Interest Rate Volatility

I. Volatility in fixed income markets

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Outline

Fixed income markets

- Debt and debt related markets, also known as *fixed income* markets, account for the lion share of the world's financial markets.
- Instruments trading in these markets fall into the categories of *cash instruments*, or *bonds*, and derivative instruments, such as interest rate swaps, bond futures and credit default swaps.
- Bonds are debt instruments issued by various entities (such as sovereigns, corporations, municipalities, US Government Sponsored Enterprizes, etc), with the purposes of raising funds in the capital markets.
- Derivatives are synthetic instruments which extract specific features of commonly traded cash instruments.
- Depending on their purpose, fixed income instruments may exhibit very complex risk profiles. The value of a fixed income instrument may depend on the level and term structure of interest rates, credit characteristics of the underlying entity, foreign exchange levels, or prepayment speeds of the collateral pool of loans.
- In these lectures we focus on modeling volatility of interest rate derivatives and its implications for quantifying risk inherent to these instruments.

LIBOR and OIS

- Like all other financial markets, fixed income markets fluctuate, and the main driver of this variability is the current perception of future interest rates.
- Market participants have adopted the convention that a *rally* refers to falling interest rates, while a *sell off* refers to rising rates.
- Each fixed income instrument is a stream of known or contingent *cash flows*. Such cash flows are typically periodic, and are computed as a fixed or floating *coupon*, applied to a principal (or a notional principal).
- Much of the activity in the world capital markets is tied to the LIBOR rates. They are widely used as benchmarks for short term (overnight to 1 year) interest rates.
- A LIBOR (= London Interbank Offered Rate) rate is the interest rate at which banks offer (at least in principle) unsecured deposits to each other.
- Daily *fixings* of LIBOR are published by Thompson Reuters on behalf of the British Banking Association (BBA) on each London business day.
- These fixings are calculated from the quotes provided by a panel of participating banks. The details on the composition of the panels and how the fixings are calculated can found on the web site www.bbalibor.com of the BBA.

LIBOR based instruments

- There is a variety of *vanilla* LIBOR based derivative instruments that are actively trading both on exchanges and over the counter (OTC):
  - (i) LIBOR futures,
  - (ii) forward rate agreements,
  - (iii) interest rate swaps.
- Each of these instruments serves as an underlying for an option.
- Below we summarize the mechanics of these instruments without discussing their economic significance.

Forward rate agreements

- *Forward rate agreements* (FRAs) are OTC transactions, i.e. they are arranged between the counterparties without an involvement of an exchange.
- In a FRA transaction, counterparty A agrees to pay counterparty B LIBOR settling t years from now applied to a specified notional amount (say, \$100 mm). In exchange, counterparty B pays counterparty A a pre-agreed interest rate (say, 3.05%) applied to the same notional.
- The contract matures on an anniversary T (say, 3 months) of the settlement date, and interest is computed on an act/360 day count basis. Anniversary dates generally follow the modified following business day convention.
- FRAs are quoted in terms of the annualized forward interest rate applied to the accrual period of the transaction.

Eurodollar futures

- *Eurodollar futures*, known also as the LIBOR futures, are *exchange traded* futures contracts on the 3 month LIBOR rate. They trade on the Chicago Mercantile Exchange, which also clears and settles the trades.
- In many ways, Eurodollar futures are similar to FRAs, except that their terms, such as contract sizes and settlement dates are standardized.
- Each of the contracts assumes a notional principal of $1,000,000. Interest on these contracts is computed on an act/360 day count basis assuming 90 day accrual period.
- In order to make a Eurodollar future resemble a bond, the market has adopted the convention according to which the forward rate R underlying the contract is quoted in terms of the "price" defined as

$$100 \times (1 - R).$$

- For example, if $R = 2.32\%$, the quoted price of the contract is 97.68. Unlike a FRA, the Eurodollar future quoted price is linear in the underlying rate.

Eurodollar futures

Ticker	Price	Open Int	Volume
EDZ11	99.355	992506	120246
EDF12	99.415	32443	10324
EDG12	99.385	7623	2849
EDH12	99.355	933497	184855
EDJ12	99.340	426	0
EDK12	99.325	255	0
EDM12	99.310	1063728	158419
EDU12	99.295	801723	151663
EDZ12	99.290	796494	146512
EDH13	99.300	753118	94834
EDM13	99.290	593587	77793
EDU13	99.260	478959	86254
EDZ13	99.190	544016	69813
EDH14	99.085	374708	58436
EDM14	98.930	307804	47007
EDU14	98.760	217849	43230
EDZ15	98.580	182137	36593
EDH16	98.415	137627	21723
EDM16	98.250	111012	17310
EDU16	98.085	76774	18580
EDZ16	97.905	55982	18628
EDH17	97.760	47385	14877
EDM17	97.620	37211	9331
EDU17	97.490	38411	9577

Table: 1. Snapshot of the Eurodollar futures market

Swaps

- A fixed for floating *interest rate swap* (or simply: a swap) is an OTC transaction in which two counterparties agree to exchange periodic interest payments on a pre-specified *notional* amount.
- One counterparty (the fixed *payer*) agrees to pay periodically the other counterparty (the fixed *receiver*) a fixed coupon (say, 3.35% per annum) in exchange for receiving periodic LIBOR applied to the same notional.
- *Spot starting* swaps based on LIBOR begin on a start date 2 business days from the current date and mature and pay interest on anniversary dates that use the same modified following business day conventions as the LIBOR index.
- Interest is usually computed on an act/360 day basis on the floating side of the swap and on 30/360 day basis in the fixed side of the pay. Typically, fixed payment dates ("coupon dates") are semiannual (every 6 months), and floating payment dates are quarterly (every 3 months) to correspond to a 3 month LIBOR.
- In addition to spot starting swaps, *forward starting* swaps are routinely traded.
- In a forward starting swap, the first accrual period can be any business day beyond spot. Swaps (spot and forward starting) are quoted in terms of the fixed coupon.

Swaps

- Table 2 below contains a snapshot of the swap rate market. Each rate is the break even rate on the swap of indicated tenor paying coupon semiannually on the 30/360 basis, versus receiving 3 month LIBOR.

Tenor	Rate (%)
2Y	0.690%
3Y	0.798%
4Y	1.008%
5Y	1.248%
7Y	1.690%
10Y	2.106%
12Y	2.298%
15Y	2.478%
20Y	2.599%
25Y	2.660%
30Y	2.694%

Table: 2. Snapshot of the swap market taken on 12/13/2011

OIS

- In the wake of the 2008 credit crunch, LIBOR's credibility as a funding rate was put to question.
- Part of the issue is the *de facto* absence of the interbank unsecured lending market which raises doubts over the validity of the quotes submitted by the participating banks to the BBA.
- As a result, rates referred to as the OIS rates, linked to the overnight rate controlled by the local (to the currency) central bank became increasingly important as benchmark funding rates. OIS stands for *overnight indexed swap*.
- An overnight indexed swap is a fixed for floating interest rate swap where the floating rate is based on a short term (overnight), daily compounded (rather quarterly, as in a LIBOR swap) interest rate.
- By default, both the fixed and floating legs accrue based on the act/360 basis. OIS swaps tend to be of short maturity, ranging from a few days to five years.
- In the USD market, OIS rates are calculated by reference to daily *fed funds effective rate*.

LIBOR / OIS basis

- The *LIBOR / OIS spread*, defined as the difference between the 3 month LIBOR and 3 month OIS rates, is an important indicator of stress in the capital markets.
- A wider spread is an indication of a decreased willingness to lend by major banks, while a tighter spread indicates easier availability of credit. The LIBOR / OIS spread is a gauge of market participants' view of the credit worthiness of other financial institutions and the general availability of funds for lending purposes.
- The LIBOR / OIS spread has been historically hovering around 10 basis points. However, at times of elevated credit stress, the basis between LIBOR and OIS can be quite volatile.
- In the midst of the financial crisis that started in 2007, the spread between LIBOR and OIS was wildly volatile and peeked at an all-time high of 364 basis points in October 2008, indicating a severe credit crunch.

Multi-curve paradigm and valuation of swaps

- We now turn to the problem of the valuation of non-contingent (but not necessarily known) future cash flows. The building blocks required are:
  (i) *Discount factors*, which allow one to calculate present value of cash received in the future.
  (ii) *Forward rates*, which allow one to make assumptions as to the future levels of interest rates.
- Until 2008, it was common practice to use LIBOR as both the discount rate, i.e. the interest rate used for calculating the discount factors, as well as the index rate, i.e. the rate used as the forward rate.
- Since then, in the wake of the financial crisis, the industry has been steadily moving away from this practice, and adopted the *multicurve* paradigm to swap valuation.
- Since OIS is a better indicator of the costs of funding, it is used for discounting, while LIBOR is the index rate. It remains to be seen whether the fed funds effective rate will retain this role in the USD market.

Zero coupon bonds

- A *zero coupon bond* (or *discount bond*) is a cash instrument which pays a predefined principal amount, say \$1, at a specified future date. More precisely, a zero coupon bond is characterized by two dates, the settlement date S which marks the start of the accrual period, and the maturity date $T > S$ on which the payment is made. Its value at settlement is thus the present value (abbreviated PV) of \$1 guaranteed to be paid at time T.
- In practice we are interested in the value $P(t, S, T)$ of the forward zero coupon bond for any *valuation date* $t \leq S$. It is thus the time t value of zero coupon bond (whose face value is \$1) which settles on the date S years from now and matures in T years. The forward zero coupon bond $P(t, S, T)$ is also called the (forward) *discount factor*.
- There is a useful no arbitrage relationship involving $P(t, S, T)$, namely:

$$P(t, S, T) = \frac{P(t, t, T)}{P(t, t, S)} . \tag{1}$$

- Throughout these presentations we adopt the following convention. If the valuation date is today, $t = 0$, then we denote the price of the zero coupon bond by $P_0(S, T)$, i.e.

$$P_0(S, T) \equiv P(0, S, T). \tag{2}$$

Forward rates

- The OIS forward rate $F(t, S, T)$ for start S and maturity T, as observed at time t, is defined as
$$F(t, S, T) = \frac{1}{\delta}\Big(\frac{1}{P(t, S, T)} - 1\Big), \tag{3}$$
where δ denotes the day count factor for the period $[S, T]$.
- The discount factor $P(t, S, T)$ can be expressed in terms of $F(t, S, T)$ by means of the formula:
$$P(t, S, T) = \frac{1}{1 + \delta F(t, S, T)}\ .$$
- The LIBOR forward rate for start S and maturity T, as observed at time t, is denoted by $L(t, S, T)$.
- The LIBOR / OIS spread $B(t, S, T)$ is given by
$$B(t, S, T) = L(t, S, T) - F(t, S, T). \tag{4}$$

Forward rates

- Discount factors can be expressed in terms of the continuously compounded *instantaneous forward rate* $f(t, s)$. For all practical purposes, we can think about $f(t, s)$ as the forward overnight OIS rate.
- In terms of $f(t, s)$,

$$P(t, S, T) = \exp\Big(-\int_S^T f(t, s)ds\Big). \tag{5}$$

- This equation is merely the definition of $f(t, s)$, and expresses the discount factor as the result of continuous discounting of the value of a dollar between the value and maturity dates.
- Conversely, the instantaneous forward rate can be computed from the discount factor:

$$\begin{aligned} f(t, s) &= -\frac{1}{P(t, S, T)} \frac{\partial P(t, S, T)}{\partial T}\Big|_{T=s} \\ &= -\frac{\partial}{\partial T} \log P(t, S, T)\Big|_{T=s}. \end{aligned} \tag{6}$$

Valuation of swaps

- We consider a swap which settles at $T_0 \geq 0$ and matures at T. We assume that the notional principal is \$1.
- Let $T_1^c < \ldots < T_{n_c}^c = T$ denote the coupon dates of the swap, and let $0 \leq t \leq T_0$ denote the valuation date. The PV of the interest payments on the fixed leg of a swap is calculated by adding up the PVs of all future cash flows:

$$P_0^{\text{fix}}(t) = \sum_{j=1}^{n_c} \alpha_j C P_0(t, T_j^c), \tag{7}$$

where C is the coupon rate, $P_0(t, T_j^c)$ are the discount factors to the valuation date, and α_j are the day count fractions on the fixed leg.
- For example, on a standard USD swap paying semi-annual coupon, the α's correspond to the modified following $30/360$ business day convention. It is useful to write this formula as

$$P_0^{\text{fix}}(t) = C A_0(t), \tag{8}$$

where

$$A_0(t) = \sum_{1 \leq j \leq n_c} \alpha_j P_0(t, T_j^c), \tag{9}$$

is the *annuity function* of the swap.

Valuation of swaps

- Likewise, let $T_1^{\mathrm{f}} < \ldots < T_{n_{\mathrm{f}}}^{\mathrm{f}} = T$ denote the LIBOR payment dates of the swap. The valuation formula for the swap's floating leg reads then:

$$P_0^{\text{float}}(t) = \sum_{1 \le j \le n_{\mathrm{f}}} \delta_j L_j P_0(t, T_j^{\mathrm{f}}). \tag{10}$$

- Here

$$L_j = L_0(T_{j-1}^{\mathrm{f}}, T_j^{\mathrm{f}}) \tag{11}$$

  is the LIBOR forward rate for settlement at T_{j-1}^{f}, and δ_j is the day count fraction applying to the floating leg. In the USD, the payments are quarterly, and the δ's correspond to the modified following act/360 business day convention.
- The PV of a swap is the difference between the PVs of the fixed and floating legs:

$$P_0(t) = P_0^{\text{fix}}(t) - P_0^{\text{float}}(t)\,.$$

- A break-even (or mid-market) swap has zero PV:

$$P_0^{\text{fix}}(t) = P_0^{\text{float}}(t)\,.$$

- That uniquely determines the *break-even swap rate*:

$$S_0(T_0, T) = \frac{P_0^{\text{float}}(t)}{A_0(t)}\,. \tag{12}$$

Vanilla interest rate options

- There are several types of liquidly traded options on interest rates:
  (i) *Caps / floors*: these are options on LIBOR forwards.
  (ii) *Eurodollar options*, which are options on Eurodollar futures.
  (iii) *Swaptions*, which are options on swaps
- Below we summarize some basic facts about these instruments and market conventions for their pricing.

Caps and floors

- *Caps* and *floors* are baskets of European calls (called *caplets*) and puts (called *floorlets*) on LIBOR forward rates. They trade over the counter.
- Let us consider, for example, a 10 year spot starting cap struck at 2.50%. It consists of 39 caplets each of which expires on the 3 month anniversary of today's date.
- A caplet pays

$$\max(\text{current LIBOR fixing} - 2.50\%, 0) \times \text{act/360 day count fraction.}$$

  The payment is made at the end of the 3 month period covered by the LIBOR contract, and follows the modified business day convention.
- Notice that the very first period is excluded from the cap: this is because the current LIBOR fixing is already known and no optionality is left in that period.

Caps and floors

- In addition to spot starting caps and floors, *forward starting* instruments are traded.
- For example, a 1 year $\times$ 5 years (in the market lingo: “1 by 5”) cap struck at 2.50% consists of 16 caplets struck at 2.50%, the first of which matures one year from today.
- The final maturity of the contract is 5 years, meaning that the last caplets matures 4 years and 9 months from today (with appropriate business dates adjustments).
- Unlike the case of spot starting caps, the first period is included into the structure, as the first LIBOR fixing is unknown. Note that the total maturity of the $m \times n$ cap is n years.
- The definitions of floors are similar with the understanding that a floorlet pays

$$\max(\text{strike} - \text{current LIBOR fixing},\, 0) \times \text{act/360 day count fraction}$$

at the end of the corresponding period.

Eurodollar options

- *Eurodollar options* are standardized contracts traded at the Merc. These are short dated American style calls and puts on Eurodollar futures.
- At each time, options on the first eight quarterly Eurodollar futures contracts and on two front serial futures are listed. Their expirations coincide with the maturity dates of the underlying Eurodollar contracts.
- The exchange sets the strikes for the options spaced every 25 basis points (or 12.5 bp for the front contracts). The options are cash settled.

Strike	Calls	Puts
98.875	0.5325	0.0525
99.000	0.4175	0.0625
99.125	0.3075	0.0775
99.250	0.2025	0.0975
99.375	0.1125	0.1325
99.500	0.0450	0.1900
99.625	0.0100	0.2800
99.750	0.0025	0.3975
99.875	0.0025	0.5200

Table: 3. ED options: March 2012 expirations. Price of the underlying 99.355

Eurodollar options

- In addition to the quarterly and serial contracts, a number of *midcurve* options are listed on the Merc. These are American style calls and puts with expirations between three months and one year on longer dated Eurodollar futures.
- Their expirations do not coincide with the maturity on the underlying futures contracts, which mature one, two, or four years later.
- The prices of all Eurodollar options are quoted in *ticks*.

Strike	Calls	Puts
98.875	0.5275	0.0925
99.000	0.4200	0.1100
99.125	0.3150	0.1300
99.250	0.2175	0.1575
99.375	0.1275	0.1925
99.500	0.0650	0.2250
99.625	0.0250	0.3400
99.750	0.0075	0.4475
99.875	0.0025	0.5650

Table: 4. ED options: June 2012 expirations. Price of the underlying 99.31

Swaptions

- European *swaptions* are European calls and puts (in the market lingo they are called *receivers* and *payers*, respectively) on interest rate swaps.
- A holder of a payer swaption has the right, upon exercise, to pay fixed coupon on a swap of contractually defined terms. Likewise, a holder of a receiver swaption has the right to receive fixed on a swap.
- Swaptions are traded over the counter.
- For example, a 2.50% 1Y $\to$ 5Y ("1 into 5") receiver swaption gives the holder the right to receive 2.50% on a 5 year swap starting in 1 year. More precisely, the option holder has the right to exercise the option on the 1 year anniversary of today (with the usual business day convention adjustments) in which case they enter into a receiver swap starting two business days thereafter.
- Similarly, a 3.50% 5Y $\to$ 10Y ("5 into 10") payer swaption gives the holder the right to pay 3.50% on a 10 year swap starting in 5 years.
- Note that the total maturity of the $m \to n$ swaption is $m + n$ years.
- Since a swap can be viewed as a particular basket of underlying LIBOR forwards, a swaption is an option on a basket of forwards.

Black's model

- The market quotes the prices on interest rate options is in terms of *Black's model*.
- We assume that a forward rate $F(t)$, such as a LIBOR forward or a forward swap rate (do not confuse with the OIS rate!), follows a driftless lognormal process,
$$dF(t) = \sigma F(t)\, dW(t)\,. \tag{13}$$
Here $W(t)$ is a Wiener process, and σ is the *lognormal volatility*.
- It is understood here, that we have chosen a numeraire $\mathcal{N}$ with the property that, in the units of that numeraire, $F(t)$ is a tradable asset. The process $F(t)$ is thus a martingale, and we let Q denote the corresponding measure.
- The prices of calls and puts are given by the Black-Scholes formulas:
$$\begin{aligned} P^{\text{call}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\left[F_0 N(d_+) - KN(d_-)\right], \\ &\triangleq \mathcal{N}(0)\, B^{\text{call}}(T, K, F_0, \sigma), \\ P^{\text{put}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\left[-F_0 N(-d_+) + KN(-d_-)\right] \\ &\triangleq \mathcal{N}(0)\, B^{\text{put}}(T, K, F_0, \sigma). \end{aligned} \tag{14}$$
- Here, $N(x)$ is the cumulative normal distribution, and
$$d_\pm = \Big(\log\frac{F_0}{K} \pm \frac{1}{2}\,\sigma^2 T\Big)/\sigma\sqrt{T}\,. \tag{15}$$

Valuation of caps and floors

- A cap is a basket of options on LIBOR forward rates. Consider the OIS forward rate $F(S, T)$ spanning the accrual period $[S, T]$.
- Its time $t \leq S$ value $F(t, S, T)$ can be expressed in terms of discount factors:

$$\begin{aligned} F(t, S, T) &= \frac{1}{\delta}\left(\frac{P(t, t, S)}{P(t, t, T)} - 1\right) \\ &= \frac{1}{\delta}\,\frac{P(t, t, S) - P(t, t, T)}{P(t, t, T)}\,. \end{aligned} \tag{16}$$

- The interpretation of this identity is that $F(t, S, T)$ is a tradable asset if we use the zero coupon bond maturing in T years as numeraire.
- Indeed, the trade is as follows:
  - (i) Buy $1/\delta$ face value of the zero coupon bond for maturity S.
  - (ii) Sell $1/\delta$ face value of the zero coupon bond for maturity T.
- The value of this position in the units of $P(t, t, T)$ is $F(t, S, T)$. An OIS forward rate can thus be modeled as a martingale! We call the corresponding martingale measure the *T-forward measure* and denote it by Q_T.

Deterministic basis assumption

- Consider now a LIBOR forward $L(S, T)$ spanning the same accrual period.
- *Throughout these presentations we assume that the LIBOR / OIS spread is deterministic* (rather than stochastic). This assumption is, clearly, a gross oversimplification of reality but it has some merits. There are no liquidly trading options on this spread, and thus calibrating a model with a stochastic spread is problematic.
- From the conceptual point of view, the picture is more transparent with a deterministic spread. Namely, we know from the discussion above that

$$\begin{aligned} L(t, S, T) &= F(t, S, T) + B(t, S, T) \\ &= \frac{1}{\delta} \frac{P(t, t, S) - P(t, t, T) + \delta B(t, S, T) P(t, t, T)}{P(t, t, T)} . \end{aligned} \tag{17}$$

- This shows that the LIBOR forward is a martingale under the T-forward measure Q_T.

Deterministic basis assumption

- This assumption generalizes the deterministic shift extension idea [1] to multi-curve context (as well as multi-factor modeling). In this framework, the discount rate and index rate are allowed to be different.
- We recognize the fact that fluctuations of the funding rate and index rate do not generally share the same set of risk factors [5], and the basis spread between LIBOR rate and OIS rate is, in fact, stochastic [6], [4].
- There are, however, valid reasons (in addition to simplicity) that the common set of risk factors assumption is defensible as a working assumption:
  - (i) The basis LIBOR / OIS remains positive.
  - (ii) Liquid volatility products on the basis LIBOR / OIS are not available, and making the basis spread stochastic results in non-calibrateable and non-hedgeable parameters.

Valuation of caps and floors

- Choosing, for now, the underlying process to be lognormal (given by (13)), we conclude that the price of a call on $L(S, T)$ (or caplet) is given by

$$P^{\text{caplet}}(T, K, L_0, \sigma) = \delta P_0(0, T) B^{\text{call}}(S, K, L_0, \sigma), \tag{18}$$

  where L_0 denotes here today's value of the forward, namely $L(0, S, T) = L_0(S, T)$.
- Since a cap is a basket of caplets, its value is the sum of the values of the constituent caplets:

$$P^{\text{cap}} = \sum_{j=1}^{n} \delta_j B^{\text{call}}(T_{j-1}, K, L_j, \sigma_j) P_0(0, T_j), \tag{19}$$

  where δ_j is the day count fraction applying to the accrual period starting at T_{j-1} and ending at T_j, and L_j is the LIBOR forward rate for that period.
- Notice that, in the formula above, the date T_{j-1} has to be adjusted to accurately reflect the expiration date of the option (2 business days before the start of the accrual period). Similarly, the value of a floor is

$$P^{\text{floor}} = \sum_{j=1}^{n} \delta_j B^{\text{put}}(T_{j-1}, K, L_j, \sigma_j) P_0(0, T_j). \tag{20}$$

Valuation of swaptions

- Consider a swap that settles at T_0 and matures at T. Let $S(t, T_0, T)$ denote the corresponding (break-even) forward swap rate observed at time $t < T_0$.
- We know from the discussion above that the forward swap rate is given by

$$S(t, T_0, T) = \frac{\sum_{1 \le j \le n_f} \delta_j L_j P(t, T_{\mathrm{val}}, T_j^f)}{A(t, T_{\mathrm{val}}, T_0, T)}\,, \tag{21}$$

  where $T_{\mathrm{val}} \le T_0$ is the valuation date of the swap.
- Here, B_j is the LIBOR / OIS spread, and $A(t, T_{\mathrm{val}}, T_0, T)$ is the forward annuity function:

$$A(t, T_{\mathrm{val}}, T_0, T) = \sum_{1 \le j \le n_c} \alpha_j P(t, T_{\mathrm{val}}, T_j^c). \tag{22}$$

- We can write $S(t, T_0, T)$ as

$$S(t, T_0, T) = \frac{P(t, t, T_0) - P(t, t, T) + \sum_{1 \le j \le n_{\mathrm{f}}} \delta_j B_j P(t, t, T_j^{\mathrm{f}})}{A(t, t, T_0, T)}\,. \tag{23}$$

- The forward annuity function $A(t, t, T_0, T)$ is the time t present value of an *annuity* paying \$1 on the dates $T_1^{\mathrm{c}}, \ldots, T_{n_{\mathrm{c}}}^{\mathrm{c}}$, as observed at t.

Valuation of swaptions

- The interpretation of (23) is that $S(t, T_0, T)$ is a tradable asset if we use the annuity as numeraire (recall that we are assuming that all the LIBOR / OIS spreads are deterministic).
- Indeed, consider the following trade:
  (i) Buy \$1 face value of the zero coupon bond for maturity T_0.
  (ii) Sell \$1 face value of the zero coupon bond for maturity T.
  (iii) Buy a stream of $\delta_j B_j$ face value zero coupon bonds for each maturity T_j^{f}.
- A forward swap rate can thus be modeled as a martingale! The corresponding martingale measure is called the *swap measure*.
- Choosing, again, the lognormal process (13), we conclude that today's value of a receiver and payer swaptions are given by

$$\begin{aligned} P^{\mathrm{rec}} &= A_0(T_0, T) B^{\mathrm{put}}(T_0, K, S_0, \sigma), \\ P^{\mathrm{pay}} &= A_0(T_0, T) B^{\mathrm{call}}(T_0, K, S_0, \sigma), \end{aligned} \tag{24}$$

  respectively. Here $A_0(T_0, T) = A(0, 0, T_0, T)$, i.e.

$$A_0(T_0, T) = \sum_{1 \leq j \leq n_{\mathrm{c}}} \alpha_j P_0(0, T_j^{\mathrm{c}}) \tag{25}$$

  (all discounting is done to today), and S_0 is today's value of the forward swap rate $S_0(T_0, T)$.

Beyond Black's model

- The basic premise of Black's model, that σ is independent of T, K, and F_0, is not supported by the interest rates markets.
- In fact, option implied volatilities exhibit:
  (i) *Term structure*: At the money volatility depends on the option expiration.
  (ii) *Smile* (or *skew*): For a given expiration, there is a pronounced dependence of implied volatilities on the option strike.
- These phenomena became pronounced in the mid nineties or so and, in order to accurately value and risk manage options portfolios, refinements to Black's model are necessary.
- Modeling term structure of volatility is hard, and not much progress has been made. We will discuss some empirical facts later in this presentation.
- Our main focus will be on modeling volatility smile.

Local volatility models

- A class of models extending Black's model, called *local volatility models*, are specified as follows:
$$dF(t) = C(t, F(t))dW(t)\,, \tag{26}$$
where $C(t, F)$ is a certain effective instantaneous volatility.
- The idea is that even though the exact nature of volatility (it could be stochastic) is unknown, one can, in principle, use the market prices of options in order to recover the risk neutral probability distribution of the underlying asset.
- To see this, note that
$$\begin{aligned}\frac{d}{dK}(F-K)^+ &= \frac{d}{dK}\left((F-K)\theta(F-K)\right)\\ &= -\theta(F-K) - (F-K)\delta(F-K)\\ &= -\theta(F-K),\end{aligned}$$
and thus
$$\frac{d^2}{dK^2}(F-K)^+ = \delta(F-K).$$

Local volatility models

- This implies that

$$\frac{d^2}{dK^2}\,\mathrm{E}^{\mathrm{Q}}[(F_T - K)^+] = \mathrm{E}^{\mathrm{Q}}[\delta(F_T - K)].$$

- Let $g_T(F, F_0)$ denote the terminal probability distribution function of the forward swap rate S. From the equality above we infer that

$$\begin{aligned}\frac{d^2}{dK^2}\,\mathrm{E}^{\mathrm{Q}}[(F_T - K)^+] &= \int \delta(F - K) g_T(F, F_0) dF.\\ &= g_T(K, F_0).\end{aligned}$$

- Consequently, the terminal probability distribution can (in principle) be computed from the option prices [3].
- This, in turn, will allow us to find an effective ("local") specification $C(t, F)$ of the underlying process so that the implied volatilities match the market implied volatilities.
- Empirical studies (see e.g. [2]) show that depending on the level of rates, their dynamics may be more akin to that of a lognormal model (very low or very high rates) or a normal model (intermediate range).

Normal model

- The dynamics for the forward rate $F(t)$ in the normal model reads

$$dF(t) = \sigma dW(t), \tag{27}$$

under a suitable choice of numeraire. The parameter σ is appropriately called the *normal volatility*.

- Prices of European calls and puts are now given by:

$$\begin{aligned} P^{\text{call}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\,\sigma\sqrt{T}\big(d_+N(d_+) + N'(d_+)\big),\\ &\triangleq \mathcal{N}(0)\,B_{\text{n}}^{\text{call}}(T, K, F_0, \sigma),\\ P^{\text{put}}(T, K, F_0, \sigma) &= \mathcal{N}(0)\,\sigma\sqrt{T}\Big(d_-N(d_-) + N'(d_-)\Big)\\ &\triangleq \mathcal{N}(0)\,B_{\text{n}}^{\text{put}}(T, K, F_0, \sigma), \end{aligned} \tag{28}$$

where

$$d_\pm = \pm\frac{F_0 - K}{\sigma\sqrt{T}}. \tag{29}$$

- The normal model is (in addition to the lognormal model) an important benchmark in terms of which implied volatilities are quoted.

CEV model

- The *CEV model* (for "constant elasticity of variance"), interpolates smoothly between the normal and lognormal models:

$$dF(t) = \sigma F(t)^{\beta}\, dW(t)\,, \tag{30}$$

  where $\beta < 1$.
- In order for the dynamics to be well defined, we have to prevent $F(t)$ from becoming negative (otherwise $F(t)^{\beta}$ would turn imaginary!).
- To this end, we specify a boundary condition at $F = 0$. It can be
  (i) *Dirichlet* (absorbing): $F|_0 = 0$. Solution exists for all values of β, or
  (ii) *Neumann* (reflecting): $F'|_0 = 0$. Solution exists for $\frac{1}{2} \leq \beta < 1$.
- The CEV model requires solving a terminal value problem for a partial differential equation, namely the backward Kolmogorov equation:

$$\begin{gathered}\frac{\partial}{\partial t}\, B(t,x) + \frac{1}{2}\,\sigma^2 x^{2\beta}\,\frac{\partial^2}{\partial x^2}\, B(t,x) = 0,\\ B(T,x) = \begin{cases}(x-K)^+, & \text{for a call,}\\ (K-x)^+, & \text{for a put,}\end{cases}\end{gathered} \tag{31}$$

  This equation has to be supplemented by a boundary condition at zero x.
- Pricing formulas for the CEV model can be obtained in a closed (albeit somewhat complicated) form.

Swaption volatilities

- Table 5 contains the December 13, 2011 snapshot of the at the money swaption market. The rows in the matrix represent the swaption expiration and the columns represent the tenor of the underlying swap. Each entry in the table represents the premium of a swaption straddle expressed as a percentage of the notional on the underlying swap.

	1Y	2Y	3Y	4Y	5Y	7Y	10Y	15Y	20Y	30Y
1M	0.06%	0.11%	0.18%	0.27%	0.37%	0.67%	1.10%	1.70%	2.17%	2.94%
3M	0.10%	0.20%	0.31%	0.48%	0.68%	1.18%	1.91%	2.90%	3.69%	5.02%
6M	0.14%	0.30%	0.47%	0.74%	1.04%	1.73%	2.71%	4.06%	5.17%	6.97%
1Y	0.21%	0.35%	0.75%	1.16%	1.60%	2.51%	3.82%	5.56%	7.05%	9.45%
2Y	0.40%	0.85%	1.37%	1.94%	2.55%	3.66%	5.26%	7.38%	9.23%	12.20%
3Y	0.62%	1.26%	1.91%	2.58%	3.25%	4.50%	6.26%	8.61%	10.64%	13.77%
4Y	0.78%	1.54%	2.28%	3.02%	3.75%	5.11%	7.00%	9.52%	11.66%	15.11%
5Y	0.88%	1.74%	2.56%	3.35%	4.13%	5.58%	7.57%	10.21%	12.49%	16.15%
7Y	0.97%	1.90%	2.78%	3.63%	4.44%	5.97%	8.09%	10.81%	13.16%	16.86%
10Y	1.01%	1.96%	2.86%	3.71%	4.53%	6.08%	8.22%	10.86%	13.12%	16.71%

Table: 5. ATM swaption prices

Swaption volatilities

- Table 6 contains the implied normal volatilities corresponding to the swaption prices in Table 5.

	1Y	2Y	3Y	4Y	5Y	7Y	10Y	15Y	20Y	30Y
1M	45	46	52	59	66	86	103	112	117	120
3M	50	48	53	61	70	88	104	112	114	120
6M	49	52	56	65	75	92	105	111	115	118
1Y	51	56	65	74	83	96	106	109	113	114
2Y	71	77	84	90	95	101	105	105	107	107
3Y	93	95	99	100	102	104	105	103	105	101
4Y	101	103	106	104	104	105	105	102	102	99
5Y	106	106	109	106	106	105	105	101	101	98
7Y	103	105	106	103	102	101	101	96	95	91
10Y	101	101	100	97	96	95	93	89	88	83

Table: 6. Swaption ATM normal volatilities (in basis points)

Empirical dynamics of the ATM swaption volatilities

- Swaption volatilities exhibit a pronounced term structure both in their dependence on the expiration as well as in their dependence on the underlying tenor.
- This term structure has a persistent dynamics whose characteristics withstand market regime switches.
- We analyse the dynamics of the ATM swaption volatility matrix using recent market data, from January 01, 2003 through September 16, 2014. This time window includes the 2007 / 2008 financial crisis, as well as the pre-crisis and post-crisis periods. We include swaptions of maturities of 1M, 3M, 6M, 1Y, 2Y, 3Y, 4Y, 5Y, 7Y and 10Y, and underlying tenors ranging from 1Y to 30Y.
- Our presentation is based on the work of Shi [7].

Term structure of ATM swaption volatilities

- The term structure of ATM swaption volatilities can be well captured by the "hump function":
$$\sigma(t) = (at + b)e^{-\lambda t} + \mu. \tag{32}$$
- The parameters of this function have clear intuitive meaning:
  (i) μ is the volatility at long expiry, as $\sigma(\infty) = \mu$;
  (ii) $b + \mu$ is the instantaneous volatility level, since $\sigma(0) = b + \mu$;
  (iii) a is the slope at short expiries;
  (iv) λ is the rate of exponential decay for long expiries.
- The hump function captures well the volatility regimes over the period January 2003 through September 2014, as illustrated by the graphs below.
- On these graphs, the black circles indicate market data, and the blue line is the fitted hump function.

Term structure of ATM swaption volatilities

- Figure 1 shows the shapes of the ATM swaption volatilities during the pre-crisis period. Note the hump in the short end (around 6M - 2Y).

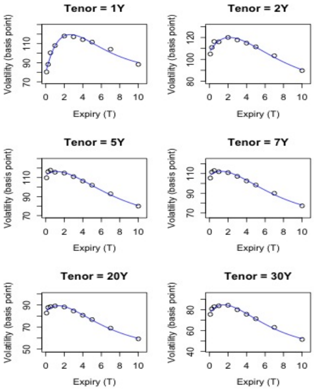

Figure: 1. Term structure ATM swaption volatilities on September 16, 2004.

Term structure of ATM swaption volatilities

- Figure 2 shows the shapes of the ATM swaption volatilities during the crisis period. Note the absence of the hump in the short end.

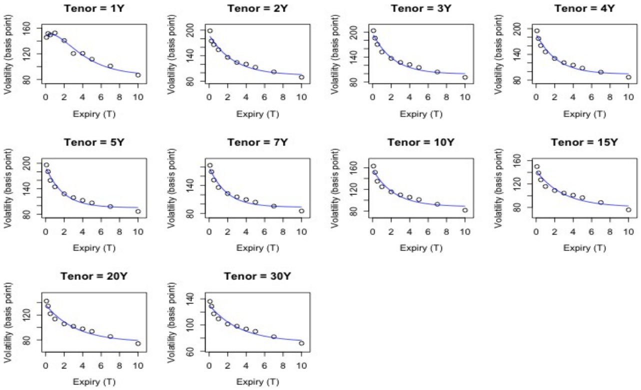

Figure: 2. Term structure ATM swaption volatilities on September 16, 2008.

Term structure of ATM swaption volatilities

- Figure 3 shows the shapes of the ATM swaption volatilities during the post-crisis period. Note that the hump has reappeared.

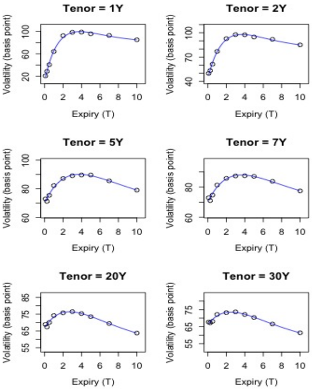

Figure: 3. Term structure ATM swaption volatilities on September 16, 2014.

Term structure of ATM swaption volatilities

- In the three graphs below we show the term structure of the parameters a, b, λ, μ as functions of the underlying tenor T.
- Each of the term structures can be fitted to a power function.
- We illustrate the dependence of the parameters on T separately for each of the market regimes: pre-crisis, crisis, and post-crisis.
- The values of the parameters are indicated by a green circle while the blue line is the fitted power function.

Term structure ATM swaption volatilities

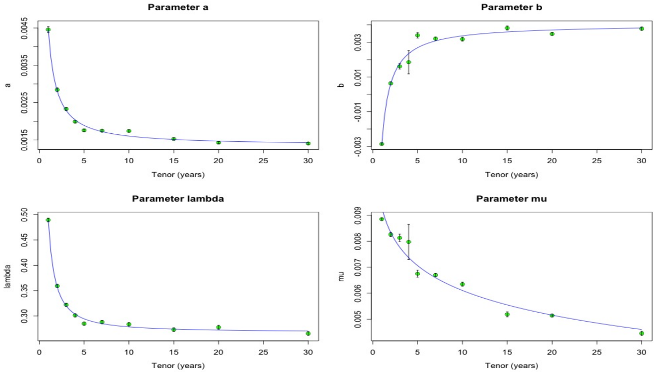

Figure: 4. Tenor dependence of the hump function parameters during the pre-crisis period.

Term structure ATM swaption volatilities

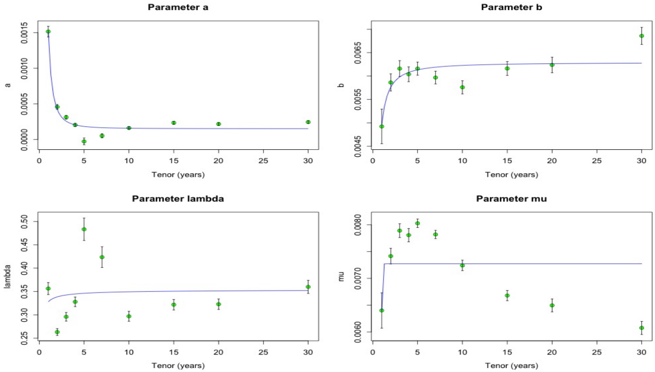

Figure: 5. Tenor dependence of the hump function parameters during the crisis period.

Term structure ATM swaption volatilities

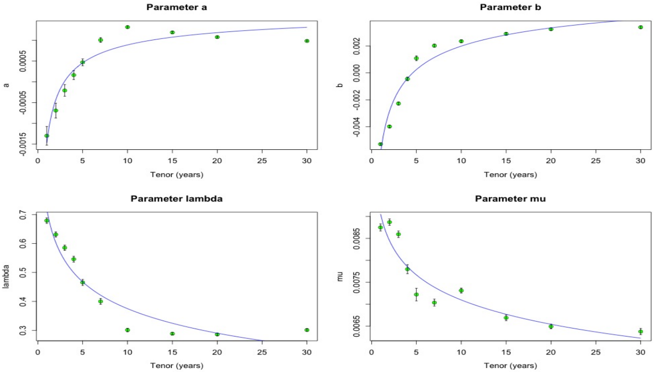

Figure: 6. Tenor dependence of the hump function parameters during the pre-crisis period.

PCA of ATM swaption volatilities

- In order to identify the factors that drive the dynamics of volatility matrix, we perform principal component analysis (PCA) on the time series of the ATM swaption volatility matrices. For this purpose, each swaption volatility matrix is regarded a vector in a 100-dimensional space.
- The analysis was performed over three time windows:
  - pre-crisis: from January 1, 2003 through December 31, 2003,
  - crisis: from January 01, 2008 to December 31, 2008,
  - post-crisis: from June 01, 2013 to May 31, 2014.
- Interestingly, the first three principal components (PC1, PC2, and PC3) together explain over 94% of the variance of the volatility matrix.
- The dominant principal component can be interpreted as parallel shift of the volatility matrix. The second and third components can be identified as tilts along the tenor and option expiration dimensions, respectively.
- This structure is present during each of the three market regimes.

PCA of ATM swaption volatilities

- During the pre-crisis period, 94.5% of the variance are explained by the three principal components, with PC1 explaining 77.1% of the variance.

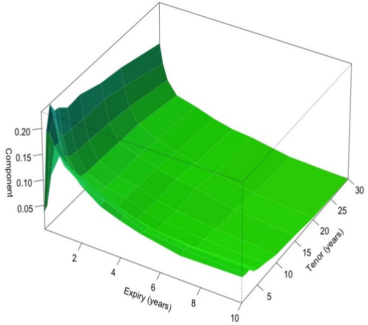

Figure: 7. PC1 of ATM swaption volatility in 2003.

PCA of ATM swaption volatilities

- PC2 explains 12.6% of the variance. It corresponds to a tilt along the tenor axis

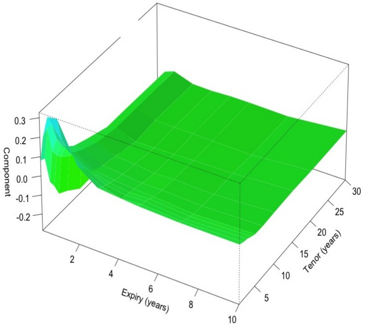

Figure: 8. PC2 of ATM swaption volatility in 2003.

PCA of ATM swaption volatilities

- PC3 explains 4.8% of the total variance. It corresponds to a tilt of the short dated volatility.

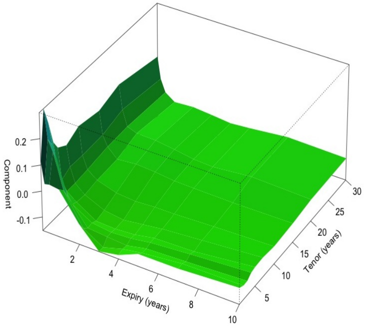

Figure: 9. PC3 of ATM swaption volatility in 2003.

PCA of ATM swaption volatilities

- The three following graphs (Figures 10, 11, and 12) show a similar picture for the crisis period. The first three components explain 80.0%, 15.0% and 2.5% of the total variance, respectively, with a total of 97.5%.

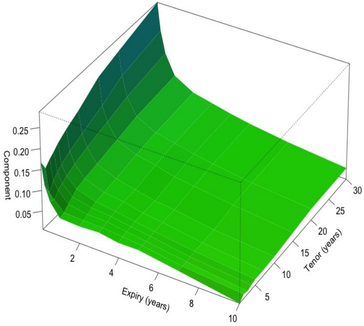

Figure: 10. PC1 of ATM swaption volatility in 2008.

PCA of ATM swaption volatilities

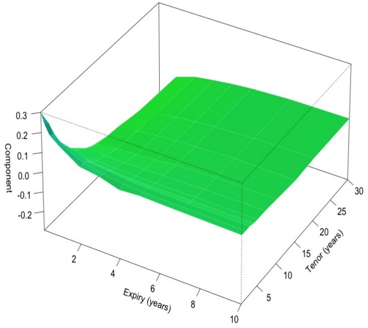

Figure: 11. PC2 of ATM swaption volatility in 2008.

PCA of ATM swaption volatilities

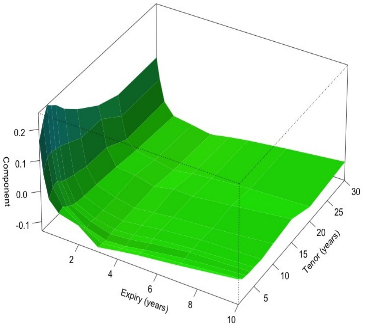

Figure: 12. PC3 of ATM swaption volatility in 2008.

PCA of ATM swaption volatilities

- Finally, Figures 13, 14, and 15 pertain to the post-crisis period. The first three components explain 77.8%, 14.5% and 4.4% of the variance, respectively, with a total of 96.8%.

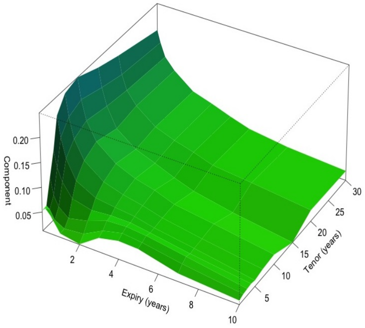

Figure: 13. PC1 of ATM swaption volatility from June 2013 to June 2014.

PCA of ATM swaption volatilities

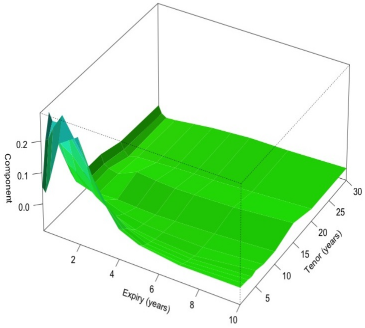

Figure: 14. PC2 of ATM swaption volatility from June 2013 to June 2014.

PCA of ATM swaption volatilities

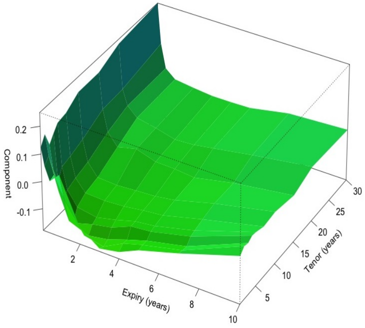

Figure: 15. PC3 of ATM swaption volatility from June 2013 to June 2014.

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