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Quantized Chaotic Dynamics and Non-commutative KS Entropy

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We study the quantization of two examples of classically chaotic dynamics, the Anosov dynamics of "cat maps" on a two dimensional torus, and the dynamics of baker's maps. Each of these dynamics is implemented as a discrete group of automorphisms of a von Neumann algebra of functions on a quantized torus. We compute the non-commutative generalization of the Kolmogorov–Sinai entropy, namely the Connes–Størmer entropy, of the generator of this group, and find that its value is equal to the classical value. This can be interpreted as a sign of persistence of chaotic behavior in a dynamical system under quantization.

I. INTRODUCTION

I.A. One of the characteristic features of chaos in classical dynamics is the positivity of the Kolmogorov–Sinai (KS) entropy. The KS entropy is a natural measure of mixing in phase space resulting from the time evolution of a dynamical system. Indeed, one can adopt the positivity of the KS entropy as a convenient way of defining chaos in a classical dynamical system. Through Pesin's theorem, this is related to another characteristic feature of chaotic evolution, namely the positivity of Lyapunov exponents.

The focus of the emerging field of "quantum chaology" [B2], [HT], [N], [V2], is the study of quantum dynamics arising from quantization of classically chaotic systems. Much emphasis has been put on understanding the semiclassical approximation to the actual quantum dynamics, and it is, in fact, a somewhat controversial issue whether "quantum chaos" exists beyond this approximation.

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In this paper we propose that a natural quantity to exhibit quantum chaos in a class of quantized dynamics is the positivity of the Connes–Størmer (CS) entropy. The CS entropy is defined in the context of von Neumann algebras, and is a natural extension of the KS entropy to the non-commutative context. We focus our attention on examples of quantized dynamics on a torus, namely the dynamics of quantized cat maps and the dynamics of quantized baker's maps, and show that in each case the CS entropy is positive and, in fact, equal to the classical value.

I.B. We begin by recalling the definition of the (classical) KS entropy. Let M be the phase space on which a probability measure ν and ν-preserving automorphism $\varphi\colon M \to M$ are defined. The latter means that φ is a measurable bijective function such that for all measurable sets $\mathcal{O}$, $\nu(\varphi(\mathcal{O})) = \nu(\mathcal{O})$. Let $\mathcal{A} = \{A_j\}$, $1 \leqslant j \leqslant n$, be a finite partition of M into measurable and pairwise disjoint (up to measure zero) subsets. The entropy of this partition is defined by

$$H(\mathcal{A}) = \sum_j \eta(\nu(A_j)), \tag{I.1}$$

where the function η is given by

$$\eta(t) = -t \log t, \qquad 0 \leqslant t \leqslant 1. \tag{I.2}$$

Clearly, H is invariant under φ,

$$H(\varphi(\mathcal{A})) = H(\mathcal{A}), \tag{I.3}$$

where $\varphi(\mathcal{A}) = \{\varphi(A_1), ..., \varphi(A_n)\}$. Now, given two such partitions, $\mathcal{A}$ and $\mathcal{B}$, we form a finer partition $\mathcal{A} \vee \mathcal{B}$ by taking the intersections of the elements of $\mathcal{A}$ with the elements of $\mathcal{B}$. The entropy is subadditive with respect to the operation $\vee$,

$$H(\mathcal{A} \vee \mathcal{B}) \leqslant H(\mathcal{A}) + H(\mathcal{B}). \tag{I.4}$$

This and (I.3) imply that the limit

$$H(\mathcal{A}, \varphi) = \lim_{n \to \infty} \frac{1}{n} H(\mathcal{A} \vee \varphi(\mathcal{A}) \vee \cdots \vee \varphi^{n-1}(\mathcal{A})) \tag{I.5}$$

exists. The KS entropy of φ is defined as the supremum of $H(\mathcal{A}, \varphi)$ over all possible choices of the finite partition $\mathcal{A}$,

$$h_{KS}(\varphi) = \sup_{\mathcal{A}} H(\mathcal{A}, \varphi). \tag{I.6}$$

This definition does not lend itself to explicit computations. However, the fundamental theorem of Kolmogorov and Sinai [CFS] states that, in fact, $h_{KS}(\varphi)$ can be computed from a single partition, provided that it is sufficiently generic. More precisely, $h_{KS}(\varphi) = H(\mathcal{A}, \varphi)$, if $\mathcal{A}$ is a partition such that the sets $\varphi^k(A_j)$, $j = 1, ..., n$, $k \in \mathbb{Z}$, generate the σ-algebra of measurable sets on M.

We will explain in Section V how Connes and Størmer generalized the theory outlined above to the non-commutative case.

I.C. For later convenience we now briefly review the definitions of the classical cat map and baker's map. For a more complete presentation and a variety of results we refer the reader to [A], [AW], and [CFS].

We consider an element $\gamma \in SL(2, \mathbb{Z})$,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{I.7}$$

with $|\mathrm{tr}(\gamma)| > 2$. Such a matrix has two eigenvalues μ_1, μ_2, with $\mu_1 \mu_2 = 1$. We label them so that $|\mu_1| > 1$, and $|\mu_2| < 1$. The action of γ on the plane $\mathbb{R}^2$ is given as usual by $(x_1, x_2) \to (y_1, y_2)$ with

$$\begin{aligned} y_1 &= a x_1 + b x_2, \\ y_2 &= c x_1 + d x_2. \end{aligned} \tag{I.8}$$

For later reference, we rewrite (I.8) in terms of the complex variable $z = (x_1 + i x_2)/\sqrt{2}$ as $z \to w$, with

$$w = \bar{\alpha} z + \beta \bar{z}, \tag{I.9}$$

where the complex parameters α and β are given by

$$\begin{aligned} \alpha &= (a + d + i(b - c))/2, \\ \beta &= (a - d + i(b + c))/2, \end{aligned} \tag{I.10}$$

and satisfy $|\alpha|^2 - |\beta|^2 = 1$. The transformation (I.8) is area preserving. Since the coefficients in (I.8) are integer, γ also defines an area preserving automorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ which we will denote by the same symbol γ. The group $\{\gamma^n\}_{n \in \mathbb{Z}}$ of automorphisms of $\mathbb{T}^2$ is called the cat dynamics (in fact, this is an example of Anosov dynamics).

The definitions above are of course meaningful without assuming that $|\mathrm{tr}(\gamma)| > 2$. The resulting dynamical systems are non-chaotic, and, as such, less relevant to the subject of this paper.

It turns out that for the cat dynamics,

$$h_{KS}(\gamma) = \log |\mu_1|, \tag{I.11}$$

where μ_1 is the eigenvalue of γ whose absolute value is larger than 1. A beautiful proof of this result in the context of symbolic dynamics is presented in [AW]. If $|\mathrm{tr}(\gamma)| \leqslant 2$, then $h_{KS}(\gamma) = 0$, showing that the corresponding dynamics is indeed non-chaotic.

The baker's map B takes a point (x_1, x_2) of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ to a point (x'_1, x'_2) of $\mathbb{T}^2$ given by

$$x'_1 = \begin{cases} 2x_1, & \text{if} \quad 0 \leqslant x_1 < 1/2; \\ 2x_1 - 1, & \text{if} \quad 1/2 \leqslant x_1 < 1, \end{cases}$$
$$x'_2 = \begin{cases} x_2/2, & \text{if} \quad 0 \leqslant x_1 < 1/2; \\ (x_2 + 1)/2, & \text{if} \quad 1/2 \leqslant x_1 < 1. \end{cases} \tag{I.12}$$

The transformation B is measure preserving. In order to prepare ground for the quantization of B, we first rewrite (I.12) in terms of generators of the algebra $L^\infty(\mathbb{T}^2)$ of essentially bounded functions on $\mathbb{T}^2$. We set $g(x_1, x_2) = e^{2\pi i x_1}$, $h(x_1, x_2) = e^{2\pi i x_2}$. Then the transformation (I.12) of $\mathbb{T}^2$ is equivalent to the following automorphism of the algebra $L^\infty(\mathbb{T}^2)$ (which, for simplicity, is denoted by the same symbol B):

$$B(g) = g^2,$$
$$B(h) = \sqrt{h}\,(2\chi_{[0,\,1/2)}(x_1) - 1), \tag{I.13}$$

where the square root $\sqrt{h}$ is defined by $\sqrt{h}\,(x_1, x_2) = e^{i\pi x_2}$, and where $\chi_{[0,\,1/2)}$ is the indicator function of the interval $[0, 1/2)$.

For the baker's map,

$$h_{KS}(B) = \log 2. \tag{I.14}$$

I.D. One of the central concepts of this paper is that of quantization of a dynamical system. Without getting involved with technicalities we would like to emphasize several points which will explain the particular conceptual framework which we chose to work with.

Quantization of a dynamical system has two components: kinematic and dynamic. The kinematic component of quantization involves the construction of a suitable quantized phase space of the system. This quantized phase space is given in terms of a non-commutative algebra $\mathfrak{A}_h$ of observables. In the language of non-commutative geometry, $\mathfrak{A}_h$ is an algebra of functions on the quantized phase space. Very much like in the classical situation, where (depending on the problem) one might be interested in the study of the algebra of continuous functions, smooth functions, compactly supported smooth functions, measurable functions, etc., specific choices of the composition of $\mathfrak{A}_h$ can be made. This may result in imposing the structure of a $\mathbb{C}^*$-algebra, a von Neumann algebra or some suitably defined locally convex algebra, on the algebra of observables.

The dynamic component of quantization consists in defining a time evolution on the quantized phase space. A natural way of doing this is to find a suitable one parameter group of automorphisms of $\mathfrak{A}_h$, where the parameter (discrete or continuous) has the meaning of time. Recall that an automorphism of an algebra $\mathfrak{R}$ is a linear one-to-one map Φ of $\mathfrak{R}$ onto itself such that $\Phi(ab) = \Phi(a)\,\Phi(b)$, for all a, $b \in \mathfrak{R}$. If $\mathfrak{R}$ is an algebra with involution, it is also required that $\Phi(a^*) = \Phi(a)^*$.

The "suitability" of the choices made, namely that of the algebra $\mathfrak{A}_\hbar$ and of the time evolution, is settled by the correspondence principle. This amounts to showing that limits of the quantized objects, as $\hbar \to 0$, yield the corresponding classical objects. Quantization is a highly non-unique procedure, and the correspondence principle is the only physical principle allowing one to decide whether a particular procedure is correct. To our taste, the most satisfying mathematical framework for quantization is that of "strict deformation quantization" proposed in [R1].

I.E. Quantization of the cat dynamics on the torus has been discussed before by a number of authors. The original reference is [HB], where a scheme is proposed using a group of unitary matrices on a finite dimensional Hilbert space. The generator of this group was determined from (i) the observation that the generating function of (I.8) is quadratic, and (ii) the assumption that, in the quadratic case, the semiclassical expressions are exact. This quantized dynamics was further studied in [K1, 2], [MO], [DGI], [BD], and [D], where a variety of beautiful number theoretic results were derived.

A similar quantization scheme for baker's dynamics was first proposed in [BV], and further refined and studied e.g. in [CTH], [SV], [S], and [BDG]. These references are concerned with questions of quantum chaology. The intrinsic simplicity of the baker's dynamics has been very useful in studying these questions.

Our approach is slightly different, even though equivalent in the sense specified at the end of previous subsection. It is based on an infinite dimensional Hilbert space. The infinite dimensionality of the Hilbert space is due to the occurrence of Θ-vacua (to use the language of quantum gauge field theory), which in turn is a consequence of the fact that the phase space of the system, namely the torus, is not simply connected. We study a non-abelian algebra, known as the algebra of functions on a quantized torus [R2], and identify a suitable group of automorphisms of this algebra as the quantized dynamics.

I.F. The paper is organized as follows. In Section II, we define the quantized linear dynamics on the plane. This will be the starting point for the construction of quantized cat dynamics. In Section III, we review the construction of the quantized torus, and show that the cat dynamics on the torus defines a group of automorphisms of the quantized torus. This group is the quantized cat dynamics on the torus. A construction of quantized baker's dynamics is described in Section IV. Section V has largely a review character. We explain the properties and construction of the CS entropy, and establish a technical lemma. Using this lemma, we compute, in Section VI, the CS entropy of the quantized dynamics on the torus.

II. Quantized Linear Dynamics on the Plane

II.A. Of the many representations of quantum mechanics we choose the Bargmann representation (see e.g. [F]), as in this representation wave functions are defined on the phase space of the system. It can also be generalized to phase spaces other than flat spaces [B1], which should be important for future extensions of the results of this paper. In the Bargmann representation, the Hilbert space of states $\mathscr{H}^2(\mathbb{C}, d\mu_\hbar)$ consists of entire functions on $\mathbb{C}$ which are square integrable with respect to the probability measure $d\mu_\hbar(z) = (\pi\hbar)^{-1} \exp\{-|z|^2/\hbar\}\, d^2z$. This Hilbert space has two remarkable properties: (i) it has a reproducing kernel, namely the function $\exp\{\bar{w}z/\hbar\} \in \mathscr{H}^2(\mathbb{C}, d\mu_\hbar)$ satisfies the equation

$$\int_{\mathbb{C}} \exp\{\bar{w}z/\hbar\}\, \phi(w)\, d\mu_\hbar(w) = \phi(z), \tag{II.1}$$

for all $\phi \in \mathscr{H}^2(\mathbb{C}, d\mu_\hbar)$, and (ii) it carries a unitary projective representation of the group of translations of $\mathbb{C}$ given by

$$U(\zeta)\, \phi(z) = \exp\left\{\frac{1}{\hbar}(\bar{\zeta}z - |\zeta|^2/2)\right\} \phi(z-\zeta), \quad \zeta \in \mathbb{C}. \tag{II.2}$$

For future reference, we note that

$$U(\zeta)\, U(\xi) = e^{i\,\mathrm{Im}(\bar{\zeta}\xi)/\hbar}\, U(\zeta+\xi). \tag{II.3}$$

The algebra of observables (or functions on the quantized plane) can be defined as an algebra generated by Toeplitz operators. A Toeplitz operator $T_\hbar(f)$ with symbol f (where f is a measurable function on $\mathbb{C}$) is defined by

$$T_\hbar(f)\, \phi(z) = \int_{\mathbb{C}} e^{z\bar{w}/\hbar} f(w)\, \phi(w)\, d\mu_\hbar(w). \tag{II.4}$$

Various restrictions on the class of symbols f may be imposed, leading to various algebras of operators on $\mathscr{H}^2(\mathbb{C}, d\mu_\hbar)$. Since the quantized plane is not the main concern of this paper, we ignore this issue, and refer the interested reader to e.g. [BC1, 2] for precise statements. See also [Z] for a related but more geometric approach. For our needs, it is only important that all bounded continuous functions are included in the class of symbols.

The Toeplitz operator with the symbol $f(z) = z$ is denoted by $A^\dagger$, and the Toeplitz operator with $f(z) = \bar{z}$ is denoted by A. These are the creation and annihilation operators obeying the usual commutation relation

$$[A, A^\dagger] = \hbar. \tag{II.5}$$

In fact, the quantization map $f \to T_\hbar(f)$ can be regarded as the anti-Wick ordering prescription, i.e., in the quantized expressions, all the annihilation operators are placed to the left of the creation operators.

II.B. To a γ as defined in the Introduction we assign the following Bogolubov transformation $(A^\dagger, A) \to (B^\dagger, B)$,

$$\begin{aligned} B^\dagger &= \bar{\alpha} A^\dagger + \beta A, \\ B &= \alpha A + \bar{\beta} A^\dagger. \end{aligned} \tag{II.6}$$

We will now show that this transformation is unitarily implementable, i.e. $B^\dagger = FA^\dagger F^{-1}$, and determine such a unitary F explicitly.

First, we note that the ground state $\omega_\gamma(z)$ for B satisfies the differential equation:

$$\hbar\alpha\omega'_\gamma(z) + \bar{\beta} z \omega_\gamma(z) = 0, \tag{II.7}$$

and so

$$\omega_\gamma(z) = |\alpha|^{-1/2} \exp\left\{-\frac{\bar{\beta} z^2}{2\hbar\alpha}\right\}, \tag{II.8}$$

where the normalizing constant has been chosen so that $\|\omega_\gamma\| = 1$. We require that F maps the function identically equal 1 (the ground state for A) to ω_γ. Then, using the Hausdorff–Baker–Campbell formula (see e.g. [F]),

$$\begin{aligned} F \exp\{\bar{w} z/\hbar\} &= F \exp\{\bar{w} A^\dagger/\hbar\}\ F^{-1} \omega_\gamma(z) \\ &= \exp\{\bar{w}(\bar{\alpha} A^\dagger + \beta A)/\hbar\}\ \omega_\gamma(z) \\ &= \exp\{(\overline{w\alpha}\, z + \bar{\alpha}\beta \bar{w}^2/2)/\hbar\}\ \exp\left\{\bar{w}\beta \frac{d}{dz}\right\} \omega_\gamma(z) \\ &= \exp\{(\overline{w\alpha}\, z + \bar{\alpha}\beta \bar{w}^2/2)/\hbar\}\ \omega_\gamma(z + \bar{w}\beta) \\ &= |\alpha|^{-1/2} \exp\{(\bar{w} z + \beta \bar{w}^2/2 - \bar{\beta} z^2/2)/\hbar\alpha\}. \end{aligned}$$

Using the fact that $\exp\{\bar{w} z/\hbar\}$ is the reproducing kernel for the measure $d\mu_\hbar$ we thus find that the action of F on $\phi \in \mathscr{H}^2(\mathbb{C}, d\mu_\hbar)$ is given by

$$\begin{aligned} F\phi(z) &= |\alpha|^{-1/2} \exp\left\{-\frac{\bar{\beta} z^2}{2\hbar\alpha}\right\} \int_{\mathbb{C}} \exp\left\{\frac{\bar{w} z}{\hbar\alpha} + \frac{\beta \bar{w}^2}{2\hbar\alpha}\right\} \phi(w)\, d\mu_\hbar(w) \\ &= T_\hbar(\omega_\gamma)\, S_{\gamma^{-1}} T_\hbar(\omega_{\gamma^{-1}})^*\, \phi(z), \end{aligned} \tag{II.9}$$

where S_γ is defined by $S_\gamma \phi(z) = |\alpha|^{1/2}\, \phi(z/\alpha)$. It is straightforward to verify that the inverse of F is given by

$$\begin{aligned} F^{-1}\phi(z) &= |\alpha|^{-1/2} \exp\left\{\frac{\bar{\beta} z^2}{2\hbar\bar{\alpha}}\right\} \int_{\mathbb{C}} \exp\left\{\frac{\bar{w} z}{\hbar\bar{\alpha}} - \frac{\beta \bar{w}^2}{2\hbar\bar{\alpha}}\right\} \phi(w)\, d\mu_\hbar(w) \\ &= T_\hbar(\omega_{\gamma^{-1}})\, S_\gamma T_\hbar(\omega_\gamma)^*\, \phi(z), \end{aligned} \tag{II.10}$$

and that F is unitary. Let us summarize the calculations above in the following theorem.

THEOREM II.1. *There exists a unique unitary operator F satisfying $FA^{\dagger}F^{-1}=B^{\dagger}$, and $F1=\omega_{\gamma}$. This operator and its inverse are given by equations* (II.9) *and* (II.10).

The group $\{F^n\}_{n\in\mathbb{Z}}$ of unitary operators on $\mathscr{H}^2(\mathbb{C}, d\mu_\hbar)$ is called the evolution group for the linear dynamics on the plane. The corresponding group of automorphisms of the algebra of observables is generated by $a\to FaF^{-1}$.

II.C. There is a simple relation between F and the unitary operators $U(\zeta)$ defined in (II.2).

THEOREM II.2. *The conjugation of $U(\zeta)$ by F is equal to $U(\gamma^{-1}\zeta)$,*

$$FU(\zeta)F^{-1}=U(\alpha\zeta-\beta\bar{\zeta}). \tag{II.11}$$

Proof. The proof is a straightforward computation. Using (II.9) and (II.10) we find that

$$\begin{aligned} FU(\zeta)\,F^{-1}\phi(z) &= |\alpha|^{-1}\exp\left\{-\frac{1}{2\hbar}\,(|\zeta|^2+\bar{\beta}z^2/\alpha-\bar{\beta}\zeta^2/\bar{\alpha})\right\} \\ &\quad\times\int_{\mathbb{C}}\exp\left\{\frac{1}{\hbar}\left(\frac{\beta\bar{w}^2}{2\alpha}+\frac{\bar{\beta}w^2}{2\bar{\alpha}}+\frac{z\bar{w}}{\alpha}+\frac{(\bar{\alpha}\bar{\zeta}-\bar{\beta}\zeta+\bar{v})\,w}{\alpha}+\frac{\bar{v}\zeta}{\alpha}+\frac{\beta\bar{v}^2}{2\bar{\alpha}}\right)\right\} \\ &\quad\times\phi(v)\;d\mu_\hbar(w)\;d\mu_\hbar(v). \end{aligned}$$

Evaluating the w-integral and using $|\alpha|^2-|\beta|^2=1$ yields

$$\begin{aligned} FU(\zeta)F^{-1}\phi(z) &= \exp\{(z\overline{(\alpha\zeta-\beta\bar{\zeta})}-|\alpha\zeta-\beta\bar{\zeta}|^2/2)/\hbar\} \\ &\quad\times\int_{\mathbb{C}}\exp\{(z-\alpha\zeta+\beta\bar{\zeta})\,\bar{v}/\hbar\}\;\phi(v)\;d\mu_\hbar(v), \end{aligned}$$

which by means of (II.1) is equal to

$$\exp\{z\overline{(\alpha\zeta-\beta\bar{\zeta})}/\hbar-|\alpha\zeta-\beta\bar{\zeta}|^2/2\hbar\}\;\phi(z-(\alpha\zeta-\beta\bar{\zeta}))=U(\gamma^{-1}\zeta)\;\phi(z),$$

as claimed. ∎

III. QUANTIZED CAT DYNAMICS ON THE TORUS

III.A. Having defined the quantized linear dynamics on the plane we now proceed to constructing the quantized cat dynamics on the torus. As explained e.g. in [KL], one can regard the quantized torus as a suitably defined quotient of the quantized plane by the group $\mathbb{Z}^2$. Namely, we define the algebra of observables on the quantized torus to be the algebra of all Toeplitz operators with continuous $\mathbb{Z}^2$-invariant symbols. Such symbols can be written as Fourier series, and so the algebra of observables is generated by $T_\hbar(f_1)$ and $T_\hbar(f_2)$, where $f_k(x_1, x_2) = \exp\{2\pi i x_k\}$. However, writing $ix_1 = i(z+\bar{z})/\sqrt{2}$, $ix_2 = (z-\bar{z})/\sqrt{2}$, we verify easily that

$$T(f_1) = e^{-\pi^2\hbar} U(-i\pi\hbar\sqrt{2}),$$
$$T(f_2) = e^{-\pi^2\hbar} U(\pi\hbar\sqrt{2}). \qquad \text{(III.1)}$$

It is thus natural to set

$$U = U(-i\hbar\pi\sqrt{2}),$$
$$V = U(\hbar\pi\sqrt{2}), \qquad \text{(III.2)}$$

and regard the operators U and V as generators of the algebra of functions on the quantized torus. Commutation relation (II.3) implies that they obey the following set of relations:

$$UU^* = U^*U = I,$$
$$VV^* = V^*V = I, \qquad \text{(III.3)}$$
$$UV = e^{i\lambda} VU,$$

where for convenience we set $\lambda = 4\pi^2\hbar$. The algebra generated by U and V with the relations above has been studied extensively by both physicists and mathematicians, and we refer the reader to [R2] for an overview and extensive list of references. In particular, it has been established that "smooth elements" in this algebra obey a strong version of the correspondence principle [R1].

III.B. For our purposes, we consider the von Neumann algebra $\mathfrak{A}_\hbar$, generated by U and V. Recall [D2] that an algebra of bounded operators $\mathfrak{R}$ on a Hilbert space $\mathscr{H}$ is called a von Neumann algebra, if (i) it is closed under taking the hermitian conjugate, and (ii) it is equal to its bicommutant, $\mathfrak{R} = \mathfrak{R}''$. Here $\mathfrak{R}'' = (\mathfrak{R}')'$, where the commutant $\mathscr{S}'$ of a set of operators $\mathscr{S}$ on $\mathscr{H}$ is defined as the set of all bounded operators on $\mathscr{H}$ which commute with all the elements of $\mathscr{S}$. The von Neumann algebra generated by a set $\mathscr{S}$ is defined as the smallest von Neumann algebra containing $\mathscr{S}$. If $\mathscr{S}$ is closed under taking the hermitian adjoint, this turns out to be $\mathscr{S}''$. In other words, $\mathfrak{A}_\hbar = \{U, U^*, V, V^*\}''$. In fact, $\mathfrak{A}_\hbar$ is isomorphic to the universal enveloping von Neumann algebra generated by U and V which obey the relations (III.3). This means, in particular, that (III.3) are the only relations between U and V. One can think of the elements of $\mathfrak{A}_\hbar$ as bounded (but not necessarily continuous) functions on the quantized torus.

The von Neumann algebra $\mathfrak{A}_\hbar$ is hyperfinite (i.e. it is a closure of an increasing subsequence of finite dimensional subalgebras) and can be equipped with a finite faithful trace. We will not reproduce here the precise definitions (see e.g. [D2]). One should just keep in mind a typical example, that of an algebra $L^\infty(M)$ of essentially bounded functions on a compact space M with a Borel probability measure $d\nu$. Such a trace is then given by

$$\tau(f) = \int_M f \, d\nu. \tag{III.4}$$

On the algebra $\mathfrak{A}_h$, a faithful normal trace is determined by

$$\tau_h\left(\sum_{j,k} \alpha_{jk} U^j V^k \right) = \alpha_{00}. \tag{III.5}$$

III.C. Let us now derive the transformation rules for U and V under conjugation by the operator F. Using Theorem II.2 and (II.3) we obtain

$$\begin{aligned} FUF^{-1} &= U(-ih\pi\sqrt{2}(\alpha+\beta)) \\ &= U(-ih\pi(a+ib)\sqrt{2}) \\ &= e^{ih\pi^2 ab} U(-ih\pi\sqrt{2}\,a)\, U(h\pi\sqrt{2}\,b) \\ &= e^{i\lambda ab/2} U^a V^b, \end{aligned}$$

and likewise

$$FVF^{-1} = e^{i\lambda cd/2} U^c V^d.$$

These expressions define an automorphism Γ_h of $\mathfrak{A}_h$. We call the group $\{\Gamma_h^n\}_{n\in\mathbb{Z}}$ of automorphisms generated by Γ_h the quantized cat dynamics on the torus.

THEOREM III.1. *The transformation*

$$\begin{aligned} \Gamma_h(U) &= e^{i\lambda ab/2} U^a V^b, \\ \Gamma_h(V) &= e^{i\lambda cd/2} U^c V^d. \end{aligned} \tag{III.6}$$

defines an automorphism of $\mathfrak{A}_h$. *The trace* τ_h *is invariant under* Γ_h, *i.e.* $\tau_h(\Gamma_h(a)) = \tau_h(a)$.

Proof. We need to show that $\Gamma_h(U)$ and $\Gamma_h(V)$ form a new set of generators. Using (III.3) we compute:

$$\Gamma_h(U)^* = e^{-i\lambda ab/2} V^{-b} U^{-a} = (e^{i\lambda ab/2} U^a V^b)^{-1} = \Gamma_h(U)^{-1}.$$

Likewise, $\Gamma_h(V)^* = \Gamma_h(V)^{-1}$. Furthermore,

$$\begin{aligned} \Gamma_h(U)\,\Gamma_h(V) &= e^{i\lambda(ab+cd)/2} U^a V^b U^c V^d \\ &= e^{i\lambda(ab+cd)/2 - i\lambda bc} U^c U^a V^b V^d \\ &= e^{i\lambda(ab+cd)/2 + i\lambda(ad-bc)} U^c V^d U^a V^b \\ &= e^{i\lambda} \Gamma_h(V)\,\Gamma_h(U). \end{aligned}$$

We also note that the inverse of $\Gamma_\hbar$ is given by

$$\begin{aligned}\Gamma_\hbar^{-1}(U) &= e^{-i\lambda bd/2} U^d V^{-b},\\ \Gamma_\hbar^{-1}(V) &= e^{-i\lambda ac/2} U^{-c} V^a.\end{aligned} \tag{III.7}$$

Finally, the $\Gamma_\hbar$-invariance is an immediate consequence of (III.5). ∎

III.D. At this point it is not quite clear that $\Gamma_\hbar$ is indeed a quantization of the classical map γ, i.e. that its classical limit $\hbar \to 0$ indeed yields γ. The goal of this subsection is to show that it is so. We let $\|\cdot\|_\hbar$ denote the operator norm on the Hilbert space $\mathscr{H}^2(\mathbb{C}, d\mu_\hbar)$.

THEOREM III.2. *Let f be a continuous $\mathbb{Z}^2$-invariant function on $\mathbb{C}$. Then*:

$$\|FT_\hbar(f)\,F^{-1} - T_\hbar(f\circ\gamma)\|_\hbar \to 0, \qquad \text{as} \quad \hbar \to 0. \tag{III.8}$$

Proof. Let $\varepsilon > 0$. We are going to show that for all sufficiently small $\hbar$,

$$\|FT_\hbar(f)F^{-1} - T_\hbar(f\circ\gamma)\|_\hbar \leqslant \varepsilon. \tag{III.9}$$

We proceed in steps.

Step 1. By the Stone–Weierstraß theorem, there is a trigonometric polynomial P such that

$$\|f - P\|_\infty \leqslant \varepsilon/3,$$

where $\|f\|_\infty = \sup_z |f(z)|$ is the usual sup-norm. Since the operator norm of a Toeplitz operator does not exceed the sup-norm of its symbol (see e.g. [B1]), $\|T_\hbar(f)\|_\hbar \leqslant \|f\|_\infty$, this yields the following inequality:

$$\|T_\hbar(f) - T_\hbar(P)\|_\hbar \leqslant \varepsilon/3. \tag{III.10}$$

Step 2. A trigonometric polynomial $P(z)$ is a linear combination of terms of the form $\exp(\bar{w}z - \bar{z}w)$. In terms of the creation and annihilation operators, for the corresponding Toeplitz operator we have:

$$T_\hbar(e^{\bar{w}z - \bar{z}w}) = e^{-wA} e^{\bar{w}A^\dagger}.$$

Conjugating the above equation by F yields:

$$\begin{aligned}FT_\hbar(e^{\bar{w}z - \bar{z}w})\,F^{-1} &= Fe^{-wA}F^{-1}Fe^{\bar{w}A^\dagger}F^{-1}\\ &= e^{-w(\alpha A + \bar{\beta}A^\dagger)} e^{\bar{w}(\bar{\alpha}A^\dagger + \beta A)}\\ &= e^{-\hbar(\alpha\bar{\beta}w^2 + \bar{\alpha}\beta\bar{w}^2)/2} e^{-\alpha w A} e^{-\bar{\beta} w A^\dagger} e^{\beta\bar{w}A} e^{\overline{\alpha w}\,A^\dagger},\end{aligned}$$

where we have used the Hausdorff–Baker–Campbell formula as in the derivation of (II.9). Commuting the third and the fourth terms gives further:

$$\begin{aligned} FT_\hbar(e^{\bar{w}z-\bar{z}w})F^{-1} &= e^{-\hbar(\alpha\bar{\beta}w^2+\bar{\alpha}\beta\bar{w}^2+2|\beta|^2|w|^2)/2}e^{-w\alpha A+\bar{w}\beta A}e^{\overline{w\alpha}A^\dagger-w\bar{\beta}A^\dagger} \\ &= e^{-\hbar(\alpha\bar{\beta}w^2+\bar{\alpha}\beta\bar{w}^2+2|\beta|^2|w|^2)/2}T_\hbar(e^{-w\alpha\bar{z}+\bar{w}\beta\bar{z}+\overline{w\alpha}z-w\bar{\beta}z}) \\ &= e^{-\hbar(\alpha\bar{\beta}w^2+\bar{\alpha}\beta\bar{w}^2+2|\beta|^2|w|^2)/2}T_\hbar(e^{\bar{w}(\bar{\alpha}z+\beta\bar{z})-w(\alpha\bar{z}+\bar{\beta}z)}) \\ &= e^{-\hbar(\alpha\bar{\beta}w^2+\bar{\alpha}\beta\bar{w}^2+2|\beta|^2|w|^2)/2}T_\hbar(e^{\bar{w}\gamma(z)-\overline{\gamma(z)}w}). \end{aligned}$$

We can thus make the following estimate:

$$\begin{aligned} &\|FT_\hbar(e^{\bar{w}z-\bar{z}w})F^{-1}-T_\hbar(e^{\bar{w}\gamma(z)-\overline{\gamma(z)}w})\|_\hbar \\ &\quad\leqslant |e^{-\hbar(\alpha\bar{\beta}w^2+\bar{\alpha}\beta\bar{w}^2+2|\beta|^2|w|^2)/2}-1|\;\|T_\hbar(e^{\bar{w}\gamma(z)-\overline{\gamma(z)}w})\|_\hbar \\ &\quad\leqslant |e^{-\hbar(\alpha\bar{\beta}w^2+\bar{\alpha}\beta\bar{w}^2+2|\beta|^2|w|^2)/2}-1|. \end{aligned}$$

Clearly, the right hand side of the above inequality goes to zero, as $\hbar\to 0$. Since P is a linear combination of finitely many terms of the above form, we can find δ (depending on P) such that for $\hbar<\delta$ we have:

$$\|FT_\hbar(P)F^{-1}-T_\hbar(P\circ\gamma)\|_\hbar\leqslant\varepsilon/3. \tag{III.11}$$

Step 3. We can now conclude the argument:

$$\begin{aligned} &\|FT_\hbar(f)\,F^{-1}-T_\hbar(f\circ\gamma)\|_\hbar \\ &\quad\leqslant\|FT_\hbar(f)\,F^{-1}-FT_\hbar(P)\,F^{-1}\|_\hbar+\|FT_\hbar(P)\,F^{-1}-T_\hbar(P\circ\gamma)\|_\hbar \\ &\qquad+\|T_\hbar(P\circ\gamma)-T_\hbar(f\circ\gamma)\|_\hbar \\ &\quad\leqslant\|T_\hbar(f)-T_\hbar(P)\|_\hbar+\varepsilon/3+\|P\circ\gamma-f\circ\gamma\|_\infty \\ &\quad\leqslant 2\,\|f-P\|_\infty+\varepsilon/3 \\ &\quad\leqslant\varepsilon, \end{aligned}$$

where we have used (III.10) and (III.11). ∎

III.E. So far the value of Planck's constant has not been restricted in any way other than it should be a positive number. In particular, the von Neumann algebra $\mathfrak{A}_\hbar$ is a well defined object for all such $\hbar$. On the other hand, its structure depends crucially on whether $\lambda/2\pi$ is a rational number or not. It is well known that physics requires $\lambda/2\pi$ to be rational. The standard informal argument, going back to Planck, is that the volume of the phase space should be an integer multiple of the elementary cell volume $2\pi\hbar$. Hence

$$\hbar=\frac{1}{2\pi N},\qquad N\in\mathbb{N}, \tag{III.12}$$

or

$$\lambda = \frac{2\pi}{N}. \tag{III.13}$$

Incidentally, this is precisely the integrality condition of geometric quantization which requires the symplectic form on the torus divided by $2\pi h$ to define a deRham cohomology class with integer coefficients. Throughout the rest of this paper, we will be assuming that the condition above is satisfied. Trivial changes in our arguments show that the conclusions below hold for arbitrary positive rational $\lambda/2\pi$.

III.F. The von Neumann algebra $\mathfrak{A}_h$ has a simple structure which is described in the theorem below. This theorem is well known, and the references to the original literature can be found in [R2]. Since the proof is not easy to extract from the original references (and for the sake of completeness), we include an elementary proof. We denote by $\mathscr{M}_N$ the (von Neumann) algebra of complex $N \times N$ matrices, while by $L^\infty(\mathbb{T}^2)$ we denote the space of all essentially bounded functions on the torus regarded as a von Neumann algebra on the Hilbert space $L^2(\mathbb{T}^2)$.

THEOREM III.3. *We have the following isomorphism of von Neumann algebras*

$$\iota\colon \mathfrak{A}_h \to L^\infty(\mathbb{T}^2) \otimes \mathscr{M}_N. \tag{III.14}$$

Under this isomorphism, the trace τ_h factorizes into a tensor product of traces,

$$\tau_h \circ \iota^{-1} = \tau \otimes (1/N)\,\mathrm{tr}, \tag{III.15}$$

where τ is given by (III.4).

Proof. It is clear from the relations (III.3) that U^N and V^N are in the center of $\mathfrak{A}_h$. Let us denote by $\mathfrak{Z}$ the von Neumann algebra generated by

$$X = U^N, \qquad \text{and} \qquad Y = V^N. \tag{III.16}$$

Obviously, $\mathfrak{Z}$ is isomorphic with $L^\infty(\mathbb{T}^2)$, with the isomorphism given by $X \to e^{2\pi i\theta_1}$ and $Y \to e^{2\pi i\theta_2}$. Consider now the following (discontinuous) functions in $L^\infty(\mathbb{T}^2)$: $f_1(\theta) = e^{2\pi i\theta_1/N}$ and $f_2(\theta) = e^{2\pi i\theta_2/N}$, and let Z_1 and Z_2 be the corresponding elements of $\mathfrak{Z}$. Then the two elements $u = Z_1{}^{-1}U$ and $v = Z_2{}^{-1}V$ obey the following set of relations:

$$\begin{aligned} uu^* &= u^*u = I, \\ vv^* &= v^*v = I, \\ uv &= e^{i\lambda}vu, \\ u^N &= v^N = I. \end{aligned} \tag{III.17}$$

This algebra has the following realization. In the Hilbert space $\mathbb{C}^N$, choose an orthonormal basis $e_1, ..., e_N$, and set $ue_j = e^{i(j-1)\lambda} e_j$, $ve_j = e_{j+1}$, where $e_{N+1} = e_1$ (by a slight abuse of notation, we denote the matrix representatives of u and v by the same symbols). A short computation shows that the only matrices commuting with u and v are scalar multiples of the identity, and thus the von Neumann algebra generated by u and v can be identified with the full matrix algebra $\mathscr{M}_N$.

We have $U = Z_1 u$, $V = Z_2 v$, and the required isomorphism is given by

$$\imath(U) = f_1 \otimes u, \quad \imath(V) = f_2 \otimes v. \tag{III.18}$$

To prove (III.15), we note that

$$(\tau \otimes (1/N)\,\mathrm{tr})(f_1^j f_2^k \otimes u^j v^k) = \int_0^1 \int_0^1 e^{2\pi i (j\theta_1 + k\theta_2)/N}\, d\theta_1\, d\theta_2 (1/N)\,\mathrm{tr}(u^j v^k). \tag{III.19}$$

Using the explicit realization of the operators u and v we see that $\mathrm{tr}(u^j v^k) = 0$, unless $k = pN$, $p \in \mathbb{Z}$. However, $\int_0^1 e^{2\pi i p\theta_2}\, d\theta_2 = 0$, for $p \neq 0$, and so (III.19) is zero for $k \neq 0$. Let $k = 0$, and $j = Np + q$, $0 \leqslant q \leqslant N-1$. If $q > 0$, then $\mathrm{tr}(u^j) = 0$. If $q = 0$, but $p \neq 0$, then $\int_0^1 e^{2\pi i p\theta_1}\, d\theta_1 = 0$. Consequently,

$$(\tau \otimes (1/N)\,\mathrm{tr})(f_1^j f_2^k \otimes u^j v^k) = \delta_{j0}\delta_{k0} = \tau_h(U^j V^k), \tag{III.20}$$

and the claim follows. ∎

Let us parenthetically remark that the corresponding result for the $\mathbb{C}^*$-algebra of functions on a quantized torus involves a bundle of full matrix algebras over the torus rather than a tensor product [R2].

III.G. It is now easy to see that, under the isomorphism above, the automorphism Γ_h becomes a tensor product of automorphisms of the factors in (III.18).

LEMMA III.4. *For $f \in L^\infty(\mathbb{T}^2)$,*

$$\imath \Gamma_h \imath^{-1}(f(\theta) \otimes I) = f(\gamma\theta + \Delta_\gamma) \otimes I, \tag{III.21}$$

where

$$\Delta_\gamma = (Nab/2, Ncd/2)$$

is a constant.

Proof. Expanding f in a Fourier series and using (III.16), we can write

$$\imath^{-1}(f \otimes I) = \sum_{m, n \in \mathbb{Z}} \hat{f}_{m,n} X^m Y^n = \sum_{m, n \in \mathbb{Z}} \hat{f}_{m,n} U^{Nm} V^{Nn},$$

and thus

$$\begin{aligned}\Gamma_h \iota^{-1}(f \otimes I) &= \sum_{m,\, n \in \mathbb{Z}} \hat{f}_{m,\, n}(e^{\pi i ab/N} U^a V^b)^{Nm}\, (e^{\pi i cd/N} U^c V^d)^{Nn} \\ &= \sum_{m,\, n \in \mathbb{Z}} \hat{f}_{m,\, n}(e^{\pi i Nab} U^{Na} V^{Nb})^m\, (e^{\pi i Ncd} U^{Nc} V^{Nd})^n \\ &= \sum_{m,\, n \in \mathbb{Z}} \hat{f}_{m,\, n}(e^{\pi i Nab} X^a Y^b)^m\, (e^{\pi i Ncd} X^c Y^d)^n,\end{aligned}$$

and the claim follows. ∎

THEOREM III.5. *We have*

$$\iota \Gamma_h \iota^{-1} = \Psi_h \otimes \Phi_h, \tag{III.22}$$

where Ψ_h *is an automorphism of* $L^\infty(\mathbb{T}^2)$ *given by*

$$\begin{aligned}\Psi_h(e^{2\pi i \theta_1}) &= e^{2\pi i (a\theta_1 + b\theta_2 + Nab/2)}, \\ \Psi_h(e^{2\pi i \theta_2}) &= e^{2\pi i (c\theta_1 + d\theta_2 + Ncd/2)},\end{aligned} \tag{III.23}$$

and where Φ_h *is an automorphism of* $\mathscr{M}_N$ *given by*

$$\begin{aligned}\Phi_h(u) &= e^{i\lambda (N+1) ab/2} u^a v^b, \\ \Phi_h(v) &= e^{i\lambda (N+1) cd/2} u^c v^d.\end{aligned} \tag{III.24}$$

Notice that in the case when *ab* and *cd* are even (this case is referred to as "quantizable" in [HB]) Ψ_h coincides with the classical map (I.8). It is thus natural to regard Ψ_h as the classical component of the dynamics, and Φ_h its purely quantum component.

Proof. The algebra $L^\infty(\mathbb{T}^2) \otimes \mathscr{M}_N$ is generated by elements of the form $f \otimes u$ and $f \otimes v$. In view of Lemma III.4, it is sufficient to compute $\iota \Gamma_h \iota^{-1}(I \otimes u)$ and $\iota \Gamma_h \iota^{-1}(I \otimes v)$. Using the notation introduced in the proof of Theorem III.3, we have

$$\begin{aligned}\Gamma_h \iota^{-1}(I \otimes u) &= \Gamma_h(Z_1^{-1})\, \Gamma_h(U) \\ &= e^{i\lambda ab/2 + i\lambda Nab/2} U^a V^b Z_1^{-a} Z_2^{-b} \\ &= e^{i\lambda (N+1) ab/2} u^a v^b.\end{aligned}$$

The calculation for $I \otimes v$ is analogous. ∎

IV. Quantized Baker's Maps

IV.A. In this section we introduce a group of automorphisms of $\mathfrak{A}_h$ which we call the quantized baker's dynamics. Our construction requires that N in (III.13) be an odd number, and we make this assumption throughout the section. This is unlike the quantization procedure proposed in [BV], [CTH], [SV], [S], and [BDG], which requires N to be even. We do not know yet whether our quantization is equivalent to it. Because of its discontinuous character, the quantized baker's dynamics can be defined in the framework of von Neumann algebras only. This should be contrasted with the cat dynamics, where we chose to work with von Neumann algebras rather than $\mathbb{C}^*$-algebras for the reason of convenience only.

First, we review some facts from operator calculus. If S is a unitary operator, then by $E_S(\sigma)$ we will denote its spectral measure. In other words, $S=\int_0^1 e^{2\pi i\sigma}\,dE_S(\sigma)$. For any real number α, we define $S^\alpha=\int_0^1 e^{2\pi i\alpha\sigma}\,dE_S(\sigma)$ (in particular, $S^{1/2}=\int_0^1 e^{\pi i\sigma}\,dE_S(\sigma)$). It follows by functional calculus that S^α is unitary, and so $S^\alpha=\int_0^1 e^{2\pi i\sigma}\,dE_{S^\alpha}(\sigma)$. It is easy to express the spectral measure E_{S^α} in terms of E_S. In particular,

$$E_{S^n}(\sigma)=\sum_{0\leqslant j\leqslant n-1} E_S\left(\frac{\sigma+j}{n}\right)-E_S\left(\frac{j}{n}\right),\qquad \text{for}\quad n\in\mathbb{N}, \tag{IV.1}$$

$$E_{S^{1/2}}(\sigma)=\begin{cases}E_S(2\sigma), & \text{if}\quad 0\leqslant\sigma<1/2;\\ I, & \text{if}\quad 1/2\leqslant\sigma<1,\end{cases} \tag{IV.2}$$

and

$$E_{S^{-1}}(\sigma)=E_S(1-\sigma). \tag{IV.3}$$

Obviously, $(S^{1/2})^2=S$. However, $(S^2)^{1/2}\neq S$. The latter fact will play a role in the following, and we state it as a lemma.

Lemma IV.1. *Let S be unitary. Then*

$$(S^2)^{1/2}=S(2E_S(1/2)-I). \tag{IV.4}$$

Proof. We use (IV.1) to compute:

$$\begin{aligned}(S^2)^{1/2}&=\int_0^1 e^{i\pi\sigma}\,dE_{S^2}(\sigma)\\ &=\int_0^1 e^{i\pi\sigma}\,dE_S\left(\frac{\sigma}{2}\right)+\int_0^1 e^{i\pi\sigma}\,dE_S\left(\frac{\sigma+1}{2}\right)\\ &=\int_0^{1/2} e^{2\pi i\sigma}\,dE_S(\sigma)-\int_{1/2}^1 e^{2\pi i\sigma}\,dE_S(\sigma)\\ &=SE_S(1/2)-S(I-E_S(1/2)).\quad\blacksquare\end{aligned}$$

IV.B. We now come back to the algebra (III.3). For a unitary $S \in \mathfrak{A}_h$ we define

$$\begin{aligned} \sqrt{S} &= (S^{-N})^{1/2}\, S^{(N+1)/2}, \\ P(S) &= E_{S^N}(1/2). \end{aligned} \tag{IV.5}$$

Clearly, $\sqrt{S}$ is a particular square root of S,

$$(\sqrt{S})^2 = S. \tag{IV.6}$$

Furthermore,

$$(\sqrt{S})^N = (S^N)^{1/2}. \tag{IV.7}$$

Note also that since N is odd and V^N is central, the following commutation relation between U and $\sqrt{V}$ holds:

$$U\sqrt{V} = -e^{i\lambda/2}\sqrt{V}\, U. \tag{IV.8}$$

Consider now the following transformation on the generators of $\mathfrak{A}_h$:

$$\begin{aligned} B_h(U) &= U^2, \\ B_h(V) &= \sqrt{V}(2P(U) - I). \end{aligned} \tag{IV.9}$$

We extend B_h to $\mathfrak{A}_h$ by requiring that $B_h(ab) = B_h(a)\, B_h(b)$ and $B_h(a^*) = B_h(a)^*$.

THEOREM IV.2. *The transformation B_h defines a τ_h-preserving *-automorphism of the von Neumann algebra $\mathfrak{A}_h$.*

Proof. We need to verify that $B_h(U)$ and $B_h(V)$ obey the same relations as U and V, and that B_h has an inverse. The former property is an immediate consequence of (IV.8), while the latter one can be established as follows. Let T be a *-antiautomorphism of $\mathfrak{A}_h$ defined by $T(U) = V$ and $T(V) = U$ (clearly, T preserves (III.3), as $T(UV) = T(V)\, T(U)$). Consider now the *-automorphism TB_hT. Using the fact that

$$B_h(V)^2 = (\sqrt{V}(2P(U) - I))^2 = V, \tag{IV.10}$$

we immediately find that

$$(TB_hT)\, B_h(U) = TB_hT(U^2) = TB_h(V^2) = TB_h(V)^2 = T(V) = U,$$

$$B_h(TB_hT)(V) = B_hTB_h(U) = B_hT(U^2) = B_h(V^2) = V.$$

It is slightly more difficult to verify the remaining two relations. We have:

$$(TB_hT)\, B_h(V) = TB_hT(\sqrt{V}(2P(U)-I))) = TB_h(\sqrt{U}(2P(V)-I)))$$
$$= T((\sqrt{U^2}(2P(B_h(V))-I))).$$

Now, according to Lemma IV.1 and (IV.3),

$$\sqrt{U^2} = (U^{-2N})^{1/2}\, U^{N+1} = U(2E_{U^{-N}}(1/2)-I)$$
$$= U(2E_{U^N}(1/2)-I) = U(2P(U)-I). \tag{IV.11}$$

Furthermore, using (IV.2) and (IV.7),

$$P(B_h(V)) = E_{B_h(V)^N}(1/2) = E_{(V^N)^{1/2}\,(2P(U)-I)}(1/2)$$
$$= E_{(V^N)^{1/2}}(1/2)\, P(U) + (I - E_{(V^N)^{1/2}}(1/2))(I-P(U))$$
$$= P(U),$$

and so

$$(TB_hT)\, B_h(V) = T(U(2P(U)-I)^2) = T(U) = V.$$

In the same fashion we verify the last relation:

$$B_h(TB_hT)(U) = B_hTB_h(V) = B_hT\,(\sqrt{V}(2P(U)-I))) = B_h(\sqrt{U}(2P(V)-I)))$$
$$= \sqrt{U^2}(2P(B_h(V))-I) = U,$$

and so $TB_hT = B_h^{-1}$.

The τ_h-invariance of B_h can be easily verified by means of (III.15) and the next theorem. ∎

The automorphism B_h of $\mathfrak{A}_h$ is called the quantized baker's map.

IV.C. The fundamental property of B_h is that it factorizes under the isomorphism (III.14).

THEOREM IV.3. *We have*

$$\iota \Gamma_h \iota^{-1} = \Psi \otimes \Phi_h, \tag{IV.12}$$

where Ψ is an automorphism of $L^\infty(\mathbb{T}^2)$ given by

$$\Psi(e^{2\pi i\theta_1}) = e^{4\pi i\theta_1},$$
$$\Psi(e^{2\pi i\theta_2}) = e^{\pi i\theta_2}(2\chi_{[0,\,1/2)}(\theta_1)-1), \tag{IV.13}$$

and where Φ_h is an automorphism of $\mathscr{M}_N$ given by

$$\begin{aligned} \Phi_h(u) &= u^2, \\ \Phi_h(v) &= v^{(N+1)/2}. \end{aligned} \tag{IV.14}$$

Observe that Ψ coincides with (I.13). As in the case of the cat dynamics, one can think about Ψ as the purely classical component of the dynamics, and about Φ_h as its purely quantum component.

Proof. Proceeding as in the proof of Lemma III.4, we readily find that

$$\imath B_h \imath^{-1}(f \otimes I) = Bf \otimes I. \tag{IV.15}$$

Furthermore,

$$\imath B_h \imath^{-1}(e^{2\pi i\theta_1/N} \otimes u) = \imath B_h(U) = \imath(U^2) = e^{4\pi i\theta_1/N} \otimes u^2.$$

Similarly,

$$\begin{aligned} \imath B_h \imath^{-1}(e^{2\pi i\theta_2/N} \otimes v) &= \imath B_h(V) = \imath((V^{-N})^{1/2}\ V^{(N+1)/2}(2P(U) - I)) \\ &= e^{i\pi\theta_2/N}[2\chi_{[0,\,1/2)}(\theta_1) - 1) \otimes v^{(N+1)/2}, \end{aligned}$$

and the proof is complete. ∎

V. Connes–Størmer Entropy

V.A. To motivate the construction of the CS entropy we first reformulate the definition of the classical KS entropy in purely algebraic (rather than measure theoretic) terms (see also [B3]). We assume that M is a compact phase space with a Borel probability measure $d\nu$ defined on it, and τ define the faithful normal trace on $L^\infty(M)$ given by (III.4). Given a partition $\mathscr{A}$ of M (defined as in the Introduction), we consider the finite dimensional subalgebra $\mathfrak{N} \subset L^\infty(M)$ which is generated by the characteristic functions χ_{A_j}. The operator of multiplication by χ_{A_j} is a projection operator and we denote it by p_j. Note that each projection p_j is minimal (i.e. is not a sum of two non-trivial projections in $\mathfrak{N}$), and $\sum_j p_j = I$. We define the entropy of the subalgebra $\mathfrak{N}$ to be

$$H(\mathfrak{N}) = \sum_j \tau(\eta(p_j)) = H(\mathscr{A}). \tag{V.1}$$

For two such subalgebras, $\mathfrak{N}_1$ and $\mathfrak{N}_2$, we let $\mathfrak{N}_1 \vee \mathfrak{N}_2$ denote the (finite dimensional) subalgebra generated by $\mathfrak{N}_1$ and $\mathfrak{N}_2$.

Now, a measure preserving automorphism φ of M defines a τ-preserving automorphism Φ of $\mathfrak{N}$,

$$\Phi f(x) = f \circ \varphi(x). \tag{V.2}$$

We set

$$H(\mathfrak{N}, \Phi) = \lim_{k \to \infty} \frac{1}{k} H(\mathfrak{N} \vee \Phi(\mathfrak{N}) \vee \cdots \vee \Phi^{k-1}(\mathfrak{N})) = H(\mathscr{A}, \varphi),$$

and define the entropy of the automorphism Φ as the supremum of this quantity over all possible choices of $\mathfrak{N}$ (this is, of course, equal to $h_{KS}(\varphi)$).

V.B. The construction above of the entropy of a measure preserving automorphism was generalized to the non-commutative case by Connes and Størmer [CS] (in the von Neumann algebraic setup), and later by Connes, Narnhofer and Thirring [CNT] (in the $\mathbb{C}^*$-algebraic setup). We choose the original Connes–Størmer construction as it suits our needs best.

Let $\mathfrak{R}$ be a von Neumann algebra, and let τ be a finite faithful normal trace on $\mathfrak{R}$. Consider a collection $\mathfrak{N}_1, ..., \mathfrak{N}_k$ of finite dimensional von Neumann subalgebras of $\mathfrak{R}$. The key difficulty to overcome here is the fact that $\mathfrak{N} \vee \mathfrak{P}$ may not be finite dimensional, even though $\mathfrak{N}$ and $\mathfrak{P}$ are. Connes and Størmer defined a function $H(\mathfrak{N}_1, ..., \mathfrak{N}_k)$ which replaces $H(\mathfrak{N}_1 \vee \cdots \vee \mathfrak{N}_k)$ but reduces to it in the commutative case. Specifically, this function satisfies the following properties:

(A) $H(\mathfrak{N}_1, ..., \mathfrak{N}_k) \leqslant H(\mathfrak{P}_1, ..., \mathfrak{P}_k)$, if $\mathfrak{N}_j \subset \mathfrak{P}_j$, for all $1 \leqslant j \leqslant k$;

(B) $H(\mathfrak{N}_1, ..., \mathfrak{N}_m, \mathfrak{N}_{m+1}, ..., \mathfrak{N}_n) \leqslant H(\mathfrak{N}_1, ..., \mathfrak{N}_m) + H(\mathfrak{N}_{m+1}, ..., \mathfrak{N}_n)$;

(C) if $\mathfrak{N}_1, ..., \mathfrak{N}_m \subset \mathfrak{N}$, then $H(\mathfrak{N}_1, ..., \mathfrak{N}_m, \mathfrak{N}_{m+1}, ..., \mathfrak{N}_n) \leqslant H(\mathfrak{N}, \mathfrak{N}_{m+1}, ..., \mathfrak{N}_n)$;

(D) if $\{p_\alpha\}$ is any family of minimal projections in $\mathfrak{N}$ such that $\sum_\alpha p_\alpha = I$, then $H(\mathfrak{N}) = \sum_\alpha \eta(\tau(p_\alpha))$;

(E) if $\mathfrak{N}_1, ..., \mathfrak{N}_k$ are pairwise commuting then $H(\mathfrak{N}_1, ..., \mathfrak{N}_k) = H((\mathfrak{N}_1 \cup \cdots \cup \mathfrak{N}_k)'')$;

(F) if Φ is an automorphism of $\mathfrak{R}$ preserving the trace τ, then $H(\Phi(\mathfrak{N}_1), ..., \Phi(\mathfrak{N}_k)) = H(\mathfrak{N}_1, ..., \mathfrak{N}_k)$.

Now, if Φ is a τ-preserving automorphism of $\mathfrak{R}$, then properties (B) and (F) imply that that the limit

$$H(\mathfrak{N}, \Phi) = \lim_{k \to \infty} \frac{1}{k} H(\mathfrak{N}, \Phi(\mathfrak{N}), ..., \Phi^{k-1}(\mathfrak{N})) \tag{V.3}$$

exists. We define the CS entropy as the supremum of the above quantity over all possible choices of the finite dimensional algebra $\mathfrak{N}$,

$$h_{CS}(\Phi) = \sup_{\mathfrak{N},\, \dim \mathfrak{N} < \infty} H(\mathfrak{N}, \Phi). \tag{V.4}$$

To be able to compute $h_{CS}(\Phi)$ we need a non-commutative version of the Kolmogorov–Sinai theorem. Such a theorem was proved in [CS] and is formulated as follows.

THEOREM V.1. *Let $\{\mathfrak{N}_k\}$ be an increasing sequence of finite dimensional von Neumann subalgebras of $\mathfrak{R}$ such that the weak closure $(\bigcup_k \mathfrak{N}_k)^-$ of $\bigcup_k \mathfrak{N}_k$ is $\mathfrak{R}$. Then*

$$h_{CS}(\Phi) = \lim_{k \to \infty} H(\mathfrak{N}_k, \Phi). \tag{V.5}$$

Recall that von Neumann algebras having the property assumed in the theorem above are called hyperfinite. This theorem was used in [CS] to compute the entropy of the non-commutative Bernoulli shift.

As expected, the CS entropy reduces to the KS entropy in the commutative case.

THEOREM V.2. *Let M be a compact space with a Borel probability measure $d\nu$ and let φ be a measure preserving automorphism of M. Consider the von Neumann algebra $\mathfrak{R} = L^\infty(M)$ with the trace τ given by* (III.4), *and the automorphism Φ of $\mathfrak{R}$ defined by* (V.2). *Then $h_{CS}(\Phi) = h_{KS}(\varphi)$.*

V.C. The actual definition of $H(\mathfrak{N}_1, ..., \mathfrak{N}_k)$ will play a role below and so we summarize it briefly.

We consider a von Neumann subalgebra $\mathfrak{N} \subset \mathfrak{R}$, and define the following inner product on $\mathfrak{N}$: $(x, y) = \tau(x^* y)$. The completion of $\mathfrak{N}$ in the norm induced by this inner product is a Hilbert space which we denote by $L^2(\mathfrak{N})$. Let $P_{\mathfrak{N}}: L^2(\mathfrak{R}) \to L^2(\mathfrak{N})$ be the orthogonal projection on $L^2(\mathfrak{N})$ and let $E_{\mathfrak{N}}$ denote the restriction of $P_{\mathfrak{N}}$ to the dense subspace $\mathfrak{R} \subset L^2(\mathfrak{R})$. This is a non-commutative version of the conditional expectation operator.

Let now $\mathscr{S}_k$ be the set of all sequences of elements of $\mathfrak{R}$, $x = \{x_{\mathbf{i}}\}$, where $\mathbf{i} \in \mathbb{N}^k$, such that:

(a) $x_{\mathbf{i}} \geqslant 0$;

(b) all but finitely many $x_{\mathbf{i}}$ are zero;

(c) $\sum_{\mathbf{i}} x_{\mathbf{i}} = I$.

For $x \in \mathscr{S}_k$ and $1 \leqslant l \leqslant k$ we set

$$x_j^l = \begin{cases} x_j, & \text{if} \quad k = 1; \\ \displaystyle\sum_{i_1 \cdots i_{l-1} i_{l+1} \cdots i_k} x_{i_1 \cdots i_{l-1} j i_{l+1} \cdots i_k}, & \text{if} \quad k \geqslant 2. \end{cases} \tag{V.6}$$

We define

$$H(\mathfrak{N}_1, ..., \mathfrak{N}_k) = \sup_{x \in \mathscr{S}_k} \left\{ \sum_{\mathbf{i} \in \mathbb{N}^k} \eta(\tau(x_{\mathbf{i}})) - \sum_{l, j} \tau(\eta(E_{\mathfrak{N}_l} x_j^l)) \right\}. \tag{V.7}$$

It now takes quite a lot of skill to establish the results stated above, and we refer the interested reader to [CS] for details.

V.D. We now formulate and prove a technical result which will be a basis for the arguments of next section.

LEMMA V.3. *Let $\mathfrak{R}_1 = L^\infty(M)$, where M is a compact space with a Borel probability measure dv and the natural faithful normal trace $\tau_1(\cdot) = \int_M (\cdot)\, dv$, let $\mathfrak{R}_2$ be a finite dimensional von Neumann algebra with a faithful normal trace τ_2, and let Ψ and Φ be trace preserving automorphisms of $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively. Consider the hyperfinite von Neumann algebra $\mathfrak{R} = \mathfrak{R}_1 \otimes \mathfrak{R}_2$ with the faithful normal trace $\tau = \tau_1 \otimes \tau_2$, and the τ-preserving automorphism $\Gamma = \Psi \otimes \Phi$ of $\mathfrak{R}$. Then $h_{CS}(\Gamma) = h_{CS}(\Psi)$.*

Proof. The proof of this lemma proceeds in steps.

Step 1. For any collection of finite dimensional subalgebras $\mathfrak{N}_1, ..., \mathfrak{N}_k \subset \mathfrak{R}_1$,

$$H(\mathfrak{N}_1 \otimes \mathfrak{R}_2, ..., \mathfrak{N}_k \otimes \mathfrak{R}_2) = H((\mathfrak{N}_1 \cup \cdots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2). \tag{V.8}$$

To prove this, note first that by property (C) of Section V,

$$H(\mathfrak{N}_1 \otimes \mathfrak{R}_2, ..., \mathfrak{N}_k \otimes \mathfrak{R}_2) \leqslant H((\mathfrak{N}_1 \cup \cdots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2), \tag{V.9}$$

as $\mathfrak{N}_j \otimes \mathfrak{R}_2 \subset (\mathfrak{N}_1 \cup \cdots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2$. To prove that

$$H(\mathfrak{N}_1 \otimes \mathfrak{R}_2, ..., \mathfrak{N}_k \otimes \mathfrak{R}_2) \geqslant H((\mathfrak{N}_1 \cup \cdots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2), \tag{V.10}$$

we proceed as follows. Let $P_1^j, ..., P_{n_j}^j$, $1 \leqslant j \leqslant n_j$, where $n_j = \dim \mathfrak{N}_j$, denote the minimal projections in $\mathfrak{N}_j$, and let $E_1, ..., E_n$ be minimal projections in $\mathfrak{R}_2$ such that $\sum_j E_j = I$. We set

$$x_{i_0 i_1 \cdots i_k} = E_{i_0} \otimes P_{i_1} \cdot \cdots \cdot P_{i_k} = (E_{i_0} \otimes P_{i_1}) \cdot \cdots \cdot (E_{i_0} \otimes P_{i_k}), \tag{V.11}$$

and observe that $\{x_{i_0 i_1 \cdots i_k}\} \in S_{k+1}$ and it forms a system of minimal projections in $(\mathfrak{N}_1 \cup \cdots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2$. Since $\eta(x_{i_0 i_1 \cdots i_k}) = 0$, property ($D$) of Section V implies that

$$\begin{aligned} H((\mathfrak{N}_1 \cup \cdots \cup \mathfrak{N}_k)'' \otimes \mathfrak{R}_2) &= \sum_{i_0 i_1 \cdots i_k} \tau(\eta(x_{i_0 i_1 \cdots i_k})) \\ &\leqslant H(\mathfrak{N}_1 \otimes \mathfrak{R}_2, ..., \mathfrak{N}_k \otimes \mathfrak{R}_2). \end{aligned}$$

Step 2. If $\mathfrak{N}$ is a finite dimensional subalgebra of $\mathfrak{R}_1$, then

$$H(\mathfrak{N}\otimes\mathfrak{R}_2)=H(\mathfrak{N})+H(\mathfrak{R}_2). \tag{V.12}$$

To prove this, note that for a projection $P\in\mathfrak{N}_1$ and a projection $E\in\mathfrak{R}_2$,

$$\tau(\eta(P\otimes E))=\tau_1(\eta(P))\,\tau_2(E)+\tau_1(P)\,\tau_2(\eta(E)). \tag{V.13}$$

Denoting by $P_1, ..., P_m$ and $E_1, ..., E_n$ systems of minimal projections in $\mathfrak{N}$ and $\mathfrak{R}_2$, respectively, and using property (D), we obtain

$$\begin{aligned} H(\mathfrak{N}\otimes\mathfrak{R}_2) &= \sum_{j,k}\tau_1(\eta(P_j))\,\tau_2(E_k)+\tau_1(P_j)\,\tau_2(\eta(E_k)) \\ &= \sum_j \tau_1(\eta(P_j))+\sum_k \tau_2(\eta(E_k)) \\ &= H(\mathfrak{N})+H(\mathfrak{R}_2). \end{aligned}$$

Step 3. Choose now an increasing sequence $\{\mathfrak{P}_n\}_{n\in\mathbb{N}}$ of finite dimensional subalgebras of $\mathfrak{R}_1$, such that $(\bigcup_n \mathfrak{P}_n)^- = \mathfrak{R}_1$. Then $\{\mathfrak{P}_n\otimes\mathfrak{R}_2\}_{n\in\mathbb{N}}$ forms an increasing sequence of finite dimensional subalgebras of $\mathfrak{R}_1\otimes\mathfrak{R}_2$, and $(\bigcup_n \mathfrak{P}_n\otimes\mathfrak{R}_2)^- = \mathfrak{R}_1\otimes\mathfrak{R}_2$. Therefore, by Theorem V.1,

$$h_{CS}(\Psi\otimes\Phi)=\lim_{n\to\infty} H(\mathfrak{P}_n\otimes\mathfrak{R}_2, \Psi\otimes\Phi). \tag{V.14}$$

By Steps 1 and 2,

$$\begin{aligned} &H(\mathfrak{P}_n\otimes\mathfrak{R}_2, \Psi(\mathfrak{P}_n)\otimes\Phi(\mathfrak{R}_2), ..., \Psi^{k-1}(\mathfrak{P}_n)\otimes\Phi^{k-1}(\mathfrak{R}_2)) \\ &\quad= H(\mathfrak{P}_n\otimes\mathfrak{R}_2, \Psi(\mathfrak{P}_n)\otimes\mathfrak{R}_2, ..., \Psi^{k-1}(\mathfrak{P}_n)\otimes\mathfrak{R}_2) \\ &\quad= H((\mathfrak{P}_n\cup\Psi(\mathfrak{P}_n)\cup\cdots\cup\Psi^{k-1}(\mathfrak{P}_n))''\otimes\mathfrak{R}_2) \\ &\quad= H((\mathfrak{P}_n\cup\Psi(\mathfrak{P}_n)\cup\cdots\cup\Psi^{k-1}(\mathfrak{P}_n))'')+H(\mathfrak{R}_2) \\ &\quad= H(\mathfrak{P}_n, \Psi(\mathfrak{P}_n), ..., \Psi^{k-1}(\mathfrak{P}_n))+H(\mathfrak{R}_2). \end{aligned}$$

But $H(\mathfrak{R}_2)$ is a constant independent of k, and so

$$H(\mathfrak{R}_1\otimes\mathfrak{R}_2, \Psi\otimes\Phi)=H(\mathfrak{R}_1, \Psi), \tag{V.15}$$

which proves the lemma. ∎

VI. Entropy of the Quantized Dynamics

VI.A. We are now ready to compute the CS entropy of the quantized cat and baker's dynamics.

THEOREM VI.1. *The CS entropy of the quantized cat dynamics on the torus is equal to the classical value,*

$$h_{CS}(\Gamma_h) = log\,|\mu_1|. \tag{VI.1}$$

Furthermore, if $|\mathrm{tr}(\gamma)| \leqslant 2$, *then* $h_{CS}(\Gamma_h) = 0$.

It is an interesting question, even if without physical significance, whether Theorem VI.1 holds without the assumption that $\lambda/2\pi$ is rational. In that case, $\mathfrak{A}_h$ is not isomorphic to a finite dimensional algebra tensored by an abelian algebra, and so Lemma V.3 cannot be applied. In the case of topological entropy, Voiculescu [V1] has recently shown that the entropy of the quantized dynamics does not exceed the classical value.

An analogous result holds for the quantized baker's map.

THEOREM VI.2. *The CS entropy of the quantized baker's map is equal to the KS entropy of the classical baker's map,*

$$h_{CS}(B_h) = \log 2. \tag{VI.2}$$

It is easy to prove the above theorems. Indeed, according to Theorem III.3 and Theorem III.5, $\mathfrak{A}_h$ and Γ_h have precisely the structure required by Lemma V.3. Hence, $h_{CS}(\Gamma_h) = h_{CS}(\Psi_h)$. It is easy to see that the map $\theta \to \gamma\theta + \Delta_\gamma$ is conjugate to the cat map $\theta \to \gamma\theta$. According to the well known theorem [CFS], conjugate maps have equal KS entropies, and so Theorem V.2 implies that $h_{CS}(\Phi) = h_{KS}(\gamma)$. Theorem VI.1 follows from (I.11).

The proof of Theorem VI.2 is analogous, with Theorem IV.3 replacing Theorem III.5, and the final conclusion following from (I.14).

VI.B. We conclude this section with a brief discussion of a dynamical system on a torus which is ergodic but is not chaotic. Consider the Kronecker map on the torus defined by

$$K: (x_1, x_2) \to (x_1 + \omega_1, x_2 + \omega_2). \tag{VI.3}$$

This map is known to be ergodic if and only if the frequencies ω_1 and ω_2 are linearly independent over $\mathbb{Z}$. The Kronecker map is, however, not chaotic, as its KS entropy is easily found to be zero [CFS].

In terms of the complex variable z, the Kronecker map reads

$$K: z \to z + \omega,$$

with $\omega = (\omega_1 + i\omega_2)/\sqrt{2}$, and so to quantize it we need to find a unitary operator implementing the following Bogolubov transformation:

$$A^\dagger \to A^\dagger + \omega I.$$

As in the case of the cat map the unitary operator is uniquely (up to a phase) determined by the above condition. In fact, an easy consequence of (II.2) is that

$$U(-\omega)\, A^{\dagger} U(-\omega)^{-1} = A^{\dagger} + \omega I,$$

and so $U(-\omega)$ is the required unitary operator.

Let now $K_\hbar$ be the automorphism of the quantum torus given by by $K_\hbar(\cdot) = U(-\omega)(\cdot)\ U(-\omega)^{-1}$. Evaluated on the generators of $\mathfrak{A}_\hbar$, $K_\hbar$ is:

$$\begin{aligned} K_\hbar(U) &= e^{2i\pi\omega_1} U, \\ K_\hbar(V) &= e^{2i\pi\omega_2} V. \end{aligned} \tag{VI.4}$$

Assume now that $\hbar = 1/2\pi N$, in which case Theorem III.3 is applicable. It is easy to see that $K_\hbar$ can be factorized, with the first factor given by the following automorphism of $L^\infty(\mathbb{T}^2)$:

$$f(\theta) \to f(\theta_1 + N\omega_1, \theta_2 + N\omega_2). \tag{VI.5}$$

Hence, the CS entropy of $K_\hbar$ is equal to the KS entropy of (VI.5) and is thus zero.

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