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Matrix Cartan Superdomains, Super Toeplitz Operators, and Quantization

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We present a general theory of non-perturbative quantization of a class of hermitian symmetric supermanifolds. The quantization scheme is based on the notion of a super Toeplitz operator on a suitable $\mathbb{Z}_2$-graded Hilbert spaces of superholomorphic functions. The quantized supermanifold arises as the $\mathbb{C}^*$-algebra generated by all such operators. We prove that our quantization framework reproduces the invariant super Poisson structure on the classical supermanifold as Planck's constant tends to zero. © 1995 Academic Press, Inc.

I. INTRODUCTION

I.A. In this paper we continue our program of non-perturbative quantization of Kähler supermanifolds by means of super Toeplitz operators. This procedure was first applied in [4] to quantize the hyperbolic unit superdisc and the flat superspace, and it rested on a $\mathbb{Z}_2$-graded

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extension of the results of [13] and [7]. Our goal here is a similar extension of the results of [6], where a unified scheme for quantization of Cartan domains was presented. The significance of Cartan domains lies in their role in classification of Hermitian symmetric spaces of non-compact type; every (irreducible) such space is equivalent to a Cartan domain. The Cartan domains fall into four infinite series (called types I, II, III and IV domains) as well as two exceptional cases. We use the term matrix domains to refer to Cartan domains of types I–III. The analysis of [6] relies on the Jordan triple approach to symmetric domains [16], which provides a unified framework for domains of all types.

I.B. The definition of a supermanifold which we adopt in this work is that of Konstant–Berezin–Leites [15, 3, 17], enhanced by the use of the projective tensor products as in [11]. Recall that a smooth supermanifold $\mathcal{M}$ is a ringed space $(M, \mathcal{O}_M)$, where M is an ordinary smooth manifold (called the base of $\mathcal{M}$), and where $\mathcal{O}_M$ is a sheaf of supercommutative algebras (over $\mathbb{R}$) satisfying the following conditions:

(*) the quotiont sheaf $\mathcal{O}_M/[\mathcal{O}_{M,1}+(\mathcal{O}_{M,1})^2]$, where $\mathcal{O}_{M,1}$ is the odd part of $\mathcal{O}_M$, is isomorphic to the sheaf of smooth functions on M;

(**) every point of M has a neighborhood U such that

$$\mathcal{O}_M \mid U \cong C^\infty(U) \otimes \bigwedge(E), \tag{I.1}$$

where $\bigwedge(E)$ is the Grassmann algebra over a finite dimensional real vector space E.

We let $C^\infty(\mathcal{M})$ denote the superalgebra of global sections of $\mathcal{O}_M$ and refer to its elements as smooth functions on $\mathcal{M}$. The definition of a complex supermanifold is analogous. The pair $(n_0 \mid n_1)$, where $n_0 = \dim_{\mathbb{C}} M$, $n_1 = \dim_{\mathbb{C}} E$, is called the (complex) dimension of $\mathcal{M}$. We equip each $\mathcal{O}_M(U)$ with the usual topology of a Frechet space. Then $\mathcal{O}_M$ becomes a sheaf of nuclear Frechet algebras. A morphism in the category of supermanifolds is a pair $(\varphi, \varphi^\#)$ where $\varphi: M \to N$ is a smooth map of the base manifolds and where $\varphi^\#: \mathcal{O}_N \to \varphi_* \mathcal{O}_M$ is a continuous map of sheaves of algebras over N ($\varphi_* \mathcal{O}_M$ denotes the direct image of $\mathcal{O}_M$ under φ). A direct product $\mathcal{M} \times \mathcal{N}$ of two supermanifolds is a product object in the category of supermanifolds. Clearly, $\mathcal{M} \times \mathcal{N} = (M \times N, \mathcal{O}_M \hat{\otimes}_\pi \mathcal{O}_N)$, where $\hat{\otimes}_\pi$ is the completed projective tensor product.

I.C. In this paper we will be concerned with Poisson supermanifolds, i.e., supermanifolds for which $C^\infty(\mathcal{M})$ is a Poisson superalgebra [3, 15]. This means that $C^\infty(\mathcal{M})$ is equipped with a bilinear mapping

$$\{\cdot, \cdot\}: C^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M}), \tag{I.2}$$

called a super Poisson bracket, which satisfies the conditions

$$\{f, g\} = (-1)^{p(f)p(g)+1}\{g, f\}, \tag{I.3}$$

$$(-1)^{p(f)p(h)}\{f, \{g, h\}\} + (-1)^{p(h)p(g)}\{h, \{f, g\}\} + (-1)^{p(g)p(f)}\{g, \{h, f\}\} = 0, \tag{I.4}$$

$$\{f, gh\} = \{f, g\}h + (-1)^{p(f)p(g)}g\{f, h\}, \tag{I.5}$$

where $f, g, h \in \mathscr{C}^\infty(\mathscr{M})$, and where $p(f) \in \{0, 1\}$ is the parity of the (homogeneous) element $f \in C^\infty(\mathscr{M})$. Conditions (I.3) and (I.4) say that $C^\infty(\mathscr{M})$ is a Lie superalgebra, while condition (I.5) says that the super Poisson bracket obeys the super Leibniz rule. Poisson supermanifolds arise in physics as phase spaces for classical systems involving both bosons and fermions. In the examples discussed in this paper, $\mathscr{M}$ is supersymplectic (in fact, super-Kähler), i.e., it comes equipped with a supersymplectic (by which we mean even, closed, and non-degenerate) two-form ω.

I.D. We plan to present a systematic approach to hermitian symmetric superspaces elsewhere. Here, we take a more modest point of view and construct explicitly three infinite series of hermitian supermanifolds which we call the matrix Cartan superdomains of types I, II, and III. Their key properties are: (i) the base of a Cartan superdomain of type (I)–(III) is an ordinary Cartan domain of the corresponding type; (ii) each Cartan superdomain is a homogeneous supermanifold [14]; i.e., it is a quotient of a Lie supergroup by an appropriate Lie subsupergroup; (iii) the isotropy supergroup of zero contains circular symmetry. Non-trivial super versions of the two exceptional domains seem not to exist. On the other hand, it is likely that a complete list of hermitian symmetric superspaces will include some "exotic" examples without classical counterparts.

The construction of superdomains in this paper can be extended to superdomains based on the type IV Cartan domains. We present this construction in a separate paper [5].

I.E. The paper is organized as follows. In Section II we explain the concept of a super Toeplitz operator and illustrate it by briefly reviewing the construction of [4]. Section III contains a brief review of some facts from super linear algebra. In Section IV we present the explicit constructions of the matrix superdomains. In Section V we describe the super analog of the Jordan triple determinant and give the corresponding Poisson structures for the Cartan superdomains. The two main results of this section, namely Theorems V.1 and V.2, are proved in Section VI. In Section VII we define the Bergman spaces of superholomorphic functions

on Cartan superdomains and define the corresponding super Toeplitz operators. We formulate a number of technical results and the two main results of this paper, which are Theorems VII.13 and VII.14. These theorems state that the map assigning to a function f the Toeplitz operator with symbol f is a (non-perturbative) quantization of the Poisson structure defined in Section V. Section VIII contains the proof of the positivity property and some other technical facts from Section V, and Section IX contains the proofs of Theorems VII.13 and VII.14.

II. Super Toeplitz Operators

II.A. A central concept of the present paper is that of a super-Toeplitz operator. A super-Toeplitz operator is a $\mathbb{Z}_2$-graded generalization of a Toeplitz operator and arises in the following context. Let $\mathscr{D} = (D, \mathcal{O}_D)$ be a complex supermanifold whose base D is a domain in $\mathbb{C}^N$. We choose global odd generators $\theta_1, \bar{\theta}_1, ..., \theta_{n_1}, \bar{\theta}_{n_1}$, and for a function $f \in C^\infty(\mathscr{D})$ we write

$$f(z, \theta_1, \bar{\theta}_1, ..., \theta_{n_1}\bar{\theta}_{n_1}) = \sum_{\alpha, \beta} f_{\alpha,\beta}(z)\, \theta^\alpha \bar{\theta}^\beta, \tag{II.1}$$

where α and β are multi-indices, $\theta^\alpha = \theta_1^{\alpha_1} \cdots \theta_{n_1}^{\alpha_{n_1}}$, and each $f_{\alpha\beta} \in C^\infty(D)$. The complex conjugation of a product of elements of $C^\infty(\mathscr{D})$ reverses the order:

$$\overline{fg} := \bar{g}\bar{f} = (-1)^{p(f)p(g)} \bar{f}\bar{g}. \tag{II.2}$$

We call a function $f \in C^\infty(\mathscr{D})$ bounded if each of the components $f_{\alpha\beta}$ together with all its derivatives is bounded. The subspace of bounded smooth functions on $\mathscr{D}$ is denoted by $B^\infty(\mathscr{D}) \subset C^\infty(\mathscr{D})$. We give $B^\infty(\mathscr{D})$ the topology of a Frechet space, which is defined by the following family of norms:

$$\|f\|_t := \sum_{|\mu| + |\nu| \leqslant t} \sum_{\alpha, \beta} \sup_{z \in D} |\partial_z^\mu \partial_z^\nu f_{\alpha\beta}(z)|, \tag{II.3}$$

where $t \geqslant 0$, and μ, ν are multi-indices of length n_0 with $|\mu| := \mu_1 + \cdots + \mu_{n_0}$. The derivatives ∂_z^μ are defined in the obvious way.

Let $d\mu$ be a volume form on $\mathscr{D}$ (a "measure") such that $\int_{\mathscr{D}} d\mu = 1$. The integral

$$(f, g) := \int_{\mathscr{D}} \overline{f(Z)}\, g(Z)\, d\mu(Z) \tag{II.4}$$

defines a sesquilinear form on $B^\infty(\mathscr{D})$. Unlike the usual forms of this type, (II.4) does not need to be positive definite (in fact, in the examples that we

study it is not positive definite). A function $f \in C^\infty(\mathscr{D})$ is called superholomorphic if $\partial_{z_j} f = \partial_{\bar{\theta}_k} f = 0$, for all j and k. The basic assumption about the measure $d\mu$ is the following positivity property (which resembles very much the reflection positivity of Euclidean field theory and statistical mechanics, see e.g. [9]).

The form (II.4) *defines an inner product on the subspace* $\mathrm{Hol}(\mathscr{D})$ *of* $B^\infty(\mathscr{D})$ *consisting of superholomorphic functions.*

We let $\mathscr{H}(\mathscr{D}, d\mu)$ denote the ($\mathbb{Z}_2$-graded) Hilbert space obtained as the completion with respect to (II.4) of $\mathrm{Hol}(\mathscr{D})$ and call it the Bergman space. Let $P: B^\infty(\mathscr{D}) \to \mathscr{H}(\mathscr{D}, d\mu)$ be a projection map. For $f \in B^\infty(\mathscr{D})$ and $\phi \in \mathscr{H}(\mathscr{D}, d\mu)$ we set

$$T(f)\phi := PM(f)\phi, \tag{II.5}$$

where $M(f)$ denotes the operator (on $B^\infty(\mathscr{D})$) of multiplication by f. The linear operator $T(f): \mathscr{H}(\mathscr{D}, d\mu) \to \mathscr{H}(\mathscr{D}, d\mu)$ is called a super-Toeplitz operator with symbol f.

II.B. To illustrate the above concepts we briefly review the construction of super Toeplitz operators arising in the quantization of the simplest hyperbolic supermanifold, namely the super unit disc (see [4] for the details and proofs). This construction will be generalized in Sections IV and V to arbitrary Cartan superdomains. The super unit disc $\mathscr{U} \equiv U^{1|1}$ is a $(1 \mid 1)$-dimensional complex supermanifold $(U, \mathscr{O}_U)$ whose base is the open unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$. We denote the odd generators of $C^\infty(\mathscr{U})$ by θ and $\bar{\theta}$.

We will use a collective notation for the generators of $C^\infty(\mathscr{U})$, namely $Z := (z, \theta)$. Consider now the following measure on $\mathscr{U}$. For $r \geqslant 1$ we set

$$d\mu_r(Z) := \frac{1}{\pi}(1 - Z\bar{Z})^{r-1}\, d^2z\, d^2\theta, \tag{II.6}$$

where $Z\bar{Z} := |z|^2 + \theta\bar{\theta}$, $d^2z = (i/2)\, dz \wedge d\bar{z}$ is the volume form on U, and $d^2\theta$ is the Berezin integral [3] with $\int \bar{\theta}\theta d^2\theta = 1$. Using the expansion

$$(1 - |z|^2 - \theta\bar{\theta})^{r-1} = (1 - |z|^2)^{r-1} - (r-1)(1 - |z|^2)^{r-2}\,\theta\bar{\theta}, \tag{II.7}$$

we compute the integral

$$\int_{\mathscr{U}} d\mu_r(Z) = \frac{r-1}{\pi}\int_U (1 - |z|^2)^{r-2}\, d^2z = (r-1)\int_0^1 (1-t)^{r-2}\, dt = 1, \tag{II.8}$$

i.e., the measure $d\mu_r$ has mass one. Using (II.7) it is easy to see that the associated sesquilinear form (II.4) is not positive definite. On the other

hand, for ϕ superholomorphic we can write $\phi(Z)=\phi_0(z)+\phi_1(z)\,\theta$, so that for such a function,

$$(\phi,\phi)=\frac{r-1}{\pi}\int_U |\phi_0(z)|^2\,(1-|z|^2)^{r-2}\,d^2z$$
$$+\frac{1}{\pi}\int_U |\phi_1(z)|^2\,(1-|z|^2)^{r-1}\,d^2z, \tag{II.9}$$

which is clearly positive. The projection map P taking bounded elements of $C^\infty(\mathcal{U})$ to $\mathcal{H}(\mathcal{U}, d\mu_r)$ is given by the integral operator

$$Pf(Z):=\int_{\mathcal{U}} K^r(Z,W)f(W)\,d\mu_r(W), \tag{II.10}$$

where

$$K^r(Z,W):=(1-Z\bar{W})^{-r} \tag{II.11}$$

is the Bergman kernel for $\mathcal{H}(\mathcal{U}, d\mu_r)$. The super Toeplitz operator, whose symbol is a bounded function $f\in C^\infty(\mathcal{U})$, is then defined by

$$(T_r(f)\,\phi)(Z):=\int_{\mathcal{U}} K^r(Z,W)f(W)\,\phi(W)\,d\mu_r(W). \tag{II.12}$$

III. Some Super Linear Algebra

III.A. Because this paper involves a good deal of explicit computation with both supermatrices and ordinary matrices, we review here our conventions. These follow those of [3]. We call a matrix with entries in a supercommuting algebra an ordinary matrix if its entries are purely even. For ordinary matrices, which will typically be denoted by lower case Roman letters, we use the standard notations of $\bar{a}$ and $a^\dagger$ to denote conjugate and transpose. Matrices with purely odd entries will be denoted by lower case Greek letters, and conjugation and transposition will be defined just as for ordinary matrices. Note, however, that

$$\overline{\alpha\beta}=-\bar{\alpha}\bar{\beta}, \qquad (\alpha\beta)^\dagger=-\beta^\dagger\alpha^\dagger. \tag{III.1}$$

Capital Roman letters will denote supermatrices. We use $*$ to denote the hermitian adjoint for these cases.

An $m\mid n\times k\mid l$ supermatrix has the form

$$A=\begin{matrix} & k & l \\ m & & \\ n & & \end{matrix}\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}, \tag{III.2}$$

where a and b are ordinary matrices and α and β have purely odd entries. If $l=0$ we will write $m \mid n \times k$ for the dimension, and if $n=0$ the dimension will be $m \times k \mid l$, i.e., single dimensions always refer to an even component. The superanalogs of conjugation and transposition are defined as follows:

$$A^c := \begin{pmatrix} \bar{a} & -\bar{\alpha} \\ \bar{\beta} & \bar{b} \end{pmatrix}, \tag{III.3}$$

$$A^T := \begin{pmatrix} a^t & \beta^t \\ -\alpha^t & b^t \end{pmatrix}. \tag{III.4}$$

Note that $T^2 \neq 1$. The hermitian adjoint of a supermatrix is given by $A^* := (A^c)^T$. We use the same symbol as for ordinary matrices because the same transformation is performed:

$$A^* = \begin{pmatrix} a^* & \beta^* \\ \alpha^* & b^* \end{pmatrix}. \tag{III.5}$$

III.B. The Berezinian [3] of a square supermatrix is defined by the formula

$$\operatorname{Ber}\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} := \frac{\det(a - \alpha b^{-1}\beta)}{\det b}. \tag{III.6}$$

We will often write supermatrices in a nonstandard form,

$$\gamma = \begin{matrix} \\ m \\ n \mid q \end{matrix} \begin{matrix} \begin{matrix} m & n \mid q \end{matrix} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{matrix}, \tag{III.7}$$

where A, B, C, and D are subsupermatrices. In this case the Berezinian is

$$\operatorname{Ber} \gamma = \operatorname{Ber} D \det(A - BD^{-1}C). \tag{III.8}$$

For convenience we state here a formula for the inverse of a matrix which we will use frequently. For any ordinary matrix or supermatrix in block form, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}. \tag{III.9}$$

The proof is obvious.

III.C. We include the following useful technical fact to illustrate the mechanics of dealing with Berezinians.

LEMMA III.1. *For an $m \times n \mid q$ supermatrix A and an $n \mid q \times m$ supermatrix B, we have*

$$\operatorname{Ber}(I_{n \mid q} - BA) = \det(I_m - AB), \tag{III.10}$$

where $I_{n \mid q}$ denotes the dimension $n \mid q$ identity supermatrix.

Proof. We write $A = (a, \alpha)$ and $B = \binom{b}{\beta}$. By definition,

$$\begin{aligned} \operatorname{Ber}(I_{n \mid q} - BA) &= \frac{\det(I_n - ba - b\alpha(I_q - \beta\alpha)^{-1}\beta\alpha)}{\det(I_q - \beta\alpha)} \\ &= \frac{\det(I_m - ab - ab\alpha\beta(I_n - \alpha\beta)^{-1})}{\det(I_q - \beta\alpha)}. \end{aligned} \tag{III.11}$$

Because the entries of α and β anticommute, we have

$$\begin{aligned} \det(I_q - \beta\alpha) &= \exp\left\{\sum_{l=0}^{\infty} \frac{1}{l} \operatorname{tr}(\beta\alpha)^l\right\} \\ &= \exp\left\{-\sum_{l=0}^{\infty} \frac{1}{l} \operatorname{tr}(\alpha\beta)^l\right\} \\ &= \det(I_n - \alpha\beta)^{-1}. \end{aligned} \tag{III.12}$$

Returning to (III.11), this implies

$$\begin{aligned} \operatorname{Ber}(I_{n \mid q} - BA) &= \det((I_n - ab)(I_n - \alpha\beta) - ab\alpha\beta) \\ &= \det(I_n - ba - \alpha\beta) \\ &= \det(I_n - BA)^{-1}. \quad \blacksquare \end{aligned} \tag{III.13}$$

Note that an immediate consequence of Lemma III.1 is that (III.8) is equivalent to

$$\operatorname{Ber} \gamma = \det A \operatorname{Ber}(D - CA^{-1}B). \tag{III.14}$$

IV. MATRIX CARTAN SUPERDOMAINS

IV.A. In this section we describe the main objects of our study, namely the matrix Cartan superdomains. Recall (see e.g. [10, 16]) that all

symmetric hermitian domains fall into four series of classical Cartan domains, with two exceptional domains. The first three classes are the matrix domains, which are defined as follows. In the formulas below, D with suitable decorations denotes a Cartan domain and $\mathrm{Aut}(D)$ denotes the Lie group of biholomorphisms of D. The definitions of all the Lie groups involved can be found in [10], whose notation we follow.

Type I. We let

$$D^{\mathrm{I}}_{m,n} := \{z \in \mathrm{Mat}_{m,n}(\mathbb{C}) : I_n - z^*z > 0\}$$
$$\cong SU(m,n)/S(U(m)\times U(n)). \tag{IV.1}$$

The group $SU(m,n)$ acts on $D^{\mathrm{I}}_{m,n}$ by the holomorphic automorphisms in the following way. We write $\gamma \in SU(m,n)$ in the block form

$$\gamma = \begin{matrix} \\ m \\ n \end{matrix} \begin{matrix} \begin{matrix} m & n \end{matrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{matrix}, \tag{IV.2}$$

where the submatrices a, b, c, and d have the dimensions indicated and satisfy

$$\begin{aligned} a^*a - c^*c &= I_m, \\ a^*b &= c^*d, \\ d^*d - b^*b &= I_n. \end{aligned} \tag{IV.3}$$

The corresponding element of $\mathrm{Aut}(D^{\mathrm{I}}_{m,n})$ is

$$\gamma: z \mapsto (az+b)(cz+d)^{-1}. \tag{IV.4}$$

Type II. We set

$$D^{\mathrm{II}}_n := \{z \in \mathrm{Mat}_{n,n}(\mathbb{C}): z' = z,\ I_n - z^*z > 0\}$$
$$\cong Sp(n)/U(n). \tag{IV.5}$$

The biholomorphic action of $Sp(n)$ on D^{II}_n is defined as follows. We write $\gamma \in Sp(n)$ as

$$\gamma = \begin{matrix} \\ n \\ n \end{matrix} \begin{matrix} \begin{matrix} n & n \end{matrix} \\ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \end{matrix}, \tag{IV.6}$$

where a, b satisfy

$$\begin{aligned} a^*a - b^t\bar{b} &= I_n, \\ a^t\bar{b} &= b^*a. \end{aligned} \tag{IV.7}$$

Then

$$\gamma: z \mapsto (az+b)(\bar{b}z+\bar{a})^{-1} \tag{IV.8}$$

is the corresponding element of $\mathrm{Aut}(D_n^{\mathrm{II}})$.

Type III. Let

$$\begin{aligned} D_n^{\mathrm{III}} &:= \{z \in \mathrm{Mat}_{n,n}(\mathbb{C}): z^t = -z,\ I_n - z^*z > 0\} \\ &\cong SO^*(2n)/U(n). \end{aligned} \tag{IV.9}$$

The action of $SO^*(2n)$ is defined as follows. We write $\gamma \in SO^*(2n)$ as a block matrix,

$$\gamma = \begin{matrix} \\ n \\ n \end{matrix} \begin{matrix} \begin{matrix} n & n \end{matrix} \\ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \end{matrix}, \tag{IV.10}$$

with a, b such that

$$\begin{aligned} a^*a - b^t\bar{b} &= I_n, \\ a^t\bar{b} &= -b^*a. \end{aligned} \tag{IV.11}$$

The corresponding element of $\mathrm{Aut}(D_n^{\mathrm{III}})$ is then

$$\gamma: z \mapsto (az+b)(-\bar{b}z+\bar{a})^{-1}. \tag{IV.12}$$

IV.B. A Cartan superdomain $\mathscr{D}$ is a supermanifold $(D, \mathscr{O})$, where D is an ordinary Cartan domain, and where $\mathscr{O}$ is a sheaf of superalgebras on D which will be defined case by case below. We define the superdomains of types I, II and III, denoted below by $\mathscr{D}^{\mathrm{I}}_{m,n\mid q}$, $\mathscr{D}^{\mathrm{II}}_{n\mid q}$, and $\mathscr{D}^{\mathrm{III}}_{n\mid q}$, respectively.

Type I. We set

$$C^\infty(\mathscr{D}^{\mathrm{I}}_{m,n\mid q}) := C^\infty(D^{\mathrm{I}}_{m,n}) \otimes \bigwedge(\mathbb{C}^{m\times q}). \tag{IV.13}$$

We organize the standard generators of $\bigwedge(\mathbb{C}^{m\times q})$ into $m\times q$ matrices $\theta = \{\theta_{ij}\}$ and $\bar{\theta} = \{\bar{\theta}_{ij}\}$, and represent the "points" of $\mathscr{D}$ as the $m \times n \mid q$ supermatrices

$$Z = (z, \theta). \tag{IV.14}$$

The matrix dimension q for the odd components is arbitrary.

We define the supermanifolds $\mathscr{D}^{\mathrm{II}}_{n\mid q}$ and $\mathscr{D}^{\mathrm{III}}_{n\mid q}$ as subsupermanifolds of the type I superdomains. This is done by imposing constraints on the generators of $C^\infty(\mathscr{D}^{\mathrm{I}}_{n,n\mid q})$, as follows.

Type II. We impose

$$z - z^t + \theta\theta^t = 0. \tag{IV.15}$$

The fermionic dimension q is again arbitrary for type II.

Type III. We require

$$z^t + z - \theta\tau_q\theta^t = 0, \tag{IV.16}$$

where τ_q is the $q\times q$ matrix

$$\tau_q := \begin{pmatrix} 0 & iI_{q/2} \\ -iI_{q/2} & 0 \end{pmatrix}. \tag{IV.17}$$

Note that q must be even for the type III superdomain.

Each of the above superdomains $\mathscr{D}$ admits an action of a Lie supergroup $\mathrm{Aut}(\mathscr{D})$ of superholomorphic automorphisms. In all cases, $\mathrm{Aut}(\mathscr{D})$ is an intersection of an orthosymplectic supergroup with the supergroup $SU(m, n\mid q)$. This supergroup is defined as follows. Its base manifold is $SU(m, n)\times SU(q)$, and its structure sheaf is generated by γ_{jk} and $\bar\gamma_{jk}$, $1\leqslant j, k\leqslant m+n+q$, with the following parity assignments:

$$p(\gamma_{jk}) = p(\bar\gamma_{jk}) = \begin{cases} 0, & \text{if } 1<j, k\leqslant m+n \text{ or } m+n<j, k\leqslant m+n+q, \\ 1, & \text{otherwise,} \end{cases} \tag{IV.18}$$

and with the following relations. We write γ as a block supermatrix

$$\gamma = \begin{array}{c} \\ m \\ n \\ q \end{array} \begin{array}{c} \begin{array}{ccc} m & n & q \end{array} \\ \begin{pmatrix} a & b & \rho \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix} \end{array}, \tag{IV.19}$$

where a, b, c, d, and e are even matrices and α, β, ρ, and δ are odd matrices of the dimensions indicated, and require that

$$\mathrm{Ber}\,\gamma = 1. \tag{IV.20}$$

The real structure on $SU(m, n \mid q)$ is defined by setting

$$\gamma^* = J\gamma^{-1}J, \tag{IV.21}$$

where

$$J = \begin{pmatrix} I_m & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & -I_q \end{pmatrix}. \tag{IV.22}$$

Equation (IV.21) is equivalent to the set of relations:

$$\begin{aligned} &a^*a - c^*c - \alpha^*\alpha = I_m, \\ &a^*b - c^*d - \alpha^*\beta = 0, \\ &a^*\rho - c^*\delta - \alpha^*e = 0, \\ &b^*b - d^*d - \beta^*\beta = I_n, \\ &b^*\rho - d^*\delta - \beta^*e = 0, \\ &\rho^*\rho - \delta^*\delta - e^*e = -I_q. \end{aligned} \tag{IV.23}$$

In view of (IV.14), we will find it convenient to rewrite (IV.19) in the non-standard form

$$\gamma = \begin{matrix} \\ m \\ n \mid q \end{matrix} \begin{matrix} m & n \mid q \end{matrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{IV.24}$$

where $A = a$, and B, C, and D are now supermatrices obeying the relations

$$\begin{aligned} &A^*A - C^*C = I_m, \\ &A^*B = C^*D, \\ &D^*D - B^*B = I_{n \mid q}. \end{aligned} \tag{IV.25}$$

Consider now the morphism $C^\infty(\mathscr{D}^I_{m,n \mid q}) \to C^\infty(SU(m, n \mid q)) \hat{\otimes}_\pi C^\infty(\mathscr{D}^I_{m,n \mid q})$ defined by

$$\gamma\colon Z \mapsto Z' := (AZ + B)(CZ + D)^{-1}, \tag{IV.26}$$

where, for simplicity, we have suppressed the tensor product symbols

(writing AZ in place of $A \otimes Z$ and so on). By the relations (IV.25) this transformation is equivalent to

$$\gamma(Z) = (ZB^* + A^*)^{-1}(ZD^* + C^*)$$
$$= (zb^* + \theta\rho^* + a^*)^{-1}(zd^* + \theta\delta^* + c^*, z\beta^* + \theta e^* + \alpha^*). \quad \text{(IV.27)}$$

Clearly Z' defines a new set of generators for $C^\infty(\mathscr{D}^I_{m,n\mid q})$.

PROPOSITION IV.1. *The above morphism defines a transitive action of* $SU(m, n \mid q)$ *on* $\mathscr{D}^I_{m,n\mid q}$. *Furthermore,*

$$\mathscr{D}^I_{m,n\mid q} \cong SU(m, n \mid q)/S(U(m) \times U(n \mid q)). \quad \text{(IV.28)}$$

Proof. The fact that $z^*z < I$ implies that $(ZB^* + A^*)$ is invertible, because A is invertible and the non-nilpotent part of ZB^* is zb^*. The result follows from the corresponding property of the underlying Cartan domain.

To prove (IV.28), we note that the isotropy subsupergroup of 0 consists of supermatrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \text{(IV.29)}$$

with

$$A^*A = I_m, \qquad D^*D = I_{n\mid q}. \quad \blacksquare \quad \text{(IV.30)}$$

IV.C. We now turn to the type II case. The Lie supergroup acting on $\mathscr{D}^{\mathrm{II}}_{n\mid q}$ is denoted by $Sp(n \mid q)$ and is defined as the intersection of $SU(n, n \mid q)$ with the orthosymplectic supergroup $SpO(n \mid q)$. The latter is defined in terms of supermatrices of the form (IV.19), where $m = n$. We require that $\mathrm{Ber}(\gamma) = 1$, and

$$\gamma^T K \gamma = K, \quad \text{(IV.31)}$$

where K is the supermatrix

$$K = \begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_q \end{pmatrix}. \quad \text{(IV.32)}$$

Solving the relations (IV.21) and (IV.31), we write the generators of $Sp(n \mid q)$ in the form

$$\gamma = \begin{array}{c} \\ n \\ n \\ q \end{array} \begin{array}{c} \begin{array}{ccc} n & n & q \end{array} \\ \begin{pmatrix} a & b & \rho \\ \bar{b} & \bar{a} & -\bar{\rho} \\ \alpha & \bar{\alpha} & e \end{pmatrix} \end{array}, \qquad \bar{e} = e, \tag{IV.33}$$

with the entries satisfying

$$\begin{aligned} &a'\bar{b} - b^*a + \alpha'\alpha = 0, \\ &a'\bar{a} - b^*b + \alpha'\bar{\alpha} = I_n, \\ &a'\bar{\rho} + b^*\rho - \alpha'e = 0, \\ &\rho'\bar{\rho} - \rho^*\rho + e'e = I_q. \end{aligned} \tag{IV.34}$$

Consider now the morphism $C^\infty(\mathscr{D}^{\mathrm{II}}_{n \mid q}) \to C^\infty(Sp(n \mid q)) \hat{\otimes}_\pi C^\infty(\mathscr{D}^{\mathrm{II}}_{n \mid q})$ defined by

$$\gamma\colon Z \mapsto Z' := (AZ + B)(CZ + D)^{-1}, \tag{IV.35}$$

where

$$A := a, \quad B := (b, \rho), \quad C := \begin{pmatrix} \bar{b} \\ \alpha \end{pmatrix}, \quad D := \begin{pmatrix} \bar{a} & -\bar{\rho} \\ \bar{\alpha} & e \end{pmatrix}. \tag{IV.36}$$

PROPOSITION IV.2. *The above morphism defines a transitive action of* $Sp(n \mid q)$ *on* $\mathscr{D}^{\mathrm{II}}_{n \mid q}$. *Furthermore,*

$$\mathscr{D}^{\mathrm{II}}_{n \mid q} \cong Sp(n \mid q)/U(n) \times SO(q). \tag{IV.37}$$

Proof. Clearly, (IV.35) is well defined by the same argument as that for Proposition IV.1. Recall that the defining relation of $\mathscr{D}^{\mathrm{II}}_{n \mid q}$ was

$$z - z^{\mathrm{t}} + \theta\theta^{\mathrm{t}} = 0. \tag{IV.38}$$

To show that this relation is preserved under the action of $Sp(n \mid q)$, we recast it as

$$(I_n, Z)\, K \begin{pmatrix} I_n \\ Z^T \end{pmatrix} = 0. \tag{IV.39}$$

Now, from (IV.27) we can write

$$(I_n, Z') = (ZB^* + A^*)^{-1}\, (I_n, Z)\, \gamma^*, \tag{IV.40}$$

so that

$$(I_n, Z')\, K \begin{pmatrix} I_n \\ Z'^T \end{pmatrix} = (ZB^* + A^*)^{-1}\, (I_n, Z)\, \gamma^* K \gamma^{*T} \begin{pmatrix} I_n \\ Z^T \end{pmatrix} ((ZB^* + A^*)^{-1})^T. \tag{IV.41}$$

Taking the adjoint and then the transpose of the relation $\gamma^T K \gamma = K$ gives $\gamma^* K \gamma^{*T} = K$, so that (IV.39) implies

$$(I_n, Z')\, K \begin{pmatrix} I_n \\ Z'^T \end{pmatrix} = 0. \tag{IV.42}$$

To prove (IV.37), we note that the isotropy supergroup of 0 consists of supermatrices

$$\gamma = \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ 0 & 0 & e \end{pmatrix}, \qquad \bar{e} = e, \tag{IV.43}$$

satisfying $a^* a = I_n$, $e^t e = I_q$, and $\det e = 1$. ∎

IV.D. The type III superdomains admit an action of the Lie supergroup $SO^*(2n \mid q)$, which is defined as the intersection of $SU(n, n \mid q)$ with the orthosymplectic supergroup $OSp(n \mid q)$. The latter is defined again in terms of supermatrices of the form (IV.19), where the submatrices have the same dimensions as in the case of $Sp(n \mid q)$. We require that $\mathrm{Ber}(\gamma) = 1$, and

$$\gamma^T L \gamma = L, \tag{IV.44}$$

where L is the supermatrix

$$L = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & \tau_q \end{pmatrix}, \tag{IV.45}$$

with τ_q defined in (IV.17). Note that $L = L^* = L^{-1}$. Solving the relations (IV.21) and (IV.44) we write the generators of $SO^*(2n \mid q)$ in the form

$$\gamma = \begin{array}{c} \\ n \\ n \\ q \end{array} \begin{array}{c} \begin{array}{ccc} n & n & q \end{array} \\ \begin{pmatrix} a & b & \rho \\ -\bar{b} & \bar{a} & \bar{\rho}\tau \\ \alpha & -\tau\bar{\alpha} & e \end{pmatrix} \end{array}, \qquad \bar{e} = \tau e \tau, \tag{IV.46}$$

with the entries satisfying

$$\begin{aligned} a'\bar{b} + b^*a + \alpha'\tau\alpha &= 0, \\ a'\bar{a} - b^*b + \alpha'\bar{\alpha} &= I_n, \\ a'\bar{\rho}\tau - b^*\rho - \alpha'\tau e &= 0, \\ \rho'\bar{\rho}\tau - \tau\rho^*\rho + e^{t\tau e} &= \tau. \end{aligned} \tag{IV.47}$$

We now consider the morphism $C^\infty(\mathscr{D}^{\mathrm{III}}_{n\mid q}) \to C^\infty(SO^*(2n \mid q)) \hat{\otimes}_\pi C^\infty(\mathscr{D}^{\mathrm{III}}_{n\mid q})$ defined by (IV.35), where

$$A := a, \quad B := (b, \rho), \quad C := \begin{pmatrix} -\bar{b} \\ \alpha \end{pmatrix}, \quad D := \begin{pmatrix} \bar{a} & \bar{\rho}\tau \\ -\tau\bar{\alpha} & e \end{pmatrix}. \tag{IV.48}$$

PROPOSITION IV.3. *The above morphism defines a transitive action of* $SO^*(2n \mid q)$ *on* $\mathscr{D}^{\mathrm{III}}_{n\mid qs}$. *Furthermore*

$$\mathscr{D}^{\mathrm{III}}_{n\mid q} \cong SO^*(2n \mid q)/U(m) \times Sp(q/2). \tag{IV.49}$$

Proof. The proof parallels the proof of Proposition IV.2. We write

$$(I_n, Z') = (ZB^* + A^*)^{-1}(I_n, Z)\gamma^*. \tag{IV.50}$$

The defining condition of $\mathscr{D}^{\mathrm{III}}_{n\mid q}$ is

$$(I_n, Z)\, L^T \begin{pmatrix} I_n \\ Z^T \end{pmatrix} = 0, \tag{IV.51}$$

which is preserved because $\gamma^* L^T (\gamma^*)^T = L^T$. To prove (IV.49), we note that the isotropy supergroup of 0 consists of supermatrices

$$\gamma = \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ 0 & 0 & e \end{pmatrix}, \qquad \bar{e} = \tau e \tau, \tag{IV.52}$$

satisfying $a^*a = I_n$, $e^t\tau e = \tau$, and $\det e = 1$. ∎

V. TRIPLE DETERMINANTS AND POISSON STRUCTURES

V.A. The construction of [6] rested on the framework of Jordan hermitian triple systems. For the purposes of this paper, we extract from

this framework the fact that the Bergman kernel of a Cartan domain is given by

$$K(z, w) = \Delta(z, w)^{-p}, \tag{V.1}$$

where $\Delta(z, w)$ is a polynomial in z and $\bar{w}$ (called the Jordan triple determinant), and where p is a positive integer called the genus of the Cartan domain, see e.g. [6] (we plan to present the theory of Jordan triples for Cartan superdomains elsewhere). We let $\mathrm{Aut}(\mathscr{D})$ denote the Lie supergroup of superholomorphic automorphisms of $\mathscr{D}$. The circular symmetry is a transformation of the form

$$(z, \theta) \to (e^{i\varphi} z, e^{i\varphi/2}\theta), \tag{V.2}$$

where φ is a real number.

V.B. For the quantization of superdomains, the central object will be an analog of the triple determinant mentioned above. We define a total genus $p = p_0 - p_1$, where p_0 is the genus of the underlying ordinary domain and p_1 is a non-negative integer which we call the fermionic genus. Also, for $\gamma \in \mathrm{Aut}(\mathscr{D})$ we define

$$\gamma'(Z)_{\mu\nu} = \frac{\partial}{\partial Z_\mu} \gamma(Z)_\nu. \tag{V.3}$$

In this definition, and throughout this paper, derivatives with respect to odd variables are left derivatives, i.e.,

$$\frac{\partial}{\partial \theta_1}(\theta_1 \theta_2) = \theta_2. \tag{V.4}$$

For future reference we note here that the chain rule takes the following forms:

$$\begin{aligned} \frac{\partial}{\partial Z_\mu} f \circ \gamma(Z) &= \sum_\rho \gamma'(Z)_{\mu\rho} \frac{\partial f}{\partial Z_\rho}(\gamma(Z)), \\ \frac{\partial}{\partial \bar{Z}_\mu} f \circ \gamma(Z) &= \sum_\rho (-1)^{\varepsilon_\mu(\varepsilon_\rho + 1)} \overline{\gamma'(Z)_{\mu\rho}} \frac{\partial f}{\partial \bar{Z}_\rho}(\gamma(Z)), \end{aligned} \tag{V.5}$$

where $\varepsilon_\mu := p(Z_\mu)$. The extra sign in the second relation occurs because

$$\frac{\partial}{\partial \bar{Z}_\mu} \overline{\gamma(Z)_\rho} = (-1)^{\varepsilon_\mu(\varepsilon_\rho + 1)} \overline{\gamma'(Z)_{\mu\rho}}. \tag{V.6}$$

THEOREM V.1. *For a Cartan superdomain there exists a polynomial* $N(Z, W)$ *in* Z *and* $\bar{W}$ *such that for all* $\gamma \in \mathrm{Aut}(\mathscr{D})$,

$$N(\gamma(Z), \gamma(W))^p = \mathrm{Ber}\, \gamma'(Z)\, N(Z, W)^p\, \overline{\mathrm{Ber}\, \gamma'(W)}. \tag{V.7}$$

Furthermore,

$$N(Z, W) = 1 - \sum_{\mu} \beta_{\mu}^{-1}\, Z_{\mu} \bar{W}_{\mu} + \textit{higher order terms}, \tag{V.8}$$

where β_{μ}^{-1} *are positive integers.*

The polynomial $N(Z, W)$ is the super analog of the Jordan triple determinant. Note that $N(Z, W)$ is invariant under the circular symmetry. The theorem below states that $N(Z, W)$ has a simple transformation property under $\mathrm{Aut}(\mathscr{D})$, a fact which will play an important role in the following.

THEOREM V.2. *There exists a unique holomorphic polynomial* $a_{\gamma}(Z)$ *such that:*

(i) *The automorphy factor* $\mathrm{Ber}\, \gamma'(Z)$ *is given by*

$$\mathrm{Ber}\, \gamma'(Z) = a_{\gamma}(Z)^p; \tag{V.9}$$

(ii) *we have the cocycle condition*

$$a_{\gamma_1 \gamma_2}(Z) = a_{\gamma_1}(\gamma_2(Z))\, a_{\gamma_2}(Z); \tag{V.10}$$

(iii) *the polynomial* $N(Z, W)$ *transforms according to*

$$N(\gamma(Z), \gamma(W)) = a_{\gamma}(Z)\, N(Z, W)\, \overline{a_{\gamma}(W)}. \tag{V.11}$$

We will prove Theorem V.1 and Theorem V.2 in the next section.

For the following we define the Lebesgue measure $dz := d^{2n_0}z = \prod_{l}^{n_0} (i/2)\, dz_l \wedge d\bar{z}_l$. We also define the Berezin integral $d\theta := d^{n_1}\theta\, d^{n_1}\bar{\theta}$ ([3], see also [12]), which is normalized so that

$$\int \prod_{l=1}^{n_1} (\bar{\theta}_l \theta_l)\, d\theta = 1. \tag{V.12}$$

Let $dZ := dz\, d\theta$. The Berezinian was defined precisely so that if $Z' = \gamma(Z)$, then

$$dZ' = \mathrm{Ber}\, \gamma'(Z)\, dZ. \tag{V.13}$$

COROLLARY V.3. *The measure*

$$d\mu(Z) := N(Z, Z)^{-p}\, dZ \tag{V.14}$$

is invariant under the action of $\mathrm{Aut}(\mathscr{D})$.

V.C. The superalgebra $C^\infty(\mathscr{D})$ of smooth functions on a Cartan superdomain can be equipped with an $\mathrm{Aut}(\mathscr{D})$-invariant super Poisson structure. This arises as follows. Let $\Omega^{k,l}(\mathscr{D})$, $k, l \in \mathbb{Z}$, denote the $C^\infty(\mathscr{D})$-modules of forms of type (k, l) on $\mathscr{D}$, and let

$$\partial: \Omega^{k,l}(\mathscr{D}) \to \Omega^{k+1,l}(\mathscr{D}) \tag{V.15}$$

and

$$\bar{\partial}: \Omega^{k,l}(\mathscr{D}) \to \Omega^{k,l+1}(\mathscr{D}) \tag{V.16}$$

denote the natural generalizations of the usual ∂ and $\bar{\partial}$ operators. We consider the even two-form defined by

$$\begin{aligned} \omega(Z) &:= \partial\bar{\partial} \log N(Z, Z) \\ &= \sum_{\mu,\nu} (-1)^{\varepsilon_\mu + 1}\, d\bar{Z}_\mu \wedge dZ_\nu \frac{\partial^2}{\partial Z_\nu\, \partial \bar{Z}_\mu} \log N(Z, Z), \end{aligned} \tag{V.17}$$

where $\varepsilon_\mu := p(Z_\mu)$. The parity conventions for forms and vector fields are $p(dZ_\mu) = \varepsilon_\mu + 1$, $p(\partial/\partial Z_\mu) = \varepsilon_\mu$.

PROPOSITION V.4. *ω is an $\mathrm{Aut}(\mathscr{D})$-invariant supersymplectic form on $\mathscr{D}$.*

Proof. To see that ω is $\mathrm{Aut}(\mathscr{D})$-invariant, we note that, as a consequence of Theorem V.2,

$$\log N(\gamma(Z), \gamma(Z)) = \log N(Z, Z) + \log a_\gamma(Z) + \log \overline{a_\gamma(Z)}. \tag{V.18}$$

Since $a_\gamma(Z)$ is holomorphic,

$$\partial\bar{\partial} \log a_\gamma(Z) = \partial\bar{\partial} \log \overline{a_\gamma(Z)} = 0, \tag{V.19}$$

and so $\gamma^*\omega = \omega$, as claimed.

Since $d = \partial + \bar{\partial}$, it follows immediately that $d\omega = 0$. It remains to show that ω is non-degenerate. Owing to that $\mathrm{Aut}(\mathscr{D})$-invariance, it is sufficient to prove that $\omega(0)$ is non-degenerate. This, however, is clear since (V.8) implies that

$$\omega(0) = \sum_\mu \beta_\mu^{-1}\, d\bar{Z}_\mu \wedge dZ_\mu. \quad \blacksquare$$

In components we write the symplectic form as

$$\omega_{\mu\nu}(Z) = (-1)^{\epsilon_\mu} \frac{\partial^2}{\partial Z_\nu \partial \bar{Z}_\mu} \log N(Z, Z)^{-1}, \tag{V.20}$$

so that $\omega(Z) = \sum_{\mu,\nu} d\bar{Z}_\mu \wedge dZ_\nu \omega_{\mu\nu}(Z)$.

We now construct the super Poisson bracket associated to ω. The Poisson bracket is defined by the inverse of ω with respect to the natural pairing

$$\Omega^{1,1} \otimes \Omega^{1,-1} \to \mathbb{C}, \tag{V.21}$$

which sends

$$d\bar{Z}_\mu \wedge dZ_\nu \otimes \frac{\partial}{\partial \bar{Z}_\sigma} \wedge \frac{\partial}{\partial Z_\rho} \mapsto \delta_{\nu\sigma}\delta_{\mu\rho}. \tag{V.22}$$

We require $\omega \otimes \omega^{-1} \mapsto 1$. Note that this corresponds to $\sum_\nu \omega_{\mu\nu}(Z)$ ${\omega^{-1}}_{\nu\rho}(Z) = \delta_{\mu\rho}$. Then the Poisson bracket is defined by

$$\{f, g\} := \omega^{-1}(Z)(df, dg). \tag{V.23}$$

According to Theorems 5.4 and 5.5 of [3], the bracket $\{\cdot, \cdot\}$ defined in this way indeed has the properties of a super Poisson bracket, as formulated in the Introduction.

Using the invariance of ω, we can write the Poisson bracket more conveniently. To each $Z \in \mathscr{D}$ we associate an element $\gamma_Z \in \mathrm{Aut}(\mathscr{D})$ such that $\gamma_Z(0) = Z$. Let $\pi \in \Omega^{-1,-1}(\mathscr{D})$ be defined by

$$\pi(Z) := \sum_{\mu,\nu} P(Z)_{\mu\nu} \frac{\partial}{\partial Z_\nu} \wedge \frac{\partial}{\partial \bar{Z}_\mu}, \tag{V.24}$$

where

$$P(Z)_{\mu\nu} := \sum_\rho \beta_\rho \overline{\gamma'_Z(0)_{\rho\mu}}\, \gamma'_Z(0)_{\rho\nu}. \tag{V.25}$$

THEOREM V.5. *The Poisson bracket associated to ω is given by*

$$\{f, g\} = \pi(df, dg). \tag{V.26}$$

Consequently, the pair $(C^\infty(\mathscr{D}), \{\cdot, \cdot\})$ is a Poisson superalgebra with an $\mathrm{Aut}(\mathscr{D})$-invariant Poisson bracket.

Proof. We make use of the invariance property by inverting ω at the origin and then pushing forward by the action of the supergroup $\mathrm{Aut}(\mathscr{D})$. Clearly,

$$\omega^{-1}(0) = \sum_{\mu} \beta_{\mu} \frac{\partial}{\partial Z_{\mu}} \wedge \frac{\partial}{\partial \bar{Z}_{\mu}}, \tag{V.27}$$

and so

$$\{f, g\}(0) = \sum_{\mu} (-1)^{\varepsilon_{\mu} p(f)} \beta_{\mu} \left[\frac{\partial f}{\partial Z_{\mu}}(0) \frac{\partial g}{\partial \bar{Z}_{\mu}}(0) - (-1)^{\varepsilon_{\mu}} \frac{\partial f}{\partial \bar{Z}_{\mu}}(0) \frac{\partial g}{\partial Z_{\mu}}(0) \right], \tag{V.28}$$

where $\varepsilon_{\mu} := p(Z_{\mu})$. From the invariance of ω under $\mathrm{Aut}(\mathscr{D})$ we conclude that

$$\begin{aligned} \{f, g\}(Z) := \omega^{-1}(Z)(df, dg) &= \omega^{-1}(\gamma_z(0))(d(f \circ \gamma_Z), d(g \circ \gamma_Z)) \\ &= \omega^{-1}(0)(d(f \circ \gamma_Z), d(g \circ \gamma_Z)) \\ &= \{f \circ \gamma_Z, g \circ \gamma_Z\}(0). \end{aligned}$$

Consequently, using (V.28) we obtain that

$$\begin{aligned} \{f, g\}(Z) = \sum_{\rho, \mu, \nu} & \beta_{\rho} \overline{\gamma'_Z(0)_{\rho\mu}} \gamma'_Z(0)_{\rho\nu} \\ & \times (-1)^{\varepsilon_{\nu} p(f)} \left[\frac{\partial f}{\partial Z_{\nu}}(Z) \frac{\partial g}{\partial \bar{Z}_{\nu}}(Z) - (-1)^{\varepsilon_{\mu}\varepsilon_{\nu}} \frac{\partial f}{\partial \bar{Z}_{\mu}}(Z) \frac{\partial g}{\partial Z_{\nu}}(Z) \right]. \end{aligned} \tag{V.29}$$

In view of (V.25) we obtain

$$\begin{aligned} &\{f, g\}(Z) \\ &\quad = \sum_{\mu, \nu} P(Z)_{\mu\nu} (-1)^{\varepsilon_{\mu} p(f)} \left[\frac{\partial f}{\partial Z_{\nu}}(Z) \frac{\partial g}{\partial \bar{Z}_{\mu}}(Z) - (-1)^{\varepsilon_{\mu}\varepsilon_{\nu}} \frac{\partial f}{\partial \bar{Z}_{\mu}}(Z) \frac{\partial g}{\partial Z_{\nu}}(Z) \right] \\ &\quad = \pi(Z)(df, dg), \end{aligned}$$

as claimed. ∎

COROLLARY V.6. *The inverse of ω is given by*

$$\omega^{-1}_{\mu\nu}(Z) = P_{\mu\nu}(Z), \tag{V.30}$$

and as a consequence

$$\operatorname{Ber}\omega(Z) = N(Z, Z)^{-p} \prod_{\mu} \beta_{\mu}^{-1}, \tag{V.31}$$

where $\omega_{\mu\nu}$ *is viewed as a supermatrix.*

Proof. The first statement is the content of the previous theorem. The definition of $P_{\mu\nu}$ then implies that

$$\operatorname{Ber}\omega(Z) = |\operatorname{Ber}\gamma_Z'(0)|^{-} \prod_{\mu} \beta_{\mu}^{-1}. \tag{V.32}$$

Applying Theorem V.1 to γ_Z yields

$$N(Z, Z)^p = N(\gamma_Z(0), \gamma_Z(0))^p = |\operatorname{Ber}\gamma_Z'(0)|^2, \tag{V.33}$$

and the second statement follows. ∎

V.D. For $\sigma \in \Omega^{-1,-1}(\mathscr{D})$ given by

$$\sigma = \sum_{\mu,\nu} f_{\mu\nu}(Z) \frac{\partial}{\partial Z_\nu} \wedge \frac{\partial}{\partial \bar{Z}_\mu}, \tag{V.34}$$

the map $\partial: \Omega^{-1,-1}(\mathscr{D}) \to \Omega^{0,-1}(\mathscr{D})$ takes σ to

$$\partial\sigma = \sum_{\mu,\nu} (-1)^{\varepsilon_\nu(\varepsilon_\mu+1)} \frac{\partial f_{\mu\nu}}{\partial Z_\nu}(Z) \frac{\partial}{\partial \bar{Z}_\mu}. \tag{V.35}$$

THEOREM V.7. *The two-vector field* $\sigma \in \Omega^{-1,-1}$ *defined by*

$$\sigma = \sum_{\mu,\nu} \frac{P_{\mu\nu}(Z)}{N(Z, Z)^p} \frac{\partial}{\partial Z_\nu} \wedge \frac{\partial}{\partial \bar{Z}_\mu} \tag{V.36}$$

satisfies $\partial\sigma = 0$.

Proof. For convenience in this proof let $\partial_\mu := \partial/\partial Z_\mu$ and likewise for $\bar{\partial}_\mu$. We start with the fact that $P_{\mu\nu} = \omega_{\mu\nu}^{-1}$, so that

$$\partial_\rho P_{\mu\nu} = -\sum_{\alpha,\beta} (-1)^{\varepsilon_\rho(\varepsilon_\mu+\varepsilon_\alpha)} P_{\mu\alpha}(\partial_\rho \omega_{\alpha\beta}) P_{\beta\nu}. \tag{V.37}$$

Thus

$$\sum_{\nu} (-1)^{\varepsilon_\nu(\varepsilon_\mu+1)} \partial_\nu P_{\mu\nu} = -\sum_{\nu,\alpha,\beta} (-1)^{\varepsilon_\nu(\varepsilon_\alpha+1)} P_{\mu\alpha}(\partial_\nu \omega_{\alpha\beta}) P_{\beta\nu}. \tag{V.38}$$

Now the statement that $\partial\omega = 0$ means that $\partial_\nu \omega_{\alpha\beta} = (-1)^{\varepsilon_\nu \varepsilon_\alpha} \partial_\alpha \omega_{\nu\beta}$, so that

$$\sum_\nu (-1)^{\varepsilon_\nu(\varepsilon_\mu+1)} \partial_\nu P_{\mu\nu} = -\sum_{\nu,\alpha,\beta} (-1)^{\varepsilon_\nu} P_{\mu\alpha}(\partial_\alpha \omega_{\nu\beta}) P_{\beta\nu}. \tag{V.39}$$

By the definitions of the supertrace and the Berezinian [3],

$$\begin{aligned}\sum_{\nu,\beta} (-1)^{\varepsilon_\nu} (\partial_\alpha \omega_{\nu\beta}) \omega^{-1}_{\beta\nu} &= \partial_\alpha \operatorname{Str} \log \omega \\ &= \partial_\alpha \log \operatorname{Ber} \omega.\end{aligned} \tag{V.40}$$

By Corollary V.6 we see that Ber ω is equal to a constant times $N(Z, Z)^{-p}$. Thus

$$\partial_\alpha \log \operatorname{Ber} \omega = -p \partial_\alpha \log N. \tag{V.41}$$

Returning to (V.39), we have

$$\sum_\nu (-1)^{\varepsilon_\nu(\varepsilon_\mu+1)} \partial_\nu P_{\mu\nu} = p \sum_\alpha P_{\mu\alpha} \partial_\alpha \log N. \tag{V.42}$$

In view of the explicit formula (V.35), the statement that $\partial\sigma = 0$ is equivalent to

$$\sum_\nu (-1)^{\varepsilon_\nu(\varepsilon_\mu+1)} \partial_\nu \frac{P_{\mu\nu}}{N^p} = 0, \tag{V.43}$$

for all μ. Using the results of the last paragraph we evaluate

$$\begin{aligned}\sum_\nu (-1)^{\varepsilon_\nu(\varepsilon_\mu+1)} \partial_\nu \frac{P_{\mu\nu}}{N^p} &= pN^{-p} \sum_\alpha P_{\mu\alpha} \partial_\alpha \log N + \sum_\nu P_{\mu\nu} \partial_\nu N^{-p} \\ &= 0. \quad \blacksquare\end{aligned} \tag{V.44}$$

VI. Proof of Theorems V.1. and V.2

VI.A. In this section we define the "super triple determinant" $N(Z, W)$ for matrix superdomains and establish Theorems V.1 and V.2. We will prove these theorems after establishing a series of propositions.

Lemma VI.1. *For $\gamma \in SU(m, n \mid q)$,*

$$\det(A^* + ZB^*) = \operatorname{Ber}(CZ + D), \tag{VI.1}$$

where A, B, C, and D are the matrix blocks of γ.

Proof. Using Lemma III.1 we have

$$\begin{aligned}\operatorname{Ber}(CZ+D) &= \operatorname{Ber} D \operatorname{Ber}(I_{n\mid q} + D^{-1}CZ) \\ &= \operatorname{Ber} D \det(I_m + ZD^{-1}C).\end{aligned} \tag{VI.2}$$

Using (IV.25) we obtain

$$\begin{aligned}I_m + ZD^{-1}C &= A^*A - C^*C + ZD^*C + ZB^*BD^{-1}C \\ &= A^*A - A^*BD^{-1}C + ZB^*A + ZB^*BD^{-1}C \\ &= (A^* + ZB^*)(A - BD^{-1}C).\end{aligned} \tag{VI.3}$$

We now combine this with the fact that

$$\operatorname{Ber}\gamma = \det(A - BD^{-1}C)\operatorname{Ber} D = 1 \tag{IV.4}$$

to see that

$$\operatorname{Ber}(CZ+D) = \det(A^* + ZB^*). \quad \blacksquare \tag{VI.5}$$

PROPOSITION VI.2. *For $\gamma \in SU(m, n \mid q)$ acting on $\mathscr{D}^{\mathrm{I}}_{m,n\mid q}$,*

$$\operatorname{Ber}\gamma'(Z) = \frac{1}{\det(A^* + ZB^*)^{m+n-q}}. \tag{VI.6}$$

Proof. The matrix of derivatives can be evaluated explicitly,

$$\frac{\partial Z'_{ij}}{\partial Z_{mn}} = (ZB^* + A^*)^{-1}_{im}(D^* - B^*Z')_{nj}.$$

In the matrix notation of (V.3) we write

$$\gamma'(Z) = [(ZB^* + A^*)^{-1}]^T \otimes (D^* - B^*Z'). \tag{VI.8}$$

Using the relations (IV.25), we see that

$$\begin{aligned}D^* - B^*Z' &= D^* - B^*(AZ+B)(CZ+D)^{-1} \\ &= [D^*CZ + D^*D - B^*AZ - B^*B](CZ+D)^{-1} \\ &= (CZ+D)^{-1}.\end{aligned} \tag{VI.9}$$

Thus the matrix of derivatives becomes

$$\gamma'(Z) = [(ZB^* + A^*)^{-1}]^T \otimes (CZ+D)^{-1}, \tag{VI.10}$$

and its Berezinian is

$$\operatorname{Ber}\gamma'(Z) = \operatorname{Det}(ZB^* + A^*)^{-(n-q)} \operatorname{Ber}(CZ + D)^{-m}. \tag{VI.11}$$

The proposition follows from Lemma VI.1. ∎

PROPOSITION VI.3. *For $\gamma \in \mathrm{Sp}(n \mid q)$ acting on $\mathscr{D}^{\mathrm{II}}_{n \mid q}$,*

$$\operatorname{Ber}\gamma'(Z) = \frac{1}{\det(A^* + ZB^*)^{n+1-q}}. \tag{VI.12}$$

Proof. First, we study the case when $Z = 0$. Choosing coordinates Z_{ij}, where either $1 \leqslant i < j \leqslant n$ or $j > n$, the supermatrix $\gamma'(0)$ is given by

$$\gamma'(0)_{kl,\,ij} = \frac{\partial Z'_{ij}}{\partial Z_{kl}}$$

$$= \begin{matrix} & i \leqslant j \leqslant n & j > n \\ k \leqslant l \leqslant n & \dfrac{1}{1+\delta_{kl}}\left[(A^{*-1})_{ik}\, u_{lj} + (A^{*-1})_{il}\, u_{kj}\right] & \dfrac{1}{1+\delta_{kl}}\left[(A^{*-1})_{ik}\, \sigma_{lj} + (A^{*\mu 1})_{il}\, \sigma_{kj}\right] \\ l > n & -(A^{*-1})_{ik}\, \eta_{lj} & (A^{*-1})_{ik}\, v_{lj} \end{matrix}, \tag{VI.13}$$

where we have represented the block entries of D^{-1} by

$$D^{-1} = \begin{pmatrix} u & \sigma \\ \eta & v \end{pmatrix}. \tag{VI.14}$$

Writing $\gamma'(0)$ as

$$\gamma'(0) = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \tag{VI.15}$$

we need to compute

$$\operatorname{Ber} T = \det(T_1 - T_2 T_4^{-1} T_3) \det T_4^{-1}. \tag{VI.16}$$

We start by observing that

$$[T_4^{-1} T_3]_{kl,\,ij} = \delta_{ik} [v^{-1}\eta]_{lj}, \tag{VI.17}$$

so that

$$[T_1 - T_2 T_4^{-1} T_3]_{kl,ij} = \frac{1}{21 + \delta_{kl}} [(A^{*-1})_{ik} (u - \sigma v^{-1} \eta)_{lj} + (A^{*-1})_{il} (u - \sigma v^{-1} \eta)_{kj}]$$

$$\equiv \left[\bar{A}^{-1} \underset{s}{\otimes} (u - \sigma v^{-1} \eta) \right]_{kl,ij},$$

where $A \otimes_s B$ denotes the symmetric tensor product of the matrices A and B. Now from (IV.36) we see that

$$(u - \sigma v^{-1} \eta) = \bar{A}^{-1}, \tag{VI.19}$$

so we have

$$\det(T_1 - T_2 T_4^{-1} T_3) = \frac{1}{\det(\bar{A} \otimes_s \bar{A})} = (\det A^*)^{-(n+1)}. \tag{VI.20}$$

To complete the calculation of (VI.16) we have

$$\det T_4 = \det(A^*)^{-q} \det v^{-n}. \tag{VI.21}$$

In terms of D, $v = (e + \bar{\alpha}\bar{\alpha}^{-1}\bar{\rho})^{-1}$, and one easily sees from the relations that $v^t v = I_q$. The result is thus

$$\operatorname{Ber} \gamma'(0) = (\det A^*)^{(n+1-q)}. \tag{VI.22}$$

To complete the proof we consider the case where γ maps Z to $Z' \neq 0$. Let $\gamma = \gamma_2 \circ \gamma_1$, where

$$\gamma_1(Z) = 0, \qquad \gamma_2(0) = Z'. \tag{VI.23}$$

We write

$$\gamma_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \qquad i = 1, 2, \tag{VI.24}$$

and

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_2 A_1 + B_2 C_1 & A_2 B_1 + B_2 D_1 \\ C_2 A_1 + D_2 C_1 & C_2 B_1 + D_2 D_1 \end{pmatrix}. \tag{VI.25}$$

Because of (VI.23),

$$Z D_1^* + C_1^* = 0, \tag{VI.26}$$

so that

$$\begin{aligned} A^* + ZB^* &= A_1^* A_2^* + C_1^* B_2^* + ZB_1^* A_2^* + ZD_1^* B_2^* \\ &= (A_1^* + ZB_1^*) A_2^* \\ &= (A_1^* + C_1^* (D_1^*)^{-1} B_1^*) A_2^*. \end{aligned} \tag{VI.27}$$

Applying the result (VI.22) and the fact that $(A_1 - B_1 D_1^{-1} C_1)^{-1}$ is the upper right submatrix of γ_1^{-1}, we have

$$\det(A^* + ZB^*)^{-p} = \frac{\operatorname{Ber} \gamma_2'(0)}{\operatorname{Ber}(\gamma_1^{-1})'(0)} = \operatorname{Ber} \gamma_2'(0) \operatorname{Ber} \gamma_1'(Z) = \operatorname{Ber} \gamma'(Z). \quad \blacksquare \tag{VI.28}$$

PROPOSITION VI.4. *For $\gamma \in SO^*(2n \mid q)$ acting on $\mathscr{D}_{n \mid q}^{\mathrm{III}}$,*

$$\operatorname{Ber} \gamma'(Z) = \frac{1}{\det(A^* + ZB^*)^{n-1-q}}. \tag{VI.29}$$

Proof. The proof closely follows that of Proposition VI.3. In place of Eq. (VI.18), we obtain

$$\begin{aligned} [T_1 - T_2 T_4^{-1} T_3]_{kl,ij} &= \frac{1}{1+\delta_{kl}} [(A^{*-1})_{ik} (u - \sigma v^{-1} \eta)_{lj} \\ &\quad - (A^{*-1})_{il} (u - \sigma v^{-1} \eta)_{kj}] \\ &\equiv \left[\bar{A}^{-1} \underset{a}{\otimes} (u - \sigma v^{-1} \eta) \right]_{kl,ij}, \end{aligned} \tag{VI.30}$$

where $A \otimes_a B$ denotes the antisymmetric tensor product of the matrices A and B. Since $\det A^* \otimes_a A^* = (\det A^*)^{n-1}$, we thus obtain

$$\gamma'(0) = (\det A^*)^{(n-1)+q} \tag{VI.31}$$

in place of (VI.22). The second half of the proof is then identical to that above. $\blacksquare$

Based on the preceeding four propositions, for all three types we define the super triple determinant

$$N(Z, W) := \det(I_m - Z^* W) = \operatorname{Ber}(I_{n \mid q} - W^* Z), \tag{IV.32}$$

and the transformation factor

$$a_\gamma(Z) := \det(A^* + ZB^*) = \operatorname{Ber}(CZ + D). \tag{VI.33}$$

PROPOSITION VI.5. *With the above definitions,*

$$N(\gamma(Z), \gamma(W)) = a_\gamma(Z)\, N(Z, W)\, \overline{a_\gamma(W)}. \tag{VI.34}$$

Proof. The statement is that

$$\mathrm{Ber}(I_{n\mid q} - \gamma(W)^*\, \gamma(Z)) = a_\gamma(Z)\, \mathrm{Ber}(I_{n\mid q} - W^*Z)\, \overline{a_\gamma(W)}. \tag{VI.35}$$

The defining property (IV.21) of $SU(m, n \mid q)$ implies that

$$I_{n\mid q} - \gamma(W)^*\, \gamma(Z) = (CW + D)^{*-1}\, (I_{n\mid q} - W^*Z)(CZ + D)^{-1}. \tag{VI.36}$$

The proposition then follows from (VI.33). ∎

VI.B. *Proof of Theorems* V.1 *and* V.2. Theorem V.2 (i) is established in Proposition VI.2, Proposition VI.3, and Proposition VI.4 for types I, II, and III, respectively (incidentally, the fermionic genus p_1 turns out to be equal to q in all these cases). Part (iii) of Theorem V.2 and the first statement of Theorem V.1 are proved in Proposition VI.5. The second statement of Theorem V.1 is clear. In particular, we find that $\beta_\mu^{-1} = 1$ or 2 in (V.8).

It remains to prove property (ii) of Theorem V.2. Let

$$\gamma_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \tag{VI.37}$$

for $i = 1, 2$. We have

$$\begin{aligned} a_{\gamma_1\gamma_2}(Z) &= \mathrm{Ber}[(C_1A_2 + D_1C_2)\, Z + (C_1B_2 + D_1D_2)] \\ &= \mathrm{Ber}[C_1(A_2Z + B_2) + D_1(C_2Z + D_2)] \\ &= \mathrm{Ber}(C_1\gamma_2(Z) + D_1)\, \mathrm{Ber}(C_2Z + D_2) \\ &= a_{\gamma_1}(\gamma_2(Z))\, a_{\gamma_2}(Z). \quad \blacksquare \end{aligned} \tag{VI.38}$$

VI.C. For future reference, we give here explicit formulas for the group elements γ_Z. For type I, γ_Z can be written as

$$\gamma_Z := \begin{pmatrix} I_m & Z \\ Z^* & I_{n\mid q} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \tag{VI.39}$$

for any A and D which satisfy

$$\begin{aligned} AA^* &= (I_m - ZZ^*)^{-1}, \\ DD^* &= (I_{n\mid q} - Z^*Z)^{-1}, \end{aligned} \tag{VI.40}$$

and $\mathrm{Ber}\, D^* \det A = 1$.

For type II we have

$$\gamma_z := \begin{pmatrix} I_n & z^t & \sigma \\ z^* & I_n & -\bar{\sigma} \\ \theta^* & \theta^t & I_q \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ 0 & 0 & e \end{pmatrix}, \tag{VI.41}$$

where

$$\begin{aligned} aa^* &= (I_n - ZZ^*)^{-1}, \\ \sigma &= (I_n - z\bar{z})^{-1}(\theta - z\bar{\theta}), \\ ee^t &= (I_q + \sigma^t\bar{\sigma} - \sigma^*\sigma)^{-1}, \end{aligned} \tag{VI.42}$$

and where

$$\det e = \frac{\det(I_n - ZZ^*)}{\det(I_n - z\bar{z})}. \tag{VI.43}$$

Finally, for type III,

$$\gamma_z := \begin{pmatrix} I_n & -z^t & \sigma \\ z^* & I_n & \bar{\sigma}\tau \\ \theta^* & -\tau\theta^t & I_q \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ 0 & 0 & e \end{pmatrix}, \tag{VI.44}$$

where

$$\begin{aligned} aa^* &= (I_n - zz^* - \theta\theta^*)^{-1}, \\ \sigma &= (I_n + z\bar{z})^{-1}(\theta + z\bar{\theta}\tau), \\ ee^* &= (I_q + \tau\sigma^t\bar{\sigma}\tau - \sigma^*\sigma)^{-1}, \end{aligned} \tag{VI.45}$$

and where

$$\det e = \frac{\det(I_n - ZZ^*)}{\det(I_n + z\bar{z})}. \tag{VI.46}$$

VII. Quantization

VII.A. Our framework for the quantization of a Cartan superdomain $\mathscr{D}$ rests on the following perturbation of the invariant measure. We will show later that there is $r_0(\mathscr{D}) > 0$ such that the measure $N(Z, Z)^r\, d\mu(Z)$ has a finite volume for $r \geqslant r_0(\mathscr{D})$. We set

$$d\mu_r(Z) := \Lambda_r N(Z, Z)^r\, d\mu(Z) = \Lambda_r N(Z, Z)^{r-p}\, dz\, d\theta, \tag{VII.1}$$

for $r \geqslant r_0(\mathscr{D})$, where $d\mu$ is the invariant meausre of Corollary V.3 and Λ_r is chosen so that the total integral is normalized to one.

For f and $g \in B^\infty(\mathscr{D})$, we set

$$(f, g)_r := \int_{\mathscr{D}} \overline{f(Z)}\, g(Z)\, d\mu_r(Z). \tag{VII.2}$$

This form is not positive definite and so it does not define an inner product on $B^\infty(\mathscr{D})$. The crucial property of $(\cdot, \cdot)_r$ is, however, that its restriction to the subspace of superholomorphic functions is positive definite. In fact, a more general property holds (which we will need). We consider the superspace $B^\infty_*(\mathscr{D})$ of functions f for which $\partial f/\partial\bar{\theta}_j = 0$. Observe that this notion is not invariant under superholomorphic changes of coordinates on $\mathscr{D}$. The following theorem will be proven in the next section.

THEOREM VII.1. *There exists $r_0(\mathscr{D}) > 0$ such that for all $r \geqslant r_0(\mathscr{D})$, the sesquilinear form $(\cdot, \cdot)_r$ defines an inner product on $B^\infty_*(\mathscr{D})$.*

Consider the space $\mathrm{Hol}(\mathscr{D})$ of superholomorphic functions in $B^\infty(\mathscr{D})$. As a consequence of the above theorem, $(\cdot, \cdot)_r$ is an inner product on this space. The completion of $\mathrm{Hol}(\mathscr{D})$ in the norm induced by this inner product forms a Hilbert space, which we denote by $\mathscr{H}_r(\mathscr{D})$.

VII.B. In this subsection we state some facts concerning the measure $d\mu_r$ that will be useful later.

PROPOSITION VII.2. *The form* (VII.1) *has the transformation property*

$$d\mu_r(\gamma(Z)) = [a_\gamma(Z)\, \overline{a_\gamma(Z)}]^r\, d\mu_r(Z), \tag{VII.3}$$

for and $\gamma \in \mathrm{Aut}(\mathscr{D})$.

Proof. This is a direct consequence of Theorem V.2. ∎

PROPOSITION VII.3. *There is a constant $C > 0$ such that for r sufficiently large*

$$\int N(Z, Z)^{r-p}\, d\theta = C r^{n_1} \Delta(z, z)^{r-p_0} [1 + O(r^{-1})], \tag{VII.4}$$

uniformly in z, where $\Delta(z, z)$ is the triple determinant of the underlying domain.

Proposition VII.3 will be established in Section VIII.

PROPOSITION VII.4. *The normalization constant Λ_r has the behavior*

$$\Lambda_r = Cr^{n_0 - n_1}[1 + O(r^{-1})], \tag{VII.5}$$

as $r \to \infty$.

Proof. The statement follows immediately from Proposition VII.3 and Lemma 3.1(i) of [6]. ∎

VII.C. The Hilbert space $\mathscr{H}_r(\mathscr{D})$ carries a natural projective unitary representation of $\mathrm{Aut}(\mathscr{D})$. This is given by $\gamma \to U(\gamma)$, where

$$U(\gamma^{-1})\,\phi(Z) = a_\gamma(Z)^r\,\phi(\gamma(Z)). \tag{VII.6}$$

Clearly, each $U(\gamma^{-1})$ is unitary because of Proposition VII.2. We see that U is a projective representation as follows.

For notational convenience in the following argument, we write $a(\gamma, Z)$ in place of $a_\gamma(Z)$. For $\gamma_1, \gamma_2 \in \mathrm{Aut}(\mathscr{D})$, define the function

$$\lambda(\gamma_1, \gamma_2)(Z) := \frac{1}{2\pi i}\{\log a(\gamma_2^{-1}\gamma_1^{-1}, Z) - \log a(\gamma_1^{-1}, Z) - \log a(\gamma_2^{-1}, \gamma_1^{-1}(Z))\}. \tag{VII.7}$$

THEOREM VII.5. *The function $\lambda(\gamma_1, \gamma_2)$ defined above has the following properties*:

(i) *$\lambda(\gamma_1, \gamma_2)(Z)$ does not depend on Z. Thus $\lambda(\gamma_1, \gamma_2)$ is a function on* $\mathrm{Aut}(\mathscr{D}) \times \mathrm{Aut}(\mathscr{D})$.

(ii) *We have the following cocycle condition*:

$$\lambda(\gamma_1, \gamma_2\gamma_3) + \lambda(\gamma_2, \gamma_3) - \lambda(\gamma_1\gamma_2, \gamma_3) - \lambda(\gamma_1, \gamma_2) = 0. \tag{VII.8}$$

(iii) $\lambda(\gamma_1, \gamma_2) \in \{-1, 0, 1\}$.

Proof. (i) We take the gradient of $\lambda(\gamma_1, \gamma_2)(Z)$ as follows:

$$2\pi i\,\nabla\lambda(\gamma_1, \gamma_2)(Z) = \frac{1}{a(\gamma_2^{-1}, \gamma_1^{-1}(Z)}\nabla a(\gamma_2^{-1}\gamma_1^{-1}, Z) - \frac{1}{a(\gamma_1^{-1}, Z)}\nabla a(\gamma_1^{-1}, Z)$$
$$- \frac{1}{a(\gamma_2^{-1}, \gamma_1^{-1}(Z))}\nabla a(\gamma_2^{-1}, \gamma_1^{-1}(Z)).. \tag{VII.9}$$

By (V.10),

$$a(\gamma_2^{-1}\gamma_1^{-1}, Z) = a(\gamma_2^{-1}, \gamma_1^{-1}(Z))\,a(\gamma_1^{-1}, Z). \tag{VII.10}$$

We thus see that

$$\frac{1}{a(\gamma_2^{-1}\gamma_1^{-1}, Z)}\nabla a(\gamma_2^{-1}\gamma_1^{-1}, Z) = -\frac{1}{a(\gamma_1^{-1}, Z)}\nabla a(\gamma_1^{-1}, Z) - \frac{1}{a(\gamma_2^{-1}, \gamma_1^{-1}(Z))}\nabla a(\gamma_2^{-1}, \gamma_1^{-1}(Z)). \tag{VII.11}$$

(ii) Consider the first two terms in (VII.8). In view of (i), we can evaluate either λ at any point. We choose to evaluate the first λ at Z and the second at $\gamma_1^{-1}(Z)$. The sum of these terms is thus

$$\lambda(\gamma_1, \gamma_2\gamma_3)(Z) + \lambda(\gamma_2, \gamma_3)(\gamma_1^{-1}(Z)) = \frac{1}{2\pi i}\{\log a(\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1}, Z) - \log a(\gamma_1^{-1}, Z) - \log a(\gamma_2^{-1}, \gamma_1^{-1}(Z)) - \log a(\gamma_3^{-1}, \gamma_2^{-1}(\gamma_1^{-1}(Z)))\}. \tag{VII.12}$$

By adding and subtracting $(1/2\pi i)\log a((\gamma_1\gamma_2)^{-1}, Z)$, we see that (VII.12) is in fact equal to

$$\lambda(\gamma_1\gamma_2, \gamma_3) + \lambda(\gamma_1, \gamma_2). \tag{VII.13}$$

To prove (iii), we set $Z = 0$ in (VII.7) and use (V.10). ∎

COROLLARY VII.6. *Formula* (VII.6) *defines a projective unitary representation of* $\mathrm{Aut}(\mathscr{D})$ *on* $\mathscr{H}_r(\mathscr{D})$.

Proof. Set

$$\sigma(\gamma_1, \gamma_2) := \exp\{2\pi i r\lambda(\gamma_1, \gamma_2)\}. \tag{VII.14}$$

As a consequence of Theorem V.2,

$$U(\gamma_1\gamma_2) = \sigma(\gamma_1, \gamma_2)\, U(\gamma_1)\, U(\gamma_2). \tag{VII.15}$$

The cocycle condition,

$$\sigma(\gamma_2, \gamma_3)\,\sigma(\gamma_1\gamma_2, \gamma_3)^{-1}\,\sigma(\gamma_1, \gamma_2\gamma_3)\,\sigma(\gamma_1, \gamma_2)^{-1} = 1, \tag{VII.16}$$

follows from Theorem VII.5 (ii), which shows that (VII.6) is consistent with associativity. The unitary is a consequence of Proposition VII.2. ∎

VII.D. A fundamental component of our construction is the Bergman (or reproducing) kernel for the space $\mathscr{H}_r(\mathscr{D})$. Let

$$K^r(Z, W) := N(Z, W)^{-r}. \tag{VII.17}$$

PROPOSITION VII.7. *The kernel function* (VII.17) *has the reproducing property, i.e., for* $\phi \in \mathscr{H}_r(\mathscr{D})$,

$$\phi(Z) = \int_{\mathscr{D}} K^r(Z, W)\, \phi(W)\, d\mu_r(W). \tag{VII.18}$$

Proof. We compute the right-hand side of (VII.18) by making the substitution $W = \gamma_z(Y)$, where γ_z is an element of $\mathrm{Aut}(\mathscr{D})$ such that $\gamma_z(0) = Z$. This yields

$$\begin{aligned}
&\int_{\mathscr{D}} K^r(Z, W)\, \phi(W)\, d\mu_r(W) \\
&\quad = \int_{\mathscr{D}} K^r(Z, \gamma_z(Y))\, \phi(\gamma_z(Y))\, d\mu_r(\gamma_z(Y)) \\
&\quad = \int_{\mathscr{D}} [a_{\gamma_z}(0)\, \overline{a_{\gamma_z}(Y)}]^{-r}\, \phi(\gamma_z(Y))[a_{\gamma_z}(Y)\, \overline{a_{\gamma_z}(Y)}]^r\, d\mu_r(Y) \\
&\quad = \int_{\mathscr{D}} a_{\gamma_z}(0)^{-r}\, a_{\gamma_z}(Y)^r\, \phi(\gamma_z(Y))\, d\mu_r(Y).
\end{aligned} \tag{VII.19}$$

We apply the simple fact that for ψ holomorphic,

$$\int_{\mathscr{D}} \psi(W)\, d\mu_r(W) = \psi(0), \tag{VII.20}$$

which is a consequence of circular symmetry, and obtain

$$\int_{\mathscr{D}} a_{\gamma_z}(0)^{-r}\, a_{\gamma_z}(Y)^r\, \phi(\gamma_z(Y))\, d\mu_r(Y) = \phi(Z). \quad \blacksquare \tag{VII.21}$$

For $g \in B^\infty(\mathscr{D})$, we define the projection P by

$$Pg(Z) := \int_{\mathscr{D}} K^r(Z, W)\, g(W)\, d\mu_r(W). \tag{VII.22}$$

Clearly, $Pg \in \mathscr{H}_r(\mathscr{D})$, and $Pg = g$ for $g \in \mathscr{H}_r(\mathscr{D})$.

PROPOSITION VII.8.

$$\int_{\mathscr{D}} \bar{Z}_\mu Z_\rho \, d\mu_r(Z) = \frac{\beta_\mu}{r} \delta_{\mu\rho}, \tag{VII.23}$$

where β_μ *is the constant of Theorem* V.1.

Proof. Let

$$A_{\mu\rho} = \int_{\mathscr{D}} \bar{Z}_\mu Z_\rho \, d\mu_r(Z). \tag{VII.24}$$

Because of Theorem VII.1 the matrix $A_{\mu\rho}$ is invertible. Using Proposition VII.7 we can also write

$$\int_{\mathscr{D}} \bar{Z}_\mu Z_\rho \, d\mu_r(Z) = \int_{\mathscr{D}\times\mathscr{D}} \overline{Z_\mu} K^r(Z, W) \, W_\rho \, d\mu_r(Z) \, d\mu_r(W). \tag{VII.25}$$

The circular symmetry and the expansion of $N(Z, W)$ in Theorem V.1 imply that the right-hand side of (VII.25) is given by

$$\sum_\nu \frac{r}{\beta_\nu} \int_{\mathscr{D}\times\mathscr{D}} \overline{Z_\mu} Z_\nu \overline{W_\nu} W_\rho \, d\mu_r(Z) \, d\mu_r(W). \tag{VII.26}$$

In terms of A this implies

$$A_{\mu\rho} = \sum_\nu A_{\mu\nu} \frac{r}{\beta_\nu} A_{\nu\rho}. \tag{VII.27}$$

We apply A^{-1} to both sides of this equation and obtain (VII.23). ∎

VII.E. As described in Section II we define super-Toeplitz operators $T_r(f)$ on $\mathscr{H}_r(\mathscr{D})$, for $f \in B^\infty(\mathscr{D})$, by setting

$$T_r(f)\,\phi(Z) := \int_{\mathscr{D}} K^r(Z, W) f(W) \, \phi(W) \, d\mu_r(W). \tag{VII.28}$$

The map $f \mapsto T_r(f)$ will be the quantization map in our scheme. We first establish some basic properties of the super-Toeplitz operators.

First of all, observe that

$$T_r(f\circ\gamma) = U(\gamma)^{-1} T_r(f) \, U(\gamma), \tag{VII.29}$$

where $U(\gamma)$ is defined by (VII.6).

Secondly, we have the following estimate on the norm of $T_r(f)$.

PROPOSITION VII.9. *$T_r(f)$ is a bounded operator on $\mathscr{H}_r(\mathscr{D})$. Furthermore,*

$$\|T_r(f)\| \leqslant C \sum_{\alpha,\beta} r^{-(|\alpha|+|\beta|)/2} \|f_{\alpha\beta}\|_0. \tag{VII.30}$$

In particular, a super-Toeplitz operator is bounded. We let $\mathscr{T}_r(\mathscr{D})$ denote the $\mathbb{C}^*$-algebra generated by all super-Toeplitz operators.

The above proposition follows directly from the following lemmas and proposition. To simplify the notation, in the rest of the paper we will suppress the subscript r in $\|\cdot\|_r$.

LEMMA VII.10. *For $\psi, \phi \in \mathscr{H}_r(\mathscr{D})$, and $g \in B^\infty(D)$ (an ordinary function) we have*

$$\left| \int_{\mathscr{D}} \overline{\psi(Z)}\, g(z)\, \phi(Z)\, d\mu_r(Z) \right| \leqslant \|g\|_\infty \|\psi\| \|\phi\|. \tag{VII.31}$$

Proof. Because of Theorem VII.1, we can view $(\cdot,\cdot)_r$ as an inner product on the space of functions which are holomorphic only in the odd coordinates. Thus we have

$$\left| \int_{\mathscr{D}} \overline{\psi(Z)}\, g(z)\, \phi(Z)\, d\mu_r(Z) \right| = |(\psi, g\phi)_r|. \tag{VII.32}$$

By the Schwarz inequality,

$$|(\psi, g\phi)_r| \leqslant \|\psi\| \left\{ \int_{\mathscr{D}} |g(z)|^2\, \overline{\phi(Z)}\, \phi(Z)\, d\mu_r(Z) \right\}^{1/2}. \tag{VII.33}$$

Becuase $\overline{\phi(Z)}\, \phi(Z)\, d\mu_r(Z)$ is a positive measure, we can extract the sup norm of $g(z)$, giving

$$\left| \int_{\mathscr{D}} \overline{\psi(Z)}\, g(Z)\, \phi(Z)\, d\mu_r(Z) \right| \leqslant \|g\|_\infty \|\psi\| \|\phi\|. \quad \blacksquare \tag{VII.34}$$

LEMMA VII.11. *For any odd generator θ_μ,*

$$\|T_r(\theta_\mu)\| \leqslant C r^{-1/2}, \tag{VII.35}$$

for r sufficiently large.

Lemma VII.11 will be proven in Section VIII.

PROPOSITION VII.12. *For $\psi, \phi \in \mathscr{H}_r(\mathscr{D})$, and $f \in B^\infty(\mathscr{D})$, we have*

$$\left| \int_{\mathscr{D}} \overline{\psi(Z)} f(Z) \phi(Z) \, d\mu_r(Z) \right| \leqslant C \sum_{\alpha, \beta} r^{-(|\alpha| + |\beta|)/2} \| f_{\alpha\beta} \|_\infty \, \|\psi\| \, \|\phi\|. \tag{VII.36}$$

In particular,

$$\left| \int_{\mathscr{D}} \overline{\psi(Z)} f(Z) \phi(Z) \, d\mu_r(Z) \right| \leqslant C \| f \|_0 \, \|\psi\| \, \|\phi\|, \tag{VII.37}$$

where $\| \cdot \|_0$ is the norm defined in (II.3).

Proof. The statement follows immediately from Lemma VII.10 and Lemma VII.11. ∎

VII.F. To conclude this section, we make the statement that the map $B^\infty(\mathscr{D}) \to \mathscr{T}_r(\mathscr{D})$, given by T_r, is a deformation quantization. This statement consists of the following theorems, which will be proved in Section V.

THEOREM VII.13. *For $f \in B^\infty(\mathscr{D})$ bounded, we have*

$$\lim_{r \to \infty} \| T_r(f) \| = \| f_{00} \|_0. \tag{VII.38}$$

In other words, the classical limit wipes out the fermions. This is not surprising as fermions do not exist in classical mechanics.

THEOREM VII.14. *For $f, g \in B^\infty(\mathscr{D})$, where the components $f_{\alpha\beta}$ are compactly supported, there is a constant $C = C(f, g)$, such that*

$$\left\| T_r(f) \, T_r(g) - T_r(fg) + \frac{1}{r} \sum_{\mu, \nu} (-1)^{\varepsilon_\mu p(f)} \, T_r\left(P_{\mu\nu} \frac{\partial f}{\partial Z_\nu} \frac{\partial g}{\partial \bar{Z}_\mu} \right) \right\|_r \leqslant C r^{-2}, \tag{VII.39}$$

for r sufficiently large.

As a consequence of this theorem, we conclude that $\mathscr{T}_r(\mathscr{D})$ is a quantum deformation of the Poisson algebra $B^\infty(\mathscr{D})$, with r^{-1} playing the role of Planck's constant. The assumption that f has compact support is certainly not optimal, but some kind of decay of at least one symbol at the boundary is clearly needed in our proof. On the other hand, it is easy to verify that the estimate holds for any polynomial f and g.

THEOREM VII.15. *Under the assumptions of Theorem* VII.14,

$$\|r[T_r(f), T_r(g)] + T_r(\{f, g\})\|_r \leqslant Cr^{-1}, \tag{VII.40}$$

for r sufficiently large.

Proof. The proof follows immediately from Theorem VII.14 and from the definition (V.29) of the super Poisson bracket. ∎

VIII. POSITIVITY AND OTHER PROPERTIES

VIII.A. Theorem VII.1 will be proved after two lemmas are established below.

DEFINITION VIII.1. Let B_+ be the cone in $B^\infty(\mathscr{D})$ generated by functions of the form $g = \bar{f}f$, with $f \in B_*^\infty(\mathscr{D})$.

LEMMA VIII.2.

(i) B_+ *is a multiplicative cone.*

(ii) $\exp B_+ \in B_+$.

(iii) *For* $g \in B_+$ *nilpotent (i.e., g contains no term which involves only the even variables),* $(1+g)^\lambda \in B_+$ *for every* $\lambda > n_1$.

Proof. Property (i) follows from the fact that

$$\bar{f}\bar{f}\bar{g}g = (-1)^{p(f)p(g)}\bar{f}\bar{g}fg = \overline{\bar{f}g}fg. \tag{VIII.1}$$

For (ii) we see that for $f \in B_+$,

$$\exp f = \sum_{n \geqslant 0} \frac{1}{n!} f^n, \tag{VIII.2}$$

which is in B_+ by (i). For (iii) we note that

$$(1+g)^\lambda = \sum_{l=0}^{n_1} \frac{\lambda(\lambda-1)\cdots(\lambda-l+1)}{l!} g^l. \quad \blacksquare \tag{VIII.3}$$

LEMMA VIII.3. *For the matrix superdomains,*

$$N(Z, Z)^\lambda \in B_+, \tag{VIII.4}$$

for $\lambda \geqslant n_1$.

Proof. Using the properties of the Berezinian we can rewrite

$$N(Z, Z) = \det(I_m - zz^* - \theta\theta^*). \tag{VIII.5}$$

If we let $X = (I_m - zz^*)^{-1/2}$, then

$$N(Z, Z)^\lambda = \det X^{-2\lambda} \det(I_m - X\theta\theta^* X)^\lambda. \tag{VIII.6}$$

The first factor on the right-hand side is clearly in B_+. Since B_+ is a multiplicative cone, and becuase of item (iii) of Lemma VIII.2, we will be done if we can show that

$$\det(I_m - X\theta\theta^* X) \in 1 + B_+. \tag{VIII.7}$$

To prove (VIII.7) we make use of the fact that for any square matrix A,

$$\det(I - A) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \operatorname{tr}(\wedge^n A). \tag{VIII.8}$$

Now, for an odd matrix η,

$$\begin{aligned} \operatorname{tr}(\wedge^n \eta\eta^*) &= (-1)^{n(n+1)/2} \operatorname{tr}[(\wedge^n \eta^*)(\wedge^n \eta)] \\ &= (-1)^{n^2} \operatorname{tr}[(\wedge^n \eta)^* (\wedge^n \eta)]. \end{aligned} \tag{VIII.9}$$

Applying this to (VIII.7) we find

$$\det(I_m - X\theta\theta^* X) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \operatorname{tr}[(\wedge^n X\theta)^* (\wedge^n X\theta)]. \tag{VIII.10}$$

Since $\operatorname{tr} A^*A$ is clearly in B_+ for any matrix of functions in $B_*^\infty(\mathscr{D})$, this completes the proof. ∎

Proof of Theorem VII.1. From (V.12) it follows that $\int_{\mathscr{D}} g\, dZ \geqslant 0$, for $g \in B_+$ such that $g_\alpha(z)$ is integrable when $\alpha = (1, 1, ..., 1)$. Thus Lemma VIII.2 (i) and Lemma VIII.3 establish that $(\cdot, \cdot)_r$ is non-negative for r sufficiently large.

It remains to show that the form is strictly positive. Suppose that there exists $f \in B_*^\infty(\mathscr{D})$ such that

$$\int_{\mathscr{D}} \overline{f(Z)} f(Z) \det(I_m - zz^* - \theta\theta^*)^\lambda\, dZ = 0. \tag{VIII.11}$$

If we change variables $\theta \to \theta' = (I_m - zz^*)^{-1/2}\theta$, then this becomes

$$\int_{\mathscr{D}} \overline{f(Z')} f(Z') \det(I_m - zz^*)^{\lambda - q} \det(I_m - \theta'\theta'^*)^{\lambda} dZ'. \quad \text{(VIII.12)}$$

We now perform the integral over z. Since the measure on z is strictly positive we see that the existence of an f satisfying (VIII.11) is equivalent to the existence of $g \in \bigwedge(\mathbb{C}^{mq})$ such that

$$\int_{\mathscr{D}} \overline{g(\theta)}\, g(\theta) \det(I_m - \theta\theta^*)^{\lambda} d\theta = 0. \quad \text{(VIII.13)}$$

We can assume that g is a homogeneous polynomial of degree k, since homogeneous polynomials of different degree will be orthogonal. We make the expansion $g(\theta) = \sum_{|\alpha| = k} x_\alpha \theta^\alpha$, where the sum ranges over multi-indices of length k. If we let A be the matrix

$$A_{\alpha\beta} = \int \overline{\theta^\alpha} \theta^\beta \det(I_m - \theta\theta^*)^{\lambda} d\theta, \quad \text{(VIII.14)}$$

then (VIII.13) is the statement that $x^*Ax = 0$. Consider the leading order of the expansion of A in powers by λ:

$$\begin{aligned} A_{\alpha\beta} &= \int \overline{\theta^\alpha} \theta^\beta e^{\lambda \operatorname{tr} \theta^*\theta} d\theta + O(\lambda^{mq-k-1}) \\ &= \int \overline{\theta^\alpha} \theta^\beta \frac{(\lambda \operatorname{tr} \theta^*\theta)^{mq-k}}{(mq-k)!} d\theta + O(\lambda^{mq-k-1}) \\ &= \lambda^{mq-k}[\delta_{\alpha\beta} + O(1/\lambda)]. \end{aligned} \quad \text{(VIII.15)}$$

We conclude that for λ sufficiently large, A is strictly positive definite. Hence $x^*Ax = 0$ implies $x = 0$, i.e., (VIII.13) implies $g = 0$. ∎

VIII.B. In this subsection we establish some facts concerning integration over purely odd matrices. These facts will be used to prove the remaining technical assumptions of Section VII in the next subsection.

LEMMA VIII.4. *Let η represent an $m \times 1$ column vector of odd variables and let S_m^k denote the set of ordered subsets of $\{1, ..., m\}$ of cardinality k. For $\alpha \in S_m^k$ and $\beta \in S_m^l$,*

$$\int \overline{\eta_{\alpha_1} \cdots \eta_{\alpha_k}} \eta_{\beta_1} \cdots \eta_{\beta_l} (1 - \eta^*\eta)^{\lambda} d\eta = \varepsilon_{\alpha\beta} \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - m + k + 1)}, \quad \text{(VIII.16)}$$

where $\varepsilon_{\alpha\beta}=0$ *unless* α *is a permutation of* β *and in this case is given by the sign of the relative permutation.*

Proof. It is clear that β must be a permutation of α for the integral to be nonzero, since $(1-\eta^*\eta)^\lambda$ contains only pairs of the form $\bar{\eta}_j\eta_j$. By permuting the set β into the set α and kepping track of the sign, we find

$$\int \overline{\eta_{\alpha_1}\cdots\eta_{\alpha_k}}\eta_{\beta_1}\cdots\eta_{\beta_l}(1-\eta^*\eta)^\lambda\,d\eta=\varepsilon_{\alpha\beta}\int \overline{\eta_{\alpha_1}\cdots\eta_{\alpha_k}}\eta_{\alpha_1}\cdots\eta_{\alpha_k}(1-\eta^*\eta)^\lambda\,d\eta. \tag{VIII.17}$$

Now we can simply compute

$$\begin{aligned}&\int \overline{\eta_{\alpha_1}\cdots\eta_{\alpha_k}}\eta_{\alpha_1}\cdots\eta_{\alpha_k}(1-\eta^*\eta)^\lambda\,d\eta\\&\quad=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+k+1)(m-k)!}\int \overline{\eta_{\alpha_1}\cdots\eta_{\alpha_k}}\eta_{\alpha_1}\cdots\eta_{\alpha_k}(\eta^*\eta)^{m-k}\,d\eta\\&\quad=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+k+1)}.\ \blacksquare\end{aligned} \tag{VIII.18}$$

LEMMA VIII.5. *For* $m\times q$ *odd matrices* θ, *we have*

$$\int \det(I_m-\theta\theta^*)^\lambda\,d\theta=\prod_{0\le k\le m-1}\frac{\Gamma(\lambda-k+q)}{\Gamma(\lambda-k)}, \tag{VIII.19}$$

which behaves as λ^{mq} *for* $\lambda\to\infty$.

Proof. Decompose θ into (θ',ρ), where ρ is the last column of θ. We have

$$I_m-\theta\theta^*=I_m-\theta'\theta'^*-\rho\rho^*. \tag{VIII.20}$$

We next define $\omega=(I_m-\theta'\theta'^*)^{-1/2}\rho$, so that

$$I_m-\theta\theta^*=(I_m-\theta'\theta'^*)(I_m-\omega\omega^*). \tag{VIII.21}$$

The change of variables from ρ to ω gives

$$\int \det(I_m-\theta\theta^*)^\lambda\,d\theta=\int \det(I_m-\theta'\theta'^*)^{\lambda-1}\,d\theta'\int \det(I_m-\omega\omega^*)^\lambda\,d\omega. \tag{VIII.22}$$

Applying this procedure recursively, and using the fact that

$$\det(I_m-\omega\omega^*)=(1-\omega^*\omega)^{-1}, \tag{VIII.23}$$

we get

$$\int \det(I_m - \theta\theta^*)^\lambda \, d\theta = \prod_{0 \leq k \leq m-1} \int (1-\omega^*\omega)^{(\lambda-k)} \, d\omega. \quad \text{(VIII.24)}$$

The result then follows from Lemma VIII.4 with $\alpha = \beta = \varnothing$. ∎

LEMMA VIII.6. *Let a be an invertible $m \times m$ ordinary matrix, and let η represent an $m \times 1$ column vector of odd variables. For $\alpha \in S_m^k$ and $\beta \in S_m^l$ (the sets defined in Lemma* VIII.4*), we have the integral formula*

$$\int \overline{\eta_{\alpha_1} \cdots \eta_{\alpha_k}} \eta_{\beta_1} \cdots \eta_{\beta_l} \det(aa^* - \eta\eta^*)^\lambda \, d\eta$$

$$= \frac{\delta_{kl}\Gamma(\lambda+1)}{\Gamma(\lambda-m+k+1)} \det(aa^*)^{\lambda-1} \det{}_{\beta\alpha}(aa^*), \quad \text{(VIII.25)}$$

where $\det_{\beta\alpha}$ *is the determinant minor taken over the rows β and columns α.*

Proof. The fact that k must be equal to l is clear. Let $\eta = a\omega$. Then

$$\det(aa^* - \eta\eta^*) = (1-\omega^*\omega)\det(aa^*), \quad \text{(VIII.26)}$$

and the measure transforms to $d\omega = \det(aa^*)\, d\eta$. Thus under the change of variables the left-hand side of (VIII.25) becomes

$$\det(aa^*)^{\lambda-1} \int \overline{(a\omega)_{\alpha_1} \cdots (a\omega)_{\alpha_k}} (a\omega)_{\beta_1} \cdots (a\omega)_{\beta_l} (1-\omega^*\omega)^\lambda \, d\omega. \quad \text{(VIII.27)}$$

We now apply Lemma VIII.4 to perform the integration over ω. The result is

$$\det(aa^*)^{\lambda-1} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+k+1)} \sum_{\mu,\nu \in S_m^k} \varepsilon_{\mu\nu} \bar{a}_{\alpha_1\mu_1} \cdots \bar{a}_{\alpha_k\mu_k} a_{\beta_1\nu_1} \cdots a_{\beta_k\nu_k}. \quad \text{(VIII.28)}$$

The sum in (VII.28) can be rewritten as

$$\sum_{\mu \in S_m^k} \sum_{\sigma \in S_k} \varepsilon(\sigma) \bar{a}_{\alpha_1\mu_1} \cdots \bar{a}_{\alpha_k\mu_k} a_{\beta_1\mu_{\sigma(1)}} \cdots a_{\beta_k\nu_{\sigma(k)}}$$

$$= \sum_{\mu \in S_m^k} \sum_{\sigma \in S_k} \varepsilon(\sigma) \bar{a}_{\alpha_{\sigma(1)}\mu_1} \cdots \bar{a}_{\alpha_{\sigma(k)}\mu_k} a_{\beta_1\mu_1} \cdots a_{\beta_k\nu_k}$$

$$= \sum_{\sigma \in S_k} \varepsilon(\sigma)(aa^*)_{\beta_1\alpha_{\sigma(1)}} \cdots (aa^*)_{\beta_k\alpha_{\sigma(k)}}, \quad \text{(VIII.29)}$$

where S_k denotes the set of permutations of $\{1, ..., k\}$ and $\varepsilon(\sigma)$ is the sign of the permutation σ. This final sum over σ is precisely the definition of the determinant minor $\det_{\beta\alpha}(aa^*)$. ∎

VIII.C. We turn now to the proofs of Proposition VII.3 and Lemma VII.11. We can in fact replace Proposition VII.3 by the following, stronger statement.

PROPOSITION VIII.7.

$$\int N(Z, Z)^{r-p}\, d^{2n|}\theta = C_r r^{n|} \Delta(z, z)^{r-p_0}, \tag{VIII.30}$$

where $\Delta(z, z) := \det(I_m - zz^*)$ *is the triple determinant of the underlying domain and where*

$$C_r = \prod_{0 \leq k \leq m-1} \frac{\Gamma(r-p_0-k)}{\Gamma(r-p-k)}. \tag{VIII.31}$$

Proof. The function $N(Z, Z)$ has the form

$$\begin{aligned} N(Z, Z) &= \det(I_m - zz^* - \theta\theta^*) \\ &= \det(I_m - zz^*) \det(I_m - (I_m - zz^*)^{-1}\theta\theta^*). \end{aligned} \tag{VIII.32}$$

By changing variables $\theta \to \theta' = (I_m - zz^*)^{-1/2}\theta$, we obtain

$$\begin{aligned} \int N(Z, Z)^{r-p}\, d\theta &= \det(I_m - zz^*)^{r-p_0+q} \\ &\quad \times \int \det(I_m - (I_m - zz^*)^{-1}\theta\theta^*)^{r-p}\, d^{2mq}\theta \\ &= \det(I_m - zz^*)^{r-p_0} \int \det(I_m - \theta'\theta'^*)^{r-p}\, d^{2mq}\theta'. \end{aligned} \tag{VIII.33}$$

The proof follows from Lemma VIII.5. ∎

Proof of Lemma VII.11. We need to compare $\|\phi\|$ to $\|\theta_{ij}\phi\|$. To do this we start by integraing over all of the odd variables except for the jth column. Denote the jth column by η, so that $\theta_{ij} = \eta_i$, and denote the remaining odd variables by θ'. Let λ denote $r-p$. The integral over θ' is

$$\begin{aligned} \|\phi\|^2 &= \Lambda_r \int_{\mathcal{O}} \bar{\phi}\phi \det(I - zz^* - \theta'\theta'^* - \eta\eta^*)^{\lambda}\, d\theta'\, dn\, dz \\ &= \Lambda_r \int_{\mathcal{O}} \int \Psi(z, \eta) \det(I - zz^* - \eta\eta^*)^{\lambda}\, d\eta\, dz, \end{aligned} \tag{VIII.34}$$

$\Psi(z, \eta)$ is also in B_+. Thus Ψ must have the form

$$\Psi = \sum_j \overline{X_j(z, \eta)}\, X_j(z, \eta), \tag{VIII.39}$$

where $X_j \in \bigwedge^k(\mathbb{C}^m)$ can be written

$$X_j(z, \eta) = \sum_{\nu \in S_m^k} X_{j,\nu}(z)\, \eta_{\nu_1} \cdots \eta_{\nu_k}. \tag{VIII.40}$$

Let $\mathbb{C}^m$ be given the inner product $(\mu, v) = u^*(I_m - zz^*)v$. The natural extension of an inner product to an exterior algebra is simply the determinant minor, i.e.,

$$(\eta_{\mu_1} \cdots \eta_{\mu_k}, \eta_{\mu_1} \cdots \eta_{\mu_k}) = \det_{\mu\nu}(I_m - zz^*). \tag{VIII.41}$$

This means that we can write the integrand of (VIII.37) as

$$\det(I_m - zz^*)^{\lambda - 1} \sum_j \|X_j\|^2_{\bigwedge^k(\mathbb{C}^m)}, \tag{VIII.42}$$

and the integrand of (VIII.38) as

$$\det(I_m - zz^*)^{\lambda - 1} \sum_j \|\eta_i X_j\|^2_{\bigwedge^{k+1}(\mathbb{C}^m)}. \tag{VIII.43}$$

The problem then reduces to computing the norm of the operator $\sigma_i : \bigwedge^k(\mathbb{C}^m) \to \bigwedge^{k+1}(\mathbb{C}^m)$ which maps $X \mapsto \eta_i X$. Let $\omega = a\eta$ where $aa^* = (I_m - zz^*)$. Then the ω's generate an orthonormal basis for $\bigwedge(\mathbb{C}^m)$. The Hilbert–Schmidt norm of σ_i is easily computed:

$$\begin{aligned} \|\sigma_i\|_2^2 &= \sum_{\mu \in \hat{S}^k_m} \|\eta_i \omega_{\mu_1} \cdots \omega_{\mu_k}\|^2_{\bigwedge^{k+1}(\mathbb{C}^m)} \\ &= \sum_{\mu \in \hat{S}^k_m} \sum_{l \not\in \mu} \bar{a}_{il} a_{il} \\ &= \binom{m-1}{k} (I_m - zz^*)_{ii} \\ &\leqslant \binom{m-1}{k}. \end{aligned} \tag{VIII.44}$$

Since the estimate is independent of z, we can use it inside the integral in (VIII.38). We conclude that the kth summand of (VIII.38) can be bounded by

$$\binom{m-1}{k} \frac{1}{\lambda - m + k + 1}, \tag{VIII.45}$$

times the kth summand of (VIII.37). ∎

IX. Proof of Deformation Estimates

IX.A. In this section we prove Theorem VII.13 and Theorem VII.14 for a generic Cartan superdomain of type I–III.

Proof of Theorem VII.13. From Lemma VII.10 and Lemma VII.11 we have

$$\|T_r(f)\| \leqslant \|f_{00}\|_\infty + O(r^{-1/2}), \tag{IX.1}$$

as $r \to \infty$, i.e., $\limsup_{r\to\infty} \|T_r(f)\| \leqslant \|f_{00}\|_\infty$. We will show below that

$$\|f_{00}\|_\infty \leqslant \|T_r(f)\| + o(1), \tag{IX.2}$$

as $r \to \infty$, i.e., $\liminf_{r\to\infty} \|T_r(f)\| \geqslant \|f_{00}\|_\infty$, and the claim will follow.

To prove (IX.2), we set $Z = (z, 0)$ and write

$$\begin{aligned} f(Z) &= f_{00}(z) \\ &= (\phi_0, T_r(f \circ \gamma_Z)\,\phi_0) + \left\{ f_{00}(z) - \int_{\mathscr{D}} f(\gamma_Z(W))\, d\mu_r(W) \right\}, \end{aligned} \tag{IX.3}$$

where $\phi_0 = 1$ is the vacuum element. Using (VII.29), we rewrite the above equation as

$$\begin{aligned} f_{00}(z) = (\phi_0, U(\gamma_Z)^{-1}\, T_r(f)\, U(\gamma_Z)\,\phi_0) \\ + \left\{ f_{00}(z) - \int_{\mathscr{D}} f_{00}(w')\, d\mu_r(W) \right\} \\ + \int_{\mathscr{D}} [f(\gamma_Z(W)) - f_{00}(w')]\, d\mu_r(W), \end{aligned} \tag{IX.4}$$

where $(w', \eta') := \gamma_Z(W)$. The first term in (IX.4) can be bounded by $\|T_r(f)\|$, as $U(\gamma_Z)$ is unitary. Using Proposition VII.3, we can apply the proof of Theorem 2.1 in [6] to show that the second term is $o(1)$ uniformly in z, as $r \to \infty$. For the third term, we use Proposition VII.12 to bound

$$\begin{aligned} &\left| \int_{\mathscr{D}} [f(\gamma_Z(W)) - f_{00}(w')]\, d\mu_r(W) \right| \\ &\quad \leqslant \sum_{\alpha,\beta,\,|\alpha|=|\beta|\neq 0} r^{-(|\alpha|+|\beta|)/2} \|(f \circ \gamma_Z)_{\alpha\beta}\|_\infty, \end{aligned} \tag{IX.5}$$

and the claim follows. ∎

IX.B. In this subsection, we give two lemmas which will be needed for the proof of Theorem VII.14. For the following lemma and its proof we extend the norm $\|\cdot\|_0$ to supermatrices by taking the supremum of the norms of the elements of the matrix. We denote by $\gamma^{(k)}(W)$ the kth complex derivative of $\gamma(W)$.

LEMMA IX.1. *For each k, there exists constants $s, s' > 0$ such that*

$$\|\gamma_Z^{(k)}(W)\|_0 \leqslant C\Delta(w, w)^{-s}\,\Delta(z, z)^{-s'}, \tag{IX.6}$$

where $\Delta(z, z)$ denotes the triple determinant of the underlying Cartan domain.

Proof. For type I superdomains, the first complex derivative $\gamma_z'(W)$ was computed in the proof of Proposition VI.2 to be

$$\gamma_Z'(W) = [(WB^* + A^*)^T \otimes (CW + D)]^{-1}, \tag{IX.7}$$

where A, B, C, D are the matrix blocks of γ_Z. For types II and III the computation is essentially the same, although the tensor product will be replaced by sme partially symmetrized or antisymmetrized tensor product. This will not affect the bounds, so we proceed to analyze (IX.7).

For the following discussion, we abuse notation slightly by letting $\|A\|_0$, for a supermatrix A, denote the supremum of the $\|\cdot\|_0$ of all the entries. Each further derivative of (IX.7) will involve an extra factor of $(ZB^* + A^*)^{-1}$ or $(CZ + D)^{-1}$, times entries of B^* and C, respectively. For types II and III there will be extra factors of two, but this will not make a difference. By a conservative estimate we have

$$\|\gamma_Z^{(k)}(W)\|_0 \leqslant K[\|B\|_0\,\|C\|_0]^{k-1}\,[\|(WB^* + A^*)^{-1}\|_0\,\|(CW + D)^{-1}\|_0]^k, \tag{IX.8}$$

where K is some constant.

For the matrices B and C we have, by virtue of the conditions (IV.25), the bounds $\|B\|_0 \leqslant \|A\|_0$ and $\|C\|_0 \leqslant \|D\|_0$. Now, for all domains, A and D satisfy the relations (VI.40), which implies $\|A\|_0 \leqslant \|(I_m - ZZ^*)^{-1}\|_0$ and $\|D\|_0 \leqslant \|(I_{n|q} - Z^*Z)^{-1}\|_0$. Furthermore, up to a constant matrix, $CW + D = D^{-1}(I_{n|q} + Z^*W)$ and $WB^* + A^* = (I_m + WZ^*)\,A^*$. Thus the proof will be finished if we can establish a bound

$$\|(I_{n|q} + Z^*W)^{-1}\|_0 \leqslant K\Delta(w, w)^{-s}\,\Delta(z, z)^{-s'} \tag{IX.9}$$

(the case of $(I_m + WZ^*)^{-1}$ is similar enough that it need not be dealt with separately).

To make this bound, we observe that

$$(I_{n|q} + Z^*W)^{-1} = \begin{pmatrix} I_n & -(I_n + z^*w)^{-1} z^*\eta \\ -(I_q + \eta\theta^*)^{-1}\,\theta^*\eta & I_m \end{pmatrix} \times \begin{pmatrix} I_n + z^*(I_m + \eta\phi^*)^{-1} w & 0 \\ 0 & I_q + \theta^*(I_m + wz^*)^{-1}\,\eta \end{pmatrix}^{-1}. \tag{IX.10}$$

It is clear that the only divergent matrix elements in this expression come from the matrix elements of $(I_n + z^* w)^{-1}$. This is precisely the divergent factor in the case of ordinary domains, and so the result follows from the proof of [6], Lemma 3.2(ii). ∎

LEMMA IX.2. *For $u, v \in B^\infty(\mathscr{D})$, and $\phi \in \mathscr{H}_r(\mathscr{D})$, we have*

$$\left|\int_{\mathscr{D}} u(W)\, v(W)\, \phi(W)\, d\mu_r(W)\right| \leqslant C\|\phi\|\, \|v\|_0 \sum_{\alpha,\beta} r^{-(|\alpha|+|\beta|)/2} \left\{\int_{\mathscr{D}} |u_{\alpha\beta}(w)|^2\, d\mu_r(W)\right\}^{1/2}. \tag{IX.11}$$

Proof. We write

$$\begin{aligned}\left|\int_{\mathscr{D}} u(W)\, v(W)\, \phi(W)\, d\mu_r(W)\right| &\leqslant \sum_{\alpha,\beta,\rho,\delta} \left|\int_{\mathscr{D}} u_{\alpha\beta}(w)\, \bar{\eta}^\alpha \eta^\beta v_{\rho\delta}(w)\, \bar{\eta}^\rho \eta^\delta \phi(W)\, d\mu_r(W)\right| \\ &= \sum_{\alpha,\beta,\rho,\delta} |(\bar{u}_{\alpha\beta}(w)\, \eta^\alpha \eta^\rho,\, v_{\rho\delta}(w)\, \eta^\beta \eta^\delta \phi(W))|. \end{aligned} \tag{IX.12}$$

By virtue of Theorem VII.1 we can apply the Schwarz inequality to this expression to obtain

$$\left|\int_{\mathscr{D}} u(W)\, v(W)\, \phi(W)\, d\mu_r(W)\right| \leqslant \sum_{\alpha,\beta,\rho,\delta} \|v_{\rho\delta}\|_\infty\, \|\bar{u}_{\alpha\beta}(w)\, \eta^\alpha \eta^\rho\|\, \|\eta^\beta \eta^\delta \phi(W)\|. \tag{IX.13}$$

By Lemma VII.11 we then have

$$\begin{aligned}\left|\int_{\mathscr{D}} u(W)\, v(W)\, \phi(W)\, d\mu_r(W)\right| &\leqslant C \sum_{\alpha,\beta,\rho,\delta} r^{-(|\alpha|+|\beta|+|\rho|+|\delta|)/2} \|v_{\rho\delta}\|_\infty\, \|\bar{u}_{\alpha\beta}\|\, \|\phi\| \\ &\leqslant C\|\phi\|\, \|v\|_0 \sum_{\alpha,\beta} r^{(|\alpha|+|\beta|)/2} \|\bar{u}_{\alpha\beta}\|. \quad ∎ \end{aligned} \tag{IX.14}$$

IX.C. *Proof of Theorem* VII.14. Our procedure will be to expand

$$(\phi, T_r(f)\, T_r(g)\, \psi) = \int_{\mathscr{D}\times\mathscr{D}} \overline{\phi(Z)} f(Z)\, K^r(Z, X)\, g(X)\, \psi(X)\, d\mu_r(Z)\, d\mu_r(X), \tag{IX.15}$$

where $\psi, \phi \in \mathscr{H}_r(\mathscr{D})$, $f, g \in B^\infty(\mathscr{D})$, in a power series in r [13]. We make the substitution $X = \gamma_Z(W)$, and use the transformation properties of the Bergman kernel to rewrite (IX.15) as

$$(\phi, T_r(f)\, T_r(g)\, \psi)$$
$$= \int_{\mathscr{D}\times\mathscr{D}} \overline{\phi(Z)} f(Z) \frac{K^r(Z,Z)}{K^r(\gamma_Z(W), Z)} g(\gamma_Z(W))\, \psi(\gamma_Z(W))\, d\mu_r(Z)\, d\mu_r(W). \tag{IX.16}$$

The next step will be to expand $g(\gamma_Z(W))$ in a Taylor series. We will need to expand out to order m, where m is an integer such that $m > n_0 + 4$. The Taylor expansion for superfunctions is

$$\begin{aligned} g(\gamma_Z(W)) = g(Z) &+ \sum_{\mu,\kappa} (W_\kappa \gamma'_Z(0)_{\kappa\mu}\, \partial_\mu g(Z) + \overline{W_\kappa \gamma'_Z(0)_{\kappa\mu}}\, \bar\partial_\mu g(Z)) \\ &+ \tfrac{1}{2} \sum_{\mu,\nu,\kappa,\rho} W_\kappa \gamma'_Z(0)_{\kappa\mu}\, W_\nu \gamma'_Z(0)_{\nu\rho}\, \partial_\rho \partial_\mu g(Z) \\ &+ \tfrac{1}{2} \sum_{\mu,\kappa,\rho} W_\kappa W_\rho \Gamma_{\rho\kappa\mu}(Z)\, \partial_\mu g(Z) \\ &+ \sum_{\mu,\nu,\kappa,\rho} \overline{W_\kappa \gamma'_Z(0)_{\kappa\mu}}\, W_\rho \gamma'_Z(0)_{\rho\nu}\, \partial_\nu \bar\partial_\mu g(Z) \\ &+ \tfrac{1}{2} \sum_{\mu,\nu,\kappa,\rho} \overline{W_\kappa \gamma'_Z(0)_{\kappa\mu}}\, \overline{W_\rho \gamma'_Z(0)_{\rho\nu}}\, \bar\partial_\nu \bar\partial_\mu g(Z) \\ &+ \tfrac{1}{2} \sum_{\mu,\kappa,\rho} \overline{W_\kappa W_\rho \Gamma_{\rho\kappa\mu}(Z)}\, \bar\partial_\mu g(Z) \\ &+ \text{terms of order 3 through } m-1 \\ &+ G(Z, W), \end{aligned} \tag{IX.17}$$

where $\partial_\mu := \partial/\partial Z_\mu$ and

$$\Gamma_{\rho\kappa\mu}(Z) := \frac{\partial}{\partial W_\rho} \frac{\partial}{\partial W_\kappa} \gamma_Z(W)_\mu \Big|_{W=0}, \tag{IX.18}$$

and the mth order remainder term is given by

$$G(Z, W) := \frac{1}{(m-1)!} \int_0^1 ds (1-s)^{(m-1)} \frac{d^m}{ds^m} g(\gamma_Z(sW)). \tag{IX.19}$$

Denote by $I_{a,b}$ the contribution to the integral from the term in the expansion of g with a powers of W and b powers of $\bar W$, and let R denote the contribution of the remainder term. In evaluating these terms we will

make use of the following facts. Given a holomorphic function χ on $\mathscr{D}$, we have remarked before that

$$\int_{\mathscr{D}} \chi(W)\, d\mu_r(W) = \chi(0). \tag{IX.20}$$

Furthermore, using the circular symmetry and Proposition VII.8 we obtain

$$\begin{aligned}\int_{\mathscr{D}} \bar{W}_\mu \chi(W)\, d\mu_r(W) &= \frac{\partial \chi}{\partial W_\mu}(0) \int_{\mathscr{D}} \bar{W}_\mu W_\mu\, d\mu_r(W) \\ &= \frac{\beta_\mu}{r} \frac{\partial \chi}{\partial W_\mu}(0),\end{aligned} \tag{IX.21}$$

for any μ.

For the lowest order term in the expansion, we have

$$I_{0,0} = \int_{\mathscr{D}\times\mathscr{D}} \overline{\phi(Z)} f(Z) \frac{K^r(Z,Z)}{K^r(\gamma_Z(W), Z)} g(Z)\, \psi(\gamma_Z(W))\, d\mu_r(Z)\, d\mu_r(W). \tag{IX.22}$$

The integrand is holomorphic in W, so apply (IX.20) to get

$$\begin{aligned}I_{0,0} &= \int_{\mathscr{D}} \overline{\phi(Z)} f(Z)\, g(Z)\, \psi(Z)\, d\mu_r(Z) \\ &= (\phi, T_r(fg)\,\psi).\end{aligned} \tag{IX.23}$$

The same fact (IX.20) also clearly implies that $I_{a,b} = 0$ for $a > b$.

The next nonzero term in the expansion is thus $I_{0,1}$, which is given by

$$\begin{aligned}I_{0,1} = \sum_{\mu,\kappa} \int_{\mathscr{D}\times\mathscr{D}} &\overline{\phi(Z)} f(Z)\, \overline{W_\kappa \gamma_Z'(0)_{\kappa\mu}}\, \bar{\partial}_\mu g(Z) \frac{\psi(\gamma_Z(W))}{K^r(\gamma_Z(W), Z)} \\ &\times K^r(Z,Z)\, d\mu_r(Z)\, d\mu_r(W).\end{aligned} \tag{IX.24}$$

We now apply (IX.21), to obtain

$$\begin{aligned}I_{0,1} = \frac{1}{r} \sum_{\mu,\kappa} (-1)^{\varepsilon_\kappa (p(g)+\varepsilon_\mu)} \beta_\kappa \int_{\mathscr{D}} &\overline{\phi(Z)} f(Z)\, \overline{\gamma_Z'(0)_{\kappa\mu}}\, \bar{\partial}_\mu g(Z) \\ &\times \left[\frac{\partial}{\partial W_\kappa} \frac{\psi(\gamma_Z(W))}{K^r(\gamma_Z(W), Z)}\right]_{W=0} K^r(Z,Z)\, d\mu_r(Z),\end{aligned} \tag{IX.25}$$

where the sign arises from the permutation of elements of the integrand (keeping in mind the fact that $\overline{WT} = \bar{T}\bar{W}$). Applying the chain rule gives

$$I_{0,1} = \frac{1}{r} \sum_{\mu,\nu,\kappa} (-1)^{\varepsilon_\kappa (p(g)+\varepsilon_\mu)} \beta_\kappa \int_{\mathscr{S}} \overline{\phi(Z)} f(Z)\, \overline{\gamma'_Z(0)_{\kappa\mu}}\, \bar{\partial}_\mu g(Z)\, \gamma'_Z(0)_{\kappa\nu}$$
$$\times \partial_\nu \left[\frac{\psi(Z)}{K^r(Z,Z)} \right] K^r(Z,Z)\, d\mu_r(Z)$$
$$= \frac{1}{r} \sum_{\mu,\nu} (-1)^{(\varepsilon_\mu+\varepsilon_\nu)p(f)+\varepsilon_\nu(p(g)+\varepsilon_\mu)} \int_{\mathscr{S}} \overline{\phi(Z)}\, P_{\mu\nu}(Z) f(Z)\, \bar{\partial}_\mu g(Z)$$
$$\times \partial_\nu \left[\frac{\psi(Z)}{K^r(Z,Z)} \right] K^r(Z,Z)\, d\mu_r(Z). \tag{IX.26}$$

Noting that

$$K^r(Z,Z)\, d\mu_r(Z) = N(Z,Z)^{\ p}\, dZ, \tag{IX.27}$$

we integrate by parts as follows:

$$I_{0,1} = -\frac{1}{r} \sum_{\mu,\nu} (-1)^{\varepsilon_\mu p(f)+\varepsilon_\nu(\varepsilon_\mu+1)} \int_{\mathscr{S}} \overline{\phi(Z)}\, \partial_\nu \left[\frac{P_{\mu\nu}(Z)}{N(Z,Z)^p} f(Z)\, \bar{\partial}_\mu g(Z) \right]$$
$$\times \psi(Z)\, N(Z,Z)^p\, d\mu_r(Z)$$
$$= -\frac{1}{r} \sum_{\mu,\nu} (-1)^{\varepsilon_\mu p(f)+\varepsilon_\nu(\varepsilon_\mu+1)} \int_{\mathscr{S}} \overline{\phi(Z)}\, \partial_\nu \left[\frac{P_{\mu\nu}(Z)}{N(Z,Z)^p} \right] f(Z)\, \bar{\partial}_\mu g(Z)$$
$$\times \psi(Z)\, N(Z,Z)^p\, d\mu_r(Z)$$
$$-\frac{1}{r} \sum_{\mu,\nu} (-1)^{\varepsilon_\mu p(f)} \int_{\mathscr{S}} \overline{\phi(Z)}\, P_{\mu\nu}(Z)\, \partial_\nu f(Z)\, \bar{\partial}_\mu g(Z)\, \psi(Z)\, d\mu_r(Z)$$
$$-\frac{1}{r} \sum_{\mu,\nu} (-1)^{(\varepsilon_\mu+\varepsilon_\nu)p(f)} \int_{\mathscr{S}} \overline{\phi(Z)}\, P_{\mu\nu}(Z) f(Z)\, \partial_\nu \bar{\partial}_\mu g(Z)\, \psi(Z)\, d\mu_r(Z). \tag{IX.28}$$

Observe that, as a consequence of the assumption that r is sufficiently large, no boundary terms are present. As a consequence of Theorem V.7,

$$\sum_\nu (-1)^{\varepsilon_\nu(\varepsilon_\mu+1)} \partial_\nu \left[\frac{P_{\mu\nu}(Z)}{N(Z,Z)^p} \right] = 0. \tag{IX.29}$$

This leaves two terms in (IX.28).

Now consider the term $I_{1,1}$, which is given by

$$I_{1,1} = \sum_{\mu,\nu,\kappa,\rho} \int_{\mathscr{D}\times\mathscr{D}} \overline{\phi(Z)} f(Z) \frac{K^r(Z,Z)}{K^r(\gamma_Z(W),Z)} \overline{W_\kappa \gamma'_Z(0)_{\kappa\mu}} \times W_\rho \gamma'_Z(0)_{\rho\nu} \partial_\nu \bar{\partial}_\mu g(Z) \psi(\gamma_Z(W))\, d\mu_r(Z)\, d\mu_r(W). \quad \text{(IX.30)}$$

Using (IX.20) and (IX.21), we can perform the W integration to get

$$I_{1,1} = \frac{1}{r} \sum_{\mu,\nu} \int_{\mathscr{D}} \overline{\phi(Z)} f(Z) P_{\mu\nu}(Z) \partial_\nu \bar{\partial}_\mu g(Z) \psi(Z)\, d\mu_r(Z)$$
$$= \frac{1}{r} \sum_{\mu,\nu} (-1)^{(\varepsilon_\mu+\varepsilon_\nu)p(f)} \int_{\mathscr{D}} \overline{\phi(Z)} P_{\mu\nu}(Z) f(Z) \partial_\nu \bar{\partial}_\mu g(Z) \psi(Z)\, d\mu_r(Z). \quad \text{(IX.31)}$$

This exactly cancels the third term in (IX.28), so that we finally obtain

$$I_{0,1} + I_{1,1} = \frac{1}{r} \sum_{\mu,\nu,\kappa} (-1)^{\varepsilon_\mu p(f)+1} (\phi, T_r(P_{\mu\nu} \partial_\nu f \bar{\partial}_\mu g)\psi). \quad \text{(IX.32)}$$

All that remains to complete the proof is to bound the other terms as $r \to \infty$. Of the remaining second order terms, $I_{2,0} = 0$ and $I_{0,2}$ is given by

$$I_{0,2} = \tfrac{1}{2} \int_{\mathscr{D}\times\mathscr{D}} \overline{\phi(Z)} f(Z) K^r(Z,Z) K^r(\gamma_Z(W),Z)^{-1} \times \Bigg[\sum_{\mu,\nu,\kappa,\rho} \overline{W_\kappa \gamma'_Z(0)_{\kappa\mu}}\, \overline{W_\rho \gamma'_Z(0)_{\rho\nu}}\, \bar{\partial}_\nu \bar{\partial}_\mu g(Z) + \sum_{\mu,\kappa,\rho} \overline{W_\kappa W_\rho \Gamma_{\rho\kappa\mu}(Z)}\, \bar{\partial}_\mu g(Z) \Bigg] \psi(\gamma_Z(W))\, d\mu_r(Z)\, d\mu_r(W). \quad \text{(IX.33)}$$

We want to bound this term for large r. To do this, we first evaluate the integration over W using the principles of (IX.20) and (IX.21). For this integral we obtain

$$\int_{\mathscr{D}} K^r(\gamma_Z(W),Z)^{-1} \overline{W_\kappa W_\rho} \psi(\gamma_Z(W))\, d\mu_r(W) = \frac{1}{2} \sum_{\mu,\nu} \frac{\partial^2}{\partial W_\nu \partial W_\mu} \frac{\psi(\gamma_Z(W))}{K^r(\gamma_Z(W),Z)} \Bigg|_{W=0} \int_{\mathscr{D}} \overline{W_\kappa W_\rho} W_\mu W_\nu\, d\mu_r(W). \quad \text{(IX.34)}$$

The convergence factor comes from the integral on the right-hand side of (IX.34). We can apply the positivity property of the measure and the Schwarz inequality to give

$$\left|\int_{\mathscr{D}} \overline{W_\kappa W_\rho} W_\mu W_\nu \, d\mu_r(W)\right| \leqslant \int_{\mathscr{D}} \left(\sum_\mu \bar{W}_\mu W_\mu\right)^2 d\mu_r(W). \quad \text{(IX.35)}$$

Because of Proposition VII.3, we can apply the fact [6, Lemma 3.1 (ii)] that

$$\frac{\int_D (\sum_\mu |w_\mu|^2)^k \, \Delta(z,z)^{r-p_0} \, dz}{\int_D \Delta(z,z)^{r-p_0} \, dz} \leqslant Cr^{-k}, \quad \text{(IX.36)}$$

together with Proposition VII.12, to see that

$$\int_{\mathscr{D}} \left(\sum_\mu \bar{W}_\mu W_\mu\right)^k d\mu_r(W) \leqslant Cr^{-k}. \quad \text{(IX.37)}$$

Substituting (IX.34) into (IX.33), we convert the derivatives with respect to W at zero into derivatives with respect to Z using the chain rule. We then integrate by parts to move these derivatives off of the $\psi(Z)$, as in the analysis of $I_{0,1}$. These derivatives then act on the expression

$$f(Z) N(Z,Z)^{-p} [\overline{\gamma'_Z(0)_{\kappa\mu}} \, \overline{\gamma'_Z(0)_{\rho\nu}} \, \bar{\partial}_\nu \bar{\partial}_\mu g(z) + \overline{\Gamma_{\rho\kappa\mu}(Z)} \, \bar{\partial}_\mu g(Z)]. \quad \text{(IX.38)}$$

The derivatives of N, γ'_Z, and Γ have potential singularities. In view of Lemma IX.1 we can bound the absolute values of the components of these terms by $\Delta(z,z)^{-s}$ for some integer s. Then, since the supports of the components of the function f are restricted to some compact set S_f, we can bound the $\|\cdot\|_0$ norm of the derivatives of (IX.38) by

$$C\|f\|_t \, \|g\|_t \sup_{S_f} \Delta(z,z)^{-s}, \quad \text{(IX.39)}$$

for some t. Using this bound in conjunction with Proposition VII.12, we thus have

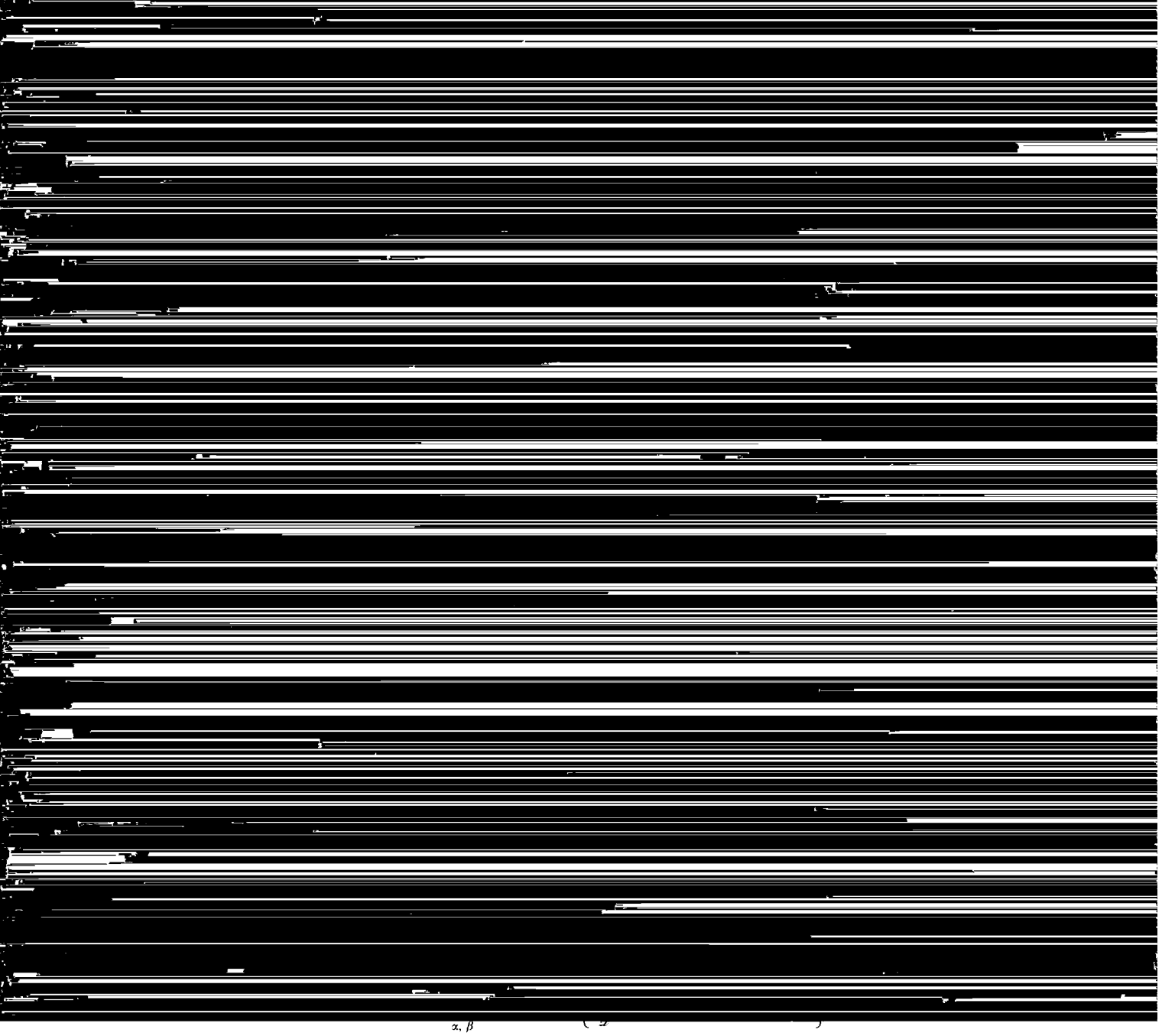
$$|I_{0,2}| \leqslant C_{S_f} r^{-2} \|f\|_t \, \|g\|_t \, \|\psi\| \, \|\phi\|, \quad \text{(IX.40)}$$

where the r^{-2} comes from the convergence factor (IX.37) and the constant C_{S_f} depends on S_f.

The same reasoning applies to the cases $I_{a,b}$ where $3 \leqslant a+b < m$. The convergence factor comes from (IX.37). The result is that

$$|I_{a,b}| \leqslant Cr^{-2} \|f\|_t \, \|g\|_t \, \|\psi\| \, \|\phi\|, \quad \text{(IX.41)}$$

for some t and for $3 \leqslant a+b \leqslant m$.



α, β

where $u(Z)$ is the function

$$u(Z) = K'(Z, Z)^{1/2} \int_{\mathscr{D}} G(Z, W)\, \psi_Z(W)\, d\mu_r(W), \qquad \text{(IX.48)}$$

and X_{S_f} is the characteristic function of the compact set S_f in which the components of f are supported. Now, to bound the components of u, we apply Lemma IX.2 to the W integration using the bound (IX.46). In this way we find

$$|u_{\alpha\beta}(z)| \leqslant C\|\psi\| \, \|g\|_t \, \Delta(z,z)^{s'} \, \|K^r(Z,Z)^{1/2}\|_0 \times \sum_{\gamma\delta} r^{-(|\gamma|+|\delta|)/2} \left[\int_{\mathscr{D}} |w|^{2(m-|\gamma|-|\delta|)} \Delta(w,w)^{-2s} \, d\mu_r(W) \right]^{1/2}. \tag{IX.49}$$

For the remaining integral over W, we have

$$\int_{\mathscr{D}} |w|^{2(m-|\gamma|-|\delta|)} \Delta(w,w)^{-2s} \, d\mu_r(W) = \int_{\mathscr{D}} |w|^{2(m-|\gamma|-|\delta|)} \, d\mu_{r'}(W), \tag{IX.50}$$

where r' and r differ by a constant. We can apply [6, Lemma 3.1(ii)] to bound this expression by a constant times $r^{-m+|\gamma|+|\delta|}$. Returning to (IX.49), since $N(Z,Z) = \Delta(z,z) + \textit{nilpotent}$, the components of $K^r(Z,Z)^{1/2}$ can be bounded by $\Delta(z,z)^{-r/2-s''}$ for some s'' (the s'' occurs when we Taylor expand $(\Delta + \textit{nilpotent})^{-1}$). We thus have

$$|u_{\alpha\beta}(z)| \leqslant Cr^{-m/2}\|\psi\| \, \|g\|_t \, \Delta(z,z)^{-r/2-s'-s''}. \tag{IX.51}$$

Applying these results to (IX.47), we find that

$$|R| \leqslant Cr^{-m/2}\|g\|_t \, \|f\|_0 \, \|\phi\| \, \|\psi\| \left| \int_{\mathscr{D}} X_{S_f}(z) \, \Delta(z,z)^{-r-2(s'+s'')} \, d\mu_r(Z) \right|^{1/2}. \tag{IX.52}$$

The θ integration in the remaining integral can be estimated using Proposition VII.3:

$$\int_{\mathscr{D}} X_{S_f}(z) \, \Delta(z,z)^{-r-2(s'+s'')} \, d\mu_r(Z) = Cr^{n_1} \Lambda_r \int_{S_f} \Delta(z,z)^{-p_0-2(s'+s'')} \, dz[1+O(r^{-1})]. \tag{IX.53}$$

The integral over S_f is finite and independent of r, so we can absorb it into the constant. According to Proposition VII.4, the normalization constant Λ_r can be bounded by a constant times $r^{n_0-n_1}$ as $r \to \infty$. Applying all of this to (IX.52), we have

$$|R| \leqslant C_{S_f} r^{-(m-n_0)/2}\|g\|_t \, \|f\|_t \, \|\phi\| \, \|\psi\|. \tag{IX.54}$$

With the fact that $m - n_0 > 4$, this completes the proof. ∎

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