# QUANTUM MAPS

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Abstract. We describe some results on quantization of discrete time dynamical systems ("quantum maps"). We focus our attention on a number of examples including the cat, Kronecker, and standard maps. Our main interest lies in studying the ergodic properties of these quantum dynamical systems.

## 1. Why Quantum Maps?

In classical dynamics, systems with a discrete time variable are referred to as *maps*. In this talk, we will describe some mathematical results concerning quantum maps (this term was coined in [2]), i.e. discrete time quantum systems. Their time evolution is not governed by the Schrödinger equation; rather it is given by a discrete unitary group acting on a Hilbert space.

There are several reasons for studying maps in classical and quantum dynamics:

- They arise as Poincaré section maps of flows;
- Often they are easier to study analytically;
- They serve as paradigms of various phenomena in ergodic theory;
- They are easier to simulate on a computer than flows;
- Interesting maps arise in applications, e.g. in statistical mechanics, the theory of quantum computation and quantum information theory, etc.

We will work within the operator algebra framework, as this is the natural setup for addressing the structural issues of quantum dynamics. Other approaches abound in the physics and mathematics literature, see e.g. [2], [5], [7], [11], and references therein. We shall focus on a somewhat restricted class of quantum dynamical systems, namely those which arise as quantizations of classical maps.

## 2. Classical dynamics

Classical mechanics is formulated in terms of a phase space $M$ which is usually assumed to be a symplectic manifold. Points $x = (q, p)$ on $M$ describe the state of the system. Their coordinates are canonical positions $q$ and canonical momenta $p$. Functions $f$ on $M$ represent classical observables. The algebra $C^\infty(M)$ of smooth functions on $M$ is equipped with a Lie structure given by the Poisson bracket $\{\cdot,\cdot\}$. There is a measure $d\mu(x)$ on $M$ which describes the distribution of states throughout $M$. It is used to define the ensemble average of an observable $f \in C^\infty(M)$:

$$\tau(f) = \int_M f(x)\, d\mu(x). \tag{1}$$

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We will consider systems for which the total volume of the phase space is finite, $\tau(M) = 1$, so that $\mu$ has the meaning of a probability distribution.

A map $T$ of the phase space $M$ to itself which

- is one to one,
- preserves the phase space volume $\mu(TA) = \mu(A)$

generates a discrete time dynamics. We think of $T$ as the evolution of the system over one time unit. Powers of $T$, $T^n$ ($n$ integer), describe the evolution of the system over $n$ time units.

**Examples.** We take $M$ to be a torus. The volume element is simply given by $d\mu(x) = dqdp$, and the observables are Fourier series in $q$ and $p$. The ensemble average of $f$ is then equal to the term $f_{00}$ in the Fourier expansion of $f$.

1. *Baker's map:*

$$q \longrightarrow q^{'} = \begin{cases} 2q & \text{if} \quad q < 1/2; \\ 2q - 1 & \text{if} \quad q \geq 1/2, \end{cases}$$

$$p \longrightarrow p^{'} = \begin{cases} p/2 & \text{if} \quad q < 1/2; \\ (p+1)/2 & \text{if} \quad q \geq 1/2. \end{cases}$$

   Clearly, this map satisfies our requirements.

2. *Cat map:*

$$\begin{aligned} q &\longrightarrow q^{'} = aq + bp, \\ p &\longrightarrow p^{'} = cq + dp. \end{aligned}$$

   Here $a, b, c, d$ are integers satisfying $ad - bc = 1$. This condition guarantees that the map preserves the phase space volume (and is one to one). We also require that $|a + d| > 2$. This means that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   has two different real eigenvalues $|\mu_1| > 1$, and $|\mu_2| < 1$. The cat map is expanding along the direction of the eigenvector corresponding to $\mu_1$, and contracting along the direction of the eigenvector corresponding to $\mu_2$.

3. *Kronecker's map:*
   This one is simply given by

$$\begin{aligned} q &\longrightarrow q^{'} = q + \alpha, \\ p &\longrightarrow p^{'} = p + \beta, \end{aligned}$$

   where $\alpha$ and $\beta$ are real numbers such that $1, \alpha, \beta$, are linearly independent over $\mathbb{Z}$. This dynamics does not have periodic orbits.

4. *Kicked maps:*
   These are maps of the following form:

$$\begin{aligned} p &\longrightarrow p^{'} = p + \gamma f(q) \qquad (\mathrm{mod}\ 1), \\ q &\longrightarrow q^{'} = q + p^{'} \qquad (\mathrm{mod}\ 1), \end{aligned}$$

   where $f(q) = \sum_{k \in \mathbb{Z}} f_k e^{2\pi i k q}$ is a continuous periodic function satisfying the condition $\sum_{k \in \mathbb{Z}} k^2 |f_k| < \infty$. The choice $f(q) = \sin 2\pi q$ gives the *standard map.*

5. *Harper's maps:*
They are similar to kicked maps except that the "kinetic part" is periodic:

$$\begin{aligned} p &\longrightarrow p' = p + \gamma_1 f(q), \qquad \text{mod } 1, \\ q &\longrightarrow q' = q + \gamma_2 v(p'), \qquad \text{mod } 1, \end{aligned}$$

where both $f, v$ are periodic and satisfy suitable regularity conditions.

## 3. Classical ergodicity

The ergodic problem in classical mechanics consists in the following: What can be learned about the (statistical) behavior of an ensemble of mechanical (deterministic) systems from the long time behavior of an individual system? Integrable systems do not exhibit any stochastic behavior as the motion takes place along periodic trajectories. Hence, ergodicity is intimately connected to classical non-integrability. We list below some fundamental concepts and results of classical ergodic theory.

1. *Recurrence theorem* (Poincaré) If $U \subset M$ has positive measure, then there is a subset $U_0$ of measure zero such that for each $x \in U \setminus U_0$ there is $k$ with the property that $T^k x \in U$
2. *Ergodic theorem* (Boltzmann, Birkhoff, von Neumann) states that for each observable $f$

$$\text{time average of } f = \text{ ensemble average of } f,$$

or, for almost all initial conditions,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(T^k x_0\right) = \int_M f(x)\, d\mu(x)$$

($f$ is an observable). This theorem holds for systems which have the following property: No non-trivial subset of the phase space is invariant under the dynamics. Baker's maps, Kronecker's map, and the cat map are all ergodic. The ergodicity of the kicked and Harper's maps is a more complicated issue. For small values of the parameter $\gamma$, they are not ergodic (a consequence of the KAM theorem). For large values of $\gamma$, the "islands of ergodicity" are getting smaller and smaller. No theorems are known, and the numerical evidence is inconclusive.
3. Stronger than ergodicity is the *mixing property*: A system is mixing if

$$\lim_{n\to\infty} \mu(A \cap T^n B) = \mu(A)\, \mu(B).$$

This property means that the dynamics spreads the set of initial states uniformly throughout the phase space. Not all ergodic systems are mixing. For instance, Kronecker's map is ergodic but not mixing. The mixing property is equivalent to the following fact about the long time behavior of the ensemble average of a product of observables:

$$\lim_{n\to\infty} \int f(x)\, g(T^n x)\, d\mu(x) = \int f(x)\, d\mu(x) \int g(x)\, d\mu(x).$$

4. Kolmogorov-Sinai (KS) entropy measures how strongly mixing is the system. It is constructed as follows. Cover the phase space $M$ with a measurable

covering $\mathcal{A} = \{A_1, \dots, A_k\}$. With this partition we associate its Shannon entropy:

$$S(\mathcal{A}) = -\sum_{j=1}^{k} \mu(A_j) \log \mu(A_j).$$

Let us now see what happens to this covering if we wait one time unit. A new partition of $M$ arises, this time given by $T\mathcal{A} = \{TA_1, \dots, TA_k\}$. To measure the resulted mixing of the phase space we compute the Shannon entropy of the partition $\mathcal{A} \vee T\mathcal{A}$ obtained by intersecting the elements of the original partition with the elements of the new partition. Keep on doing it. The limit

$$S(T, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} S\left(\mathcal{A} \vee T\mathcal{A} \vee \dots \vee T^{n-1}\mathcal{A}\right)$$

exists, and its supremum over all choices of the initial partition

$$S(T) = \sup_{\mathcal{A}} S(T, \mathcal{A})$$

is called the KS entropy. The KS entropy is a measure of chaos in a system as

- it is zero for periodic systems;
- it is zero for ergodic but not mixing systems (e.g. Kronecker's dynamics);
- it is related to the Lyapunov exponents (Pesin's theorem).

For Kronecker's map the KS entropy is zero, for bakers map,

$$S(T_{bak}) = \log 2,$$

while for the cat map,

$$S(T_{cat}) = \log|\mu_1|.$$

For kicked and Harper's maps, the KS entropy is unknown.

## 4. Quantum mechanics

4.1. **Quantization.** In quantum mechanics, the commutative world of classical mechanics is replaced by the non-commutative world of operators on Hilbert spaces (Heisenberg, Born, Jordan, Schrödinger, Dirac, von Neumann,...). The quantum phase space is no longer a set of points. Rather, it is a non-commutative space defined in terms of a non-commutative algebra of observables. In the simplest case of the quantized flat space, this algebra is generated by the canonical position and momentum operators.

Quantization of a dynamical system has two components: kinematic and dynamic. The kinematic component involves the construction of a suitable quantized phase space of the system. This quantized phase space is given in terms of a non-commutative algebra $\mathfrak{A}_\hbar$ of observables. In the language of non-commutative geometry, $\mathfrak{A}_\hbar$ is an algebra of functions on the quantized phase space. Specific choices of the structure of $\mathfrak{A}_\hbar$ can be made: a $\mathbb{C}^*$-algebra, a von Neumann algebra, or some suitably defined locally convex algebra. Throughout this talk, we will assume that $\mathfrak{A}_\hbar$ is a von Neumann algebra with a countable predual. In other words, $\mathfrak{A}_\hbar$ acts on a separable Hilbert space, an assumption usual made in physics. A classical observable $f$ is mapped onto a quantum observable $Q_\hbar(f) \in \mathfrak{A}_\hbar$.

The "suitability" of the choices made, namely that of the algebra $\mathfrak{A}_\hbar$ and of the time evolution, is settled by the correspondence principle. This amounts to showing that limits of the quantized objects, as $\hbar \to 0$, yield the corresponding classical objects. Quantization is a highly non-unique procedure, and the correspondence principle is the only physical principle allowing one to decide whether a particular procedure is correct. A natural mathematical framework for quantization is "strict deformation quantization" proposed by Rieffel [17]. The key requirement is that

$$\lim_{\hbar\to 0}\left\|\frac{1}{i\hbar}\left[Q_\hbar(f),Q_\hbar(g)\right]-Q_\hbar(\{f,g\})\right\|_\hbar=0,$$

where $f,g\in C^\infty(M)$.

The dynamic component of quantization consists in defining a time evolution on the quantized phase space. A natural way of doing this is to find a suitable one parameter group of automorphisms $\alpha_t$ of $\mathfrak{A}_\hbar$, where the parameter (discrete or continuous) has the meaning of time. For maps, $\alpha_n=\alpha^n$. In examples, $\alpha$ is often implemented by a unitary operator $F$, $\alpha(O)=F^{-1}OF$. The correspondence principle takes the form of the requirement

$$\lim_{\hbar\to 0}\left\|F^{-n}Q_\hbar(f)F^n-Q_\hbar(f\circ T^n)\right\|_\hbar=0.$$

Statements of this kind are similar to Egoroff's theorem in the theory of pseudo-differential operators.

The ensemble average of a quantum system is given by a state $\tau_\hbar$ over the algebra $\mathfrak{A}_\hbar$. For technical reasons, we will assume that this state is faithful and normal. Physically, this means that an ensemble average is given by a density matrix whose pure components form a separating set for $\mathfrak{A}_\hbar$.

**Definition 4.1.** *A quantum map is a triple $(\mathfrak{A}_\hbar,\alpha,\tau_\hbar)$ arising as a quantization of a discrete time dynamical system in the sense described above.*

This definition is somewhat tentative, and we make it here merely for the sake of convenience. We leave out, for example, the issue of whether each meaningful quantum system arises as a quantization of a classical system.

4.2. **Flat space.** In the case of a flat space, we choose to work with the Bargmann representation of the Hilbert space of states, i.e. the space of analytic functions $\varphi(z),\psi(z)$, with an inner product

$$\langle\varphi,\psi\rangle=\int_{\mathbb{C}}\overline{\varphi(z)}\psi(z)\,d\mu_\hbar(z),$$

where $d\mu_\hbar(z)=\frac{\hbar}{\pi}e^{-|z|^2/\hbar}d^2z$. We denote this Hilbert space by $\mathcal{H}^2(\mathbb{C})$. It carries a projective unitary representation of the group of translations $z\to U(z)$:

$$U(z)\varphi(w)=e^{-|z|^2/2\hbar+\overline{z}w/\hbar}\varphi(w-z).$$

For concreteness, the quantization map $Q_\hbar$ is taken to be the Toeplitz quantization. As a suitable class of symbols for the Toeplitz operators one may take almost periodic functions on the plane [3], [4].

4.3. **Quantum torus.** We study quantized discrete time systems whose classical phase space is a torus. A toroidal phase space can be quantized by replacing classical functions by unitary operators $U=e^{2\pi iq}=U\left(-i\sqrt{2}\pi\hbar\right)$ and $V=e^{2\pi ip}=U\left(\sqrt{2}\pi\hbar\right)$ acting on the Hilbert space $\mathcal{H}^2(\mathbb{C})$ defined above. They satisfy the following commutation relation

$$UV=e^{4\pi^2\hbar i}VU.$$

Since the torus is compact, Planck's constant must obey an integrality condition $\hbar = 1/2\pi N$, $N$ positive integer. The algebra of observables $\mathfrak{A}_\hbar$ is defined as the von Neumann algebra generated by $U, V$. For a summary of results concerning this algebra (with and without the integrality condition), see [18]. The ensemble average of an observable $O$ is given by the following state on $\mathfrak{A}_\hbar$:

$$\tau_\hbar(O) = \int_{\mathbb{T}^2} \left\langle \varphi, U(z)^\dagger OU(z)\varphi \right\rangle d^2z,$$

where $\varphi$ is an arbitrary normalized element of $\mathcal{H}^2(\mathbb{C})$. It is well known that $\tau_\hbar$ is, in fact, a faithful normal trace on $\mathfrak{A}_\hbar$. As $\hbar \to 0$, this trace reproduces the classical ensemble average given by (1). It is characterized by the property that

$$\tau_\hbar(U^m V^n) = \delta_{m0}\delta_{n0}.$$

**Theorem 4.1.** [17], [18] *This algebra satisfies the conditions of strict deformation quantization.*

We introduce the following notation:

$$X = U\left(-i/\sqrt{2}\right), \quad Y = U\left(1/\sqrt{2}\right),$$

and observe that

$$[X,\ Y] = 0.$$

The operators $X$ and $Y$ generate an action of the group $\mathbb{Z}^2$ on $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$. We also verify easily that,

$$\begin{aligned} [X,\ Y] &= 0, \quad [X,\ V] = 0, \\ [Y,\ U] &= 0, \quad [Y,\ V] = 0, \end{aligned}$$

and so $X$ and $Y$ are in the commutant of $\mathfrak{A}_\hbar$. Also,

$$X = U^N, \quad Y = V^N$$

We shall call a holomorphic function $\phi$ on $\mathbb{C}$ a $\mathbb{Z}^2$-*automorphic form* if

$$\begin{aligned} X\phi(z) &= e^{2\pi i\theta_1}\phi(z), \\ Y\phi(z) &= e^{2\pi i\theta_2}\phi(z), \end{aligned}$$

where $\theta = (\theta_1,\ \theta_2) \in \mathbb{T}^2$. In other words, $\mathbb{Z}^2$-automorphic forms are simultaneous generalized eigenvectors of $X$ and $Y$. Let $\mathcal{H}_\hbar(\theta)$ denote the space of all $\mathbb{Z}^2$-automorphic forms with fixed $\theta$. Clearly, $\phi \in \mathcal{H}_\hbar(\theta)$ is uniquely determined once defined on the fundamental domain $D = [0,\ 1] \times [0,\ 1] \subset \mathbb{R}$. The space $\mathcal{H}_\hbar(\theta)$ has a natural inner product defined as an integral over this domain:

$$\langle \phi_1,\ \phi_2 \rangle = \int_D \overline{\phi_1(z)}\phi_2(z)\, d\mu_\hbar(z).$$

(Note a similar integral over the entire complex plane *does not converge*, hence the $\mathbb{Z}^2$-automorphic forms are not in $\mathcal{H}^2(\mathbb{C})$.) This inner product is a $\mathbb{Z}^2$ version of the Petersson inner product. In the following theorem we construct a natural orthonormal basis for the space $\mathcal{H}_\hbar(\theta)$.

**Theorem 4.2.** (1) *The following functions are elements of* $\mathcal{H}_\hbar(\theta)$*:*

$$\phi_m^{(\theta)}(z) = C_m(\theta)\, e^{-N\pi z^2 + 2\sqrt{2}\pi(\theta_1+m)z} \sum_{k\in\mathbb{Z}} e^{-N\pi k^2 - 2\pi(\theta_1+i\theta_2+m)k + 2\sqrt{2}N\pi kz},$$

*where*

$$C_m(\theta) = (2/N)^{1/4} e^{-\pi(\theta_1+m)^2/N - 2\pi i\theta_2 m/N}.$$

*They are periodic in* $m$*,*

$$\phi_{m+N}^\theta = \phi_m^\theta,$$

*and furthermore,*

$$\phi_0^\theta, \ldots, \phi_{N-1}^\theta$$

*are orthonormal vectors in* $\mathcal{H}_\hbar(\theta)$*.*
(2) *The space* $\mathcal{H}_\hbar(\theta)$ *has dimension* $N$*. Consequently, the functions* $\phi_n^\theta$, $n = 0, \ldots, N-1$, *form an orthonormal basis for* $\mathcal{H}_\hbar(\theta)$*.*
(3) *There is an isomorphism*

$$\kappa : \mathcal{H}(\mathbb{C}) \longrightarrow \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}_\hbar(\theta)\, d\theta,$$

*such that*

$$\kappa U \kappa^{-1} \phi_m(\theta, z) = e^{2\pi i(\theta_1+m)/N} \phi_m(\theta, z),$$
$$\kappa V \kappa^{-1} \phi_m(\theta, z) = e^{2\pi i\theta_2/N} \phi_m(\theta, z).$$

This is the kinematic part of quantization of the torus.

4.4. **More complicated geometries.** Phase spaces of more complicated geometry can be quantized in an analogous way. Various techniques have been developed (geometric quantization, deformation quantization, Toeplitz quantization, ...). Explicit constructions are known, for example, for compact and non-compact Hermitian symmetric spaces, Riemann surfaces, compact Kahler manifolds, a large class of Hermitian symmetric supermanifolds, etc. An approach based on a holomorphic representation was initiated in [1] and has been developed by a number of authors.

4.5. **Quantization of maps.** Now, we quantize some of the the toroidal maps introduced before.

(1) The *cat map* is quantized by means of a single time unit evolution operator $F$ on Bargmann space such that

$$U \longrightarrow U' = F^{-1}UF = e^{2\pi^2 iab\hbar} U^a V^b,$$
$$V \longrightarrow V' = F^{-1}VF = e^{2\pi^2 icd\hbar} U^c V^d.$$

This defines an automorphism of $\mathfrak{A}_\hbar$. The evolution operator $F$ can be written down explicitly in terms of Gauss sums.

**Theorem 4.3.** [15] *The matrix elements of the operator* $F$*,*

$$\left\langle \phi_m^{(\theta^*)}, F\phi_n^{(\theta)} \right\rangle = \int_D \overline{\phi_m^{(\theta^*)}(z)}\, F\phi_n^{(\theta)}(z)\, d\mu_\hbar(z),$$

*where* $\theta^* = T^{-1}\theta - (Nbd/2, Nac/2)$, *is given by*

$$\left\langle \phi_m^{(\theta^*)}, F\phi_n^{(\theta)} \right\rangle = (Nb)^{-1/2}\, e^{i\nu/2} e^{2\pi i \left(m\widetilde{\theta}_2 - n\theta_2\right)/N} \sum_{r=0}^{|b|-1} e^{-2\pi i r\theta_2} e^{i\pi\Phi\left(m+\widetilde{\theta}_1, n+Nr+\theta_1\right)},$$

*where* $e^{i\nu} = -i\alpha/|\alpha|$, *and where*

$$\Phi(x,y) = ax^2 - 2xy + dy^2.$$

(2) *Kronecker's map* is easy to quantize [15]:

$$\begin{aligned} U &\longrightarrow U' = F^{-1}UF = e^{2\pi i\alpha}U \\ V &\longrightarrow V' = F^{-1}VF = e^{2\pi i\beta}V \end{aligned}$$

One can write down explicit expressions for $F$ in the representation given by

$$\left\langle \phi_m^{(\widetilde{\theta})}, F\phi_n^{(\theta)} \right\rangle = e^{2\pi i\beta(\theta_1 - N\alpha/2)}\delta_{mn}.$$

(3) Quantum *kicked maps* are given by the following automorphism of $\mathfrak{A}_\hbar$ [16]:

$$\begin{aligned} U &\longrightarrow U' = F^{-1}UF = e^{-2\pi^2 iab\hbar}V'U, \\ V &\longrightarrow V' = F^{-1}VF = Ve^{2\pi i\gamma\widetilde{f}(U)}, \end{aligned}$$

where

$$\widetilde{f}(U) = \sum \frac{1 - e^{-4\pi^2\hbar ik}}{4\pi^2\hbar ki} f_k U^k.$$

(4) Quantization of Harper's maps is similar, and I will skip the details.

The quantum dynamics defined above obey the correspondence principle.

**Theorem 4.4.** *For the quantum cat, Kronecker, kicked, and Harper's maps,*

$$\lim_{\hbar\to 0} \left\| F^{-n}Q_\hbar(f)F^n - Q_\hbar(f\circ T^n)\right\|_\hbar = 0.$$

## 5. Quantum ergodicity

5.1. **Quantum recurrence.** Much of classical ergodic theory can be extended to the quantum mechanical context. The first fundamental result is the Poincaré recurrence theorem. The classical recurrence theorem states that the state of a system returns arbitrarily close to the initial point if one waits sufficiently long. There is a simple quantum analog of this theorem. Let $\psi_1, \psi_2 \ldots$ , be a sequence of normalized vectors in the Hilbert space of states, and let

$$\rho = \sum_n p_n \rho_n$$

be the corresponding density matrix. Here $\rho_n = P_{\psi_n}$ is the projection operator onto the vector $\psi_n$. It is easiest to state the quantum recurrence theorem for the case of flows rather than maps.

**Theorem 5.1.** *Let* $\rho$ *be a density matrix, and let* $F$ *have a purely discrete spectrum. Then for any* $\epsilon > 0$ *there is* $T = T(\epsilon) > 0$ *such that every interval of length* $T$ *contains at least one* $\tau$ *with the property that* $\|\rho(t) - \rho\|_{HS} \leq \epsilon$.

This theorem merely states that the function $t \longrightarrow \mathrm{tr}\,(\rho\,(t)\,\rho)$ is almost periodic. It is a slight extension of a theorem proved in the fifties by Bocchieri and Loigner [6] for the case of pure states.

5.2. **Ergodicity and mixing.** We define the Hilbert space $\mathcal{K} = L^2(\mathfrak{A}, \tau)$ associated with the algebra $\mathfrak{A}$ as the completion of $\mathfrak{A}$ in the norm given by the inner product $(A, B) = \tau\left(A^\dagger B\right)$. It is natural to regard this space as the quantum version of the Koopman Hilbert space, as it reduces to the latter in the classical case. We would like to emphasize that the analogy often drawn in the literature between the classical Koopman space and the quantum mechanical Hilbert space of states $\mathcal{H}$ is misleading: it is the space $\mathcal{K}$ that is a natural scene for quantum ergodic theory. As a consequence of the time invariance of $\tau$, $\alpha$ defines a unitary operator on $\mathcal{K}$ which we will continue to denote by the same symbol. This operator is the quantum version of the classical Koopman operator.

A quantum map $(\mathfrak{A}, \alpha, \tau)$ is called:
*mixing*, if for all $A, B \in \mathfrak{A}$,

$$\lim_{N\to\infty} \tau\left(\alpha^n\left(A\right)B\right) = \tau\left(A\right)\tau\left(B\right);$$

*weak mixing*, if for all $A, B \in \mathfrak{A}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n\leq N-1} \left|\tau\left(\alpha^n\left(A\right)B\right) - \tau\left(A\right)\tau\left(B\right)\right|^2 = 0;$$

*ergodic*, if for all $A \in \mathfrak{A}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n\leq N-1} \alpha^n\left(A\right) = \tau\left(A\right)I,$$

strongly on $\mathcal{K}$.

For quantum maps, we have the usual hierarchy: mixing $\Rightarrow$ weak mixing $\Rightarrow$ ergodicity.

Ergodic, weakly mixing and mixing systems can be characterized in terms of the properties of the spectrum of the automorphism $\alpha$. We will say that $\alpha$ has continuous spectrum if 1 is its only eigenvalue and the corresponding eigenvectors are the multiples of the identity operator.

**Theorem 5.2.** (i) *A quantum map is ergodic if and only if* 1 *is an eigenvalue of* $\alpha$ *and the corresponding eigenvectors are multiples of* $I$*;*
(ii) *A quantum map is weakly mixing if and only if the spectrum of* $\alpha$ *is continuous;*
(iii) *A weakly mixing quantum map is mixing if the spectrum of* $\alpha$ *is absolutely continuous.*

Hence, quantum maps for which $\alpha$ has pure point spectrum cannot be mixing.

**Theorem 5.3.** (i) *Quantum Kronecker's dynamic is ergodic but not mixing;*
(ii) *Quantum cat dynamics is mixing.*

In fact, quantum Kronecker's maps are uniformly ergodic [15].

5.3. **Connes-Stormer entropy (quantum KS entropy).** Given an algebra of observables $\mathfrak{A}$, a faithful normal trace $\tau$, and an automorphism $\alpha$, there is a construction of an entropy associated with mixing of the quantum phase space resulting from the time evolution. This entropy, denoted here by $H(\alpha)$, is called the Connes-Stormer (CS) entropy. The construction of the CS entropy is, roughly, parallel to the construction of the KS entropy. The CS entropy is a measure of chaos in a quantum dynamical system, very much like the KS entropy is a measure of chaos in a classical system.

For simple dynamics, like the cat, Kronecker, and baker's dynamics, the CS entropy can be calculated explicitly. The result is that the quantum entropy equals the classical entropy [13].

**Theorem 5.4.** *The CS entropies of the quantized cat, Kronecker's and baker's maps are equal to the KS entropies of the corresponding classical dynamics.*

This means that, in these systems, chaotic behavior persists quantization.

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