the second term on the right hand side of (29), we shall first establish the following symmetry property of the noncentral χ^2 distribution:

$$\int_x^\infty p(y; r-2, \lambda)\, dy + \int_\lambda^\infty p(x; r, \mu)\, d\mu = 1. \tag{32}$$

Indeed, consider the function:

$$\varphi(\lambda) = \int_x^\infty p(y; r-2, \lambda)\, dy + \int_\lambda^\infty p(x; r, \mu)\, d\mu.$$

From (24),

$$\varphi(\lambda) \to 1, \quad \text{as } \lambda \to 0.$$

On the other hand, we verify readily that

$$\begin{aligned} \frac{\partial}{\partial x}\, p(x; r, \lambda) &= \frac{1}{2}\big(-p(x; r, \lambda) + p(x; r-2, \lambda) \big), \\ \frac{\partial}{\partial \lambda}\, p(x; r, \lambda) &= \frac{1}{2}\big(-p(x; r, \lambda) + p(x; r+2, \lambda) \big), \end{aligned}$$

and thus for all positive λ,

$$\begin{aligned} \frac{d}{d\lambda}\, \varphi(\lambda) &= \int_x^\infty \frac{\partial}{\partial \lambda}\, p(y; r-2, \lambda)\, dy - p(x; r, \lambda) \\ &= -\int_x^\infty \frac{\partial}{\partial y}\, p(y; r, \lambda)\, dy - p(x; r, \lambda) \\ &= 0. \end{aligned}$$

Consequently, $\varphi(\lambda) = 1$, as claimed.

Now, introducing the notation

$$\begin{aligned} \lambda &= \frac{4\nu^2 x^{1/\nu}}{\sigma^2 \tau}\,, \\ q &= \frac{4\nu^2 c^{1/\nu}}{\sigma^2 \tau}\,, \end{aligned}$$

and using (32), we find that

$$\begin{aligned} \int_c^\infty G_X^D(\tau, x)\; dX &= \int_q^\infty \left(\frac{\lambda}{z}\right)^\nu p(z; 2\nu+2, \lambda)\, dz \\ &= \int_q^\infty p(\lambda; 2\nu+2, z)\, dz \\ &= 1 - \int_\lambda^\infty p(z; 2\nu, q)\, dz \\ &= \int_0^\lambda p(z; 2\nu, q)\, dz. \end{aligned}$$

Putting everything together we obtain the following explicit representation for the pricing function of a call struck at c:

$$\begin{aligned} \mathcal{V}^D_{\text{call}}(\tau, x) = x &\left(1 - \chi^2\left(\frac{4\nu^2 c^{1/\nu}}{\sigma^2\tau};\; 2\nu+2, \frac{4\nu^2 x^{1/\nu}}{\sigma^2\tau}\right)\right) \\ &- c\chi^2\left(\frac{4\nu^2 x^{1/\nu}}{\sigma^2\tau};\; 2\nu, \frac{4\nu^2 c^{1/\nu}}{\sigma^2\tau}\right)\,. \end{aligned} \tag{33}$$

From the put call parity, the pricing function of a put struck at c is:

$$\begin{aligned}\mathcal{V}^{D}_{\text{put}}(\tau,x) = x\chi^2\left(\frac{4\nu^2 c^{1/\nu}}{\sigma^2\tau};\ 2\nu+2,\frac{4\nu^2 x^{1/\nu}}{\sigma^2\tau}\right)\\ - c\left(1-\chi^2\left(\frac{4\nu^2 x^{1/\nu}}{\sigma^2\tau};\ 2\nu,\frac{4\nu^2 c^{1/\nu}}{\sigma^2\tau}\right)\right)\ .\end{aligned} \tag{34}$$

3 Neumann boundary condition

Let us now turn to the Neumann problem. Since, as $x \to 0$,

$$\begin{aligned}\frac{\partial}{\partial x}\, g(\lambda,x) \sim\ \frac{1}{\Gamma(\nu+1)}\ \left(\frac{\nu\sqrt{2\lambda}}{\sigma}\right)^{\nu}\ \big(A_\nu(\lambda)+\cot(\nu\pi)\,B_\nu(\lambda)\big)\\ - B_\nu(\lambda)\ \frac{2\lambda\Gamma(\nu)}{\sigma^2\pi}\ \left(\frac{\sigma}{\nu\sqrt{2\lambda}}\right)^{\nu}\ x^{\frac{1}{\nu}-1}\ ,\end{aligned} \tag{35}$$

we conclude that *the Neumann problem has a solution if* $1/\nu - 1 \geq 0$. Equivalently, the Neumann problem has a solution if

$$\beta \geq \frac{1}{2}\ . \tag{36}$$

Assume first that $\nu < 1$. The coefficients A_ν and B_ν must then obey the relation:

$$A_\nu(\lambda) = -B_\nu(\lambda)\cot(\nu\pi)\,, \tag{37}$$

and thus, as a consequence of (12)

$$\begin{aligned}u(z) &= B_\nu\left(-\cot(\nu\pi)\,J_\nu(z)+Y_\nu(z)\right)\\ &= -B_\nu\ \frac{1}{\sin(\nu\pi)}\ J_{-\nu}(z)\,.\end{aligned} \tag{38}$$

This implies that $g(\lambda,x)$ is given by

$$g(\lambda,x) = -B_\nu(\lambda)\ \frac{1}{\sin(\nu\pi)}\ \sqrt{x}J_{-\nu}\left(\frac{2\nu\sqrt{2\lambda}}{\sigma}\ x^{\frac{1}{2\nu}}\right). \tag{39}$$

The computation follows now the same outline as in the Dirichlet case. Using the technique of Hankel transforms, we find that the Neumann Green's function $G^N_X(\tau,x)$ is given by

$$\begin{aligned}G^N_X(\tau,x) = \frac{\left(xX^{1-4\beta}\right)^{1/2}}{(1-\beta)\,\sigma^2\tau}\\ \times\exp\left(-\frac{x^{2(1-\beta)}+X^{2(1-\beta)}}{2\,(1-\beta)^2\,\sigma^2\tau}\right) I_{-\nu}\left(\frac{(xX)^{1-\beta}}{(1-\beta)^2\,\sigma^2\tau}\right)\end{aligned} \tag{40}$$

or, equivalently,

$$G_X^N(\tau, x) = \frac{4\nu X^{1/\nu - 1}}{\sigma^2 \tau}\, p\left(\frac{4\nu^2 X^{1/\nu}}{\sigma^2\tau};\ -2\nu + 2,\ \frac{4\nu^2 x^{1/\nu}}{\sigma^2 \tau}\right). \tag{41}$$

This is a *bona fide* probability distribution of X; a straightforward calculation shows that

$$\int_0^\infty G_X^N(\tau, x)\ dX = 1. \tag{42}$$

Finally, the pricing functions with Neumann boundary conditions at zero are given by

$$\begin{aligned} \mathcal{V}_{\text{call}}^N(\tau, x) = x\chi^2 &\left(\frac{4\nu^2 c^{1/\nu}}{\sigma^2\tau};\ -2\nu, \frac{4\nu^2 x^{1/\nu}}{\sigma^2\tau}\right) \\ &- c\left(1 - \chi^2\left(\frac{4\nu^2 x^{1/\nu}}{\sigma^2\tau};\ -2\nu + 2, \frac{4\nu^2 c^{1/\nu}}{\sigma^2\tau}\right)\right)\ , \end{aligned} \tag{43}$$

and

$$\begin{aligned} \mathcal{V}_{\text{put}}^N(\tau, x) = x &\left(1 - \chi^2\left(\frac{4\nu^2 c^{1/\nu}}{\sigma^2\tau};\ -2\nu, \frac{4\nu^2 x^{1/\nu}}{\sigma^2\tau}\right)\right) \\ &- c\chi^2\left(\frac{4\nu^2 x^{1/\nu}}{\sigma^2\tau};\ -2\nu + 2, \frac{4\nu^2 c^{1/\nu}}{\sigma^2\tau}\right)\ . \end{aligned} \tag{44}$$

References

[1] Davies, B.: *Integral Transforms and Their Applications*, Springer Verlag (2002).

[2] Guenther, R. B., and Fee, J. F.: *Partial Differential Equations of Mathematical Physics and Integral Equations*, Dover Publications (1996).

[3] Schroder, M.: Computing the constant elasticity of variance option pricing formula, *J. Finance*, **44**, 211 - 219 (1989).

[4] Watson, G. N.: *A Treatise on the Theory of Bessel Functions*, Cambridge University Press (1944).

[5] Whittaker, E. T., and Watson, G. N.: *A Course of Modern Analysis*, Cambridge University Press (1927).