Option Smile and the SABR Model of Stochastic Volatility

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March 20, 2014

Outline

LIBOR rate

- *LIBOR spot rate* is a key benchmark interest rate for capital market transactions:
  - Interest rate on a 3 month deposit, set daily in London
  - Used as index rate for setting cash flows on floating rate instruments
- *OIS rate*:
  - Based the Federal Funds effective rate, set daily
  - Used for discounting cash flows

FRAs and swaps

- *FRAs*:
  - Counterparty A agrees to pay counterparty B interest rate F on a 3 month deposit starting T years from now
  - The break-even rate on a FRA is called the LIBOR *forward rate*
  - Traded over the counter (OTC)
- *Swaps*:
  - Multi-period versions of FRAs
  - The break-even rate on a FRA is called the *swap rate*
  - Traded OTC

Options on LIBOR

The simplest *options* on a LIBOR forward rate are *caps* and *floors*.

- A cap struck at K (say 2%) is the option to receive T years from now the interest
$$(F - K)^+,$$
on a pre-specified notional principal. Here, $x^+ = \max(x, 0)$.
- Likewise, a floor struck at K is the option to receive T years from now the interest
$$(K - F)^+.$$

Other commonly traded interest rate options are options on swaps a.k.a. *swaptions*. Swaptions can be viewed as options on *forward swap rates*.

Black's model

The market standard for quoting option prices is in terms of *Black's model*. We assume that a forward rate $F(t)$, such as a LIBOR forward or a forward swap rate, follows a driftless lognormal process reminiscent of the basic Black-Scholes model,

$$dF(t) = \sigma F(t)\, dW(t)\,.$$

Here $W(t)$ is a Wiener process, and σ is the *lognormal volatility*. It is understood here, that we have chosen a numeraire $\mathcal{Z}$ with the property that, in the units of that numeraire, $F(t)$ is a tradable asset. The process $F(t)$ is thus a martingale, and we let Q denote the probability distribution.

The solution to this stochastic differential equation reads:

$$F(t) = F_0 \exp\left(\sigma W(t) - \frac{1}{2}\,\sigma^2 t\right).$$

Black's model

Consider a European call struck at K and expiring at T. Its value today is

$$\mathrm{Call}(T, K, F_0, \sigma) = \mathcal{Z}_0 \mathsf{E}[(F(T) - K)^+],$$

where E denotes expected value with respect to Q, and where $\mathcal{Z}_0$ is the current value of $\mathcal{Z}$. Explicitly,

$$\mathrm{Call}(T, K, F_0, \sigma) = \mathcal{Z}_0 \left[F_0 N(d_+) - K N(d_-) \right]$$

Here,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

is the cumulative normal distribution, and

$$d_\pm = \frac{\log \frac{F_0}{K} \pm \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} .$$

Similarly, the price of a European put is given by:

$$\mathrm{Put}(T, K, F_0, \sigma) = \mathcal{Z}_0 \left[- F_0 N(-d_+) + K N(-d_-) \right].$$

Calibrating Black's model

Black's model is calibrated to the market if

$$\text{Price}\left(T, K, F_0, \sigma\right) = \text{MktPrice}.$$

This equation can be solved for σ :

$$\sigma_{\text{impl}} = \text{ImplVol}\left(T, K, F_0, \text{MktPrice}\right).$$

σ_{impl} is known as the *implied volatility*.

Issues with Black's model

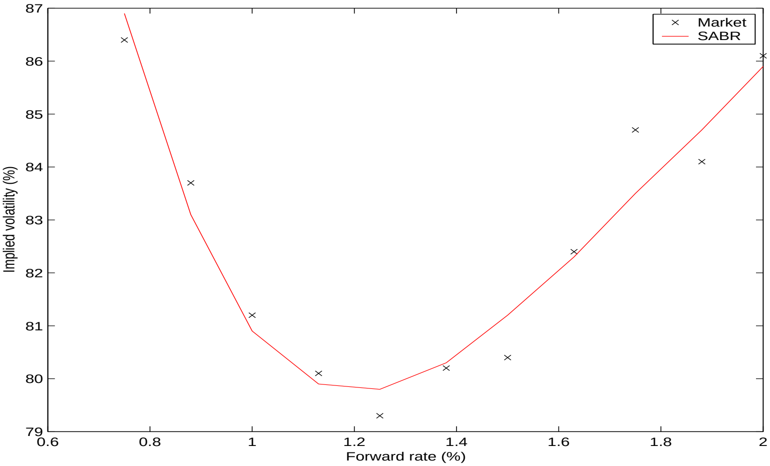
The basic premise of Black's model, that σ is independent of T, K, and F_0, is not supported by the markets. In fact, implied volatilities exhibit:

(a) *Term structure*: At the money volatility depends on the option expiration.

(b) *Smile* (or *skew*): For a given expiration, there is a pronounced dependence of implied volatilities on the option strike.

In order to accurately value and risk manage options portfolios, refinements to Black's model are necessary. Modeling term structure of volatility is hard, and not much progress has been made. We will focus on modeling volatility smile.

Volatility smile



Local volatility models

An extension of Black's model is a class of models called *local volatility models*.

- The idea is that even though the exact nature of volatility of the underlying asset is unknown, one can use an effective ("local") specification of the underlying process so that the implied volatilities match the market implied volatilities.
- Local volatility models are specified in the form

$$dF(t) = C(t, F(t))dW(t),$$

where $C(t, F)$ is an effective instantaneous volatility. It is convenient to work with a parametric specification of $C(t, F(t))$ that fits the market data.

Local volatility models

Popular local volatility models which admit analytic solutions include:

(a) Normal model
(b) Shifted lognormal model
(c) CEV model

Option price in a local volatility model has the form

$$\mathrm{Price}(T, K, F_0, \sigma) = \mathcal{Z}_0 \mathrm{B}(T, K, F_0, \sigma),$$

where $\mathrm{B}(T, K, F_0, \sigma) = \mathsf{E}[(F(T) - K)^+]$.

Normal model

The dynamics of the forward $F(t)$ in the normal model reads

$$dF(t) = \sigma dW(t).$$

The parameter σ is called the *normal volatility*. This is easy to solve:

$$F(t) = F_0 + \sigma W(t).$$

This solution exhibits a drawbacks of the normal model: $F(t)$ may become negative in finite time.
The of a European call is given by:

$$\mathrm{Call}(T, K, F_0, \sigma) = \mathcal{Z}_0 \sigma \sqrt{T} \Big(d_+ N(d_+) + N'(d_+) \Big),$$

where

$$d_+ = \frac{F_0 - K}{\sigma \sqrt{T}}.$$

Many traders think in terms of normal implied volatilities, as the normal model often seems to capture the rates dynamics better than the lognormal model.

Shifted lognormal model

The *shifted lognormal model* is a diffusion process whose volatility structure is a linear interpolation between the normal and lognormal volatilities:

$$dF(t) = (\sigma_1 F(t) + \sigma_0)\, dW(t)\,.$$

The volatility structure of the shifted lognormal model is given by the values of the parameters σ_1 and σ_0.
The price of a call is given by the following valuation formula:

$$\mathrm{Call}(T, K, F_0, \sigma_0, \sigma_1) = \mathcal{Z}_0\Big((F_0 + \sigma_0/\sigma_1)N(d_+) - (K + \sigma_0/\sigma_1)N(d_-)\Big),$$

where

$$d_\pm = \frac{\log \dfrac{\sigma_1 F_0 + \sigma_0}{\sigma_1 K + \sigma_0} \pm \dfrac{1}{2}\,\sigma_1^2 T}{\sigma_1 \sqrt{T}}\,.$$

CEV model

The *CEV model*, whose volatility structure is a power interpolation between the normal and lognormal volatilities, is given by

$$dF(t) = \sigma F(t)^{\beta} \, dW(t),$$

where $-\infty < \beta < 1$. In order for the dynamics to be well defined, we specify a boundary condition at $F = 0$:

(a) *Dirichlet* (absorbing): $F|_0 = 0$. Solution exists for all values of β, or

(b) *Neumann* (reflecting): $F'|_0 = 0$. Solution exists for $\frac{1}{2} \le \beta < 1$.

The CEV model requires solving a terminal value problem for the backward Kolmogorov equation:

$$\frac{\partial}{\partial t} \mathrm{B}(t,x) + \frac{1}{2} \sigma^2 x^{2\beta} \frac{\partial^2}{\partial x^2} \mathrm{B}(t,x) = 0,$$

$$\mathrm{B}(T,x) = \begin{cases} (x-K)^+, & \text{for a call,} \\ (K-x)^+, & \text{for a put,} \end{cases}$$

with appropriate boundary condition at zero x.

CEV model

Pricing formulas for the CEV model can be obtained in a closed (albeit somewhat complicated) form.

For example, the Dirichlet call price can be expressed in terms of the the cumulative function $\Phi(x; r, \lambda)$ of the non-central χ^2 distribution as follows:

$$
\begin{aligned}
&\mathrm{Call}(T, K, F_0, \sigma) \\
&\quad = \mathcal{Z}_0\Big(F_0\Big(1 - \Phi\big(\tfrac{\nu^2 K^{2/\nu}}{\sigma^2 T};\ \nu + 2, \tfrac{\nu^2 F_0^{2/\nu}}{\sigma^2 T}\big)\Big) - K\Phi\big(\tfrac{\nu^2 F_0^{2/\nu}}{\sigma^2 T};\ \nu, \tfrac{\nu^2 K^{2/\nu}}{\sigma^2 T}\big)\Big)
\end{aligned}
$$

where

$$
\nu = \frac{1}{1-\beta}\,, \quad \text{i.e.} \nu \geq 1\,.
$$

SABR model

The volatility skew models that we have discussed so far improve on Black's models but still fail to reflect the market dynamics. One issue is, for example, the "wing effect" exhibited by the implied volatilities of some expirations (especially shorter dated) which is not captured by these models: the implied volatilities tend to rise for high strikes forming the familiar "smile" shape. Among the attempts to move beyond the locality framework are:

(a) *Stochastic volatility models*. In this approach, we add a new stochastic factor to the dynamics by assuming that a suitable volatility parameter itself follows a stochastic process.

(b) *Jump diffusion models*. These models use a broader class of stochastic processes (for example, *Levy processes*) to drive the dynamics of the underlying asset. These more general processes allow for discontinuities ("jumps") in the asset dynamics.

SABR model

The SABR model is an extension of the CEV model in which the volatility parameter follows a stochastic process:

$$dF(t) = \sigma(t) F(t)^{\beta} dW(t),$$
$$d\sigma(t) = \alpha\sigma(t) dZ(t).$$

Here $F(t)$ is a process which, may denote a LIBOR forward or a forward swap rate, and $\sigma(t)$ is the stochastic volatility parameter. The two Brownian motions, $W(t)$ and $Z(t)$ are correlated:

$$\mathrm{E}[dW(t)\, dZ(t)] = \rho dt,$$

where ρ is assumed constant. The parameter α is the volatility of $\sigma(t)$ (the *volvol*), and is also assumed constant. The dynamics is subject to the initial condition:

$$F(0) = F_0,$$
$$\sigma(0) = \sigma_0,$$

where F_0 is the current value of the forward, and σ_0 is the current value of the volatility parameter.

SABR model

As in the case of the CEV model, the analysis of the SABR model requires solving the terminal value problem for the backward Kolmogorov equation. Namely, the valuation function $\mathrm{B}(t, x, y)$ is the solution to

$$\frac{\partial}{\partial t}\,\mathrm{B} + \frac{1}{2}\,\sigma^2\Big(x^{2\beta}\,\frac{\partial^2}{\partial x^2} + 2\alpha\rho x^{\beta}\,\frac{\partial^2}{\partial x \partial y} + \alpha^2\,\frac{\partial^2}{\partial y^2}\Big)\mathrm{B} = 0,$$

$$\mathrm{B}(T, x, y) = \begin{cases} (x-K)^+\,, & \text{for a call,} \\ (K-x)^+\,, & \text{for a put.} \end{cases}$$

This is a more difficult problem than the models discussed above. Except for the special case of $\beta = 0$, no explicit solution to this model is known. The general case can be solved approximately by means of a perturbation expansion in the parameter $\varepsilon = T\alpha^2$, where T is the maturity of the option. Luckily, this parameter is typically small and the approximate solution is actually quite accurate. Also significantly, this solution is very easy to implement in computer code, and lends itself well to risk management of large portfolios of options in real time.

The SABR model

There is no known closed form option valuation formula in the SABR model. Instead, one takes the following approach. We force the valuation formula in the SABR model to be of the form

$$\begin{aligned}\mathrm{Call}(T, K, F_0, \sigma) &= \mathcal{Z}_0\big[F_0 N(d_+) - K N(d_-)\big],\\ \mathrm{Put}(T, K, F_0, \sigma) &= \mathcal{Z}_0\big[-F_0 N(-d_+) + K N(-d_-)\big],\\ d_\pm &= \frac{\log\dfrac{F_0}{K} \pm \dfrac{1}{2}\,\sigma^2 T}{\sigma\sqrt{T}}\,,\end{aligned}$$

given by Black's model, with the implied volatility σ now a function of the SABR model parameters and the market data. This can be done by means of an asymptotic expansion.

SABR model

One can show shows that the implied volatility is then approximately given by:

$$\sigma(T, K, F_0, \sigma_0, \alpha, \beta, \rho) = \alpha \, \frac{\log(F_0/K)}{D(\zeta)} \Big\{ 1 + \Big[\frac{2\gamma_2 - \gamma_1^2 + 1/F_{\text{mid}}^2}{24} \times \Big(\frac{\sigma_0 F_{\text{mid}}^{\beta}}{\alpha} \Big)^2 + \frac{\rho\gamma_1}{4} \, \frac{\sigma_0 F_{\text{mid}}^{\beta}}{\alpha} + \frac{2 - 3\rho^2}{24} \Big] \varepsilon + \dots \Big\},$$

where F_{mid} denotes a midpoint between F_0 and K (such as $(F_0 + K)/2$), and

$$\gamma_1 = \frac{\beta}{F_{\text{mid}}} \, , \quad \gamma_2 = \frac{\beta(\beta - 1)}{F_{\text{mid}}^2} \, .$$

The "distance function" entering the formula above is given by:

$$\delta(\zeta) = \log\Big(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} + \zeta - \rho}{1 - \rho} \Big),$$

where

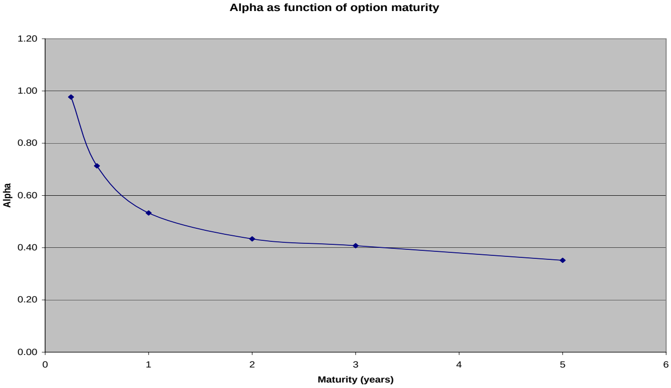
$$\zeta = \frac{\alpha}{\sigma_0} \int_K^{F_0} \frac{dx}{x^{\beta}} = \frac{\alpha}{\sigma_0(1 - \beta)} \, (F_0^{1-\beta} - K^{1-\beta}).$$

Calibration of SABR

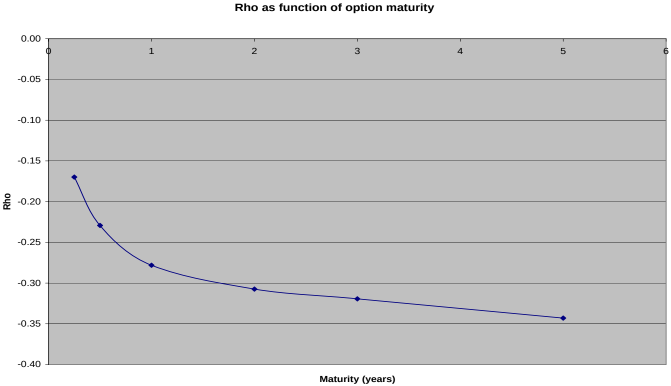
For each option maturity we need four model parameters: $\sigma_0, \alpha, \beta, \rho$. We choose them so that the model matches the market implied vols for several different strikes.

- It turns out that there is a bit of redundancy between the parameters β and ρ. As a result, one usually calibrates the model by fixing β.
- For a fixed $\beta \leq 1$, say $\beta = 0.5$, we calibrate σ_0, α, ρ. This works quite well under "normal" conditions.
- The SABR model can be calibrated to match the market in a variety of conditions.
- Calibration results show a persistent term structure of the model parameters as functions of option expiration: typical is the shape of the parameter α which starts out high for short dated options and then declines monotonically as the option expiration increases.

Calibration of SABR



Calibration of SABR



Calibration of SABR

Generally, the model behaves very well under various market conditions. However:

- In times of distress, such as 2008 - 2009, the height of the recent financial crisis, the choice of $\beta = 0.5$ occasionally led to extreme calibrations of the correlation parameters ($\rho = \pm 1$). As a result, some practitioners choose high β's, $\beta \approx 1$ for short expiry options and let it decay as option expiries move out.
- On a few days at the height of the recent financial crisis the value of α corresponding to 1 month into 1 year swaptions was as high as 4.7.

Calibration of SABR

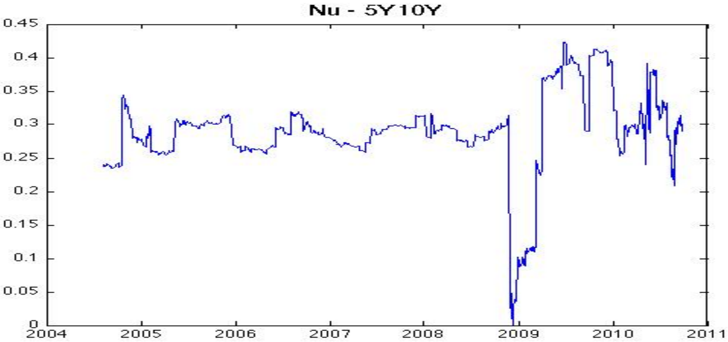


Figure: Historical values of the calibrated parameter α ($\beta = 0.5$)

Calibration of SABR

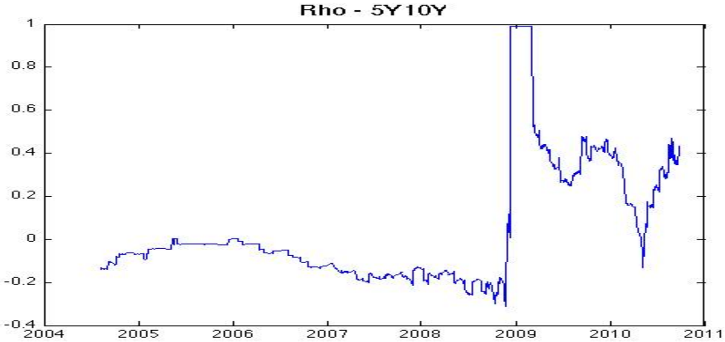


Figure: Historical values of the calibrated parameter ρ ($\beta = 0.5$)

Arbitrage in the SABR model

The explicit implied volatility formulas make the SABR model easy to implement, calibrate, and use. These implied volatility formulas are usually treated as if they were exact, even though they are derived from an asymptotic expansion which requires that $\alpha^2 T \ll 1$. The unstated argument is that instead of treating these formulas as an accurate approximation to the SABR model, they could be regarded as the exact solution to some other model which is well approximated by the SABR model. This is a valid viewpoint as long as the option prices obtained using the explicit formulas for σ are arbitrage free. There are two key requirements for arbitrage freeness of a volatility smile model:

(i) Put-call parity, which holds automatically since we are using the same implied volatility σ for both calls and puts.

(ii) The terminal probability density function implied by the call and put prices needs to be positive.

Arbitrage in the SABR model

To explore the second condition, note that call and put prices can be written quite generally as

$$\begin{aligned}\mathrm{Call}(T,K) &= \mathcal{Z}_0 \int_{-\infty}^{\infty} (F-K)^+\, q_T(F,K)dF,\\ \mathrm{Put}(T,K) &= \mathcal{Z}_0 \int_{-\infty}^{\infty} (K-F)^+\, q_T(F,K)dF,\end{aligned}$$

where $q_T(F,K)$ is the density of the risk-neutral probability Q at the exercise date. Since

$$\frac{d^2}{dx^2}\, x^+ = \delta(x),$$

it follows that

$$\begin{aligned}\frac{\partial^2}{\partial K^2}\,\mathrm{Call}(T,K) &= \frac{\partial^2}{\partial K^2}\,\mathrm{Put}(T,K)\\ &= \mathcal{Z}_0 q_T(F,K) \geq 0,\end{aligned}$$

for all K.

Arbitrage in the SABR model

The explicit implied volatility formula can represent an arbitrage free model only if

$$\begin{aligned}\frac{\partial^2}{\partial K^2}\,\mathrm{Call}(T, K, F_0, \sigma(K, \ldots)) &= \frac{\partial^2}{\partial K^2}\,\mathrm{Put}(T, K, F_0, \sigma(K, \ldots))\\ &\geq 0.\end{aligned}$$

In other words, there cannot be a “butterfly arbitrage”.

Arbitrage in the SABR model

As it turns out:

- It is possible for this requirement to be violated for very low strike and very long expiry options.
- The problem does not appear to be the quality of the option prices obtained from the explicit implied volatility formulas, because these are quite accurate. Rather, the problem seems to be that implied volatility curves are not a robust representation of option prices for low strikes.
- It is very easy to find a reasonable looking volatility curve $\sigma(K, \ldots)$ which violates the arbitrage free constraint for a range of values of K.
- Implied volatility calculations in the SABR model can be done in a way that guarantees arbitrage freeness.

Uses of SABR

- Portfolio valuation (pricing)
- Risk management
- Relative value
- Clearing and systemic risk management

Hedging under SABR

The choice of volatility model impacts not only the prices of (out of the money) options but also, at least equally significantly, their risk sensitivities. The issues one faces (among many others) are:

- How is the delta risk defined: which volatility parameter should be kept constant when taking the derivative with respect to the underlying?
- How is the vega risk defined?

An answer to these questions may have a profound impact on the portfolio P&L.

Hedging under SABR

The delta risk of an option is calculated by shifting the current value of the underlying while keeping the current value of implied volatility σ_0 fixed. In the case of an interest rate option, this amounts to shifting the relevant forward rate without changing the implied volatility:

$$F_0 \to F_0 + \Delta F_0,$$
$$\sigma_0 \to \sigma_0,$$

where ΔF_0 is, say, -1 bp. This scenario leads to the option delta:

$$\Delta = \frac{\partial \text{Price}}{\partial F_0} + \frac{\partial \text{Price}}{\partial \sigma} \frac{\partial \sigma}{\partial F_0}.$$

The first term on the right hand side in the formula above is the original Black model delta, and the second arises from the systematic change in the implied vol as the underlying changes. This formula shows that both the classic delta and vega contribute to the smile adjusted delta of an option.

Hedging under SABR

Similarly, the vega risk is calculated from

$$\begin{aligned} F_0 &\to F_0, \\ \sigma_0 &\to \sigma_0 + \Delta\sigma_0, \end{aligned}$$

to be

$$\Lambda = \frac{\partial V}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma_0} .$$

These formulas are the classic SABR risk sensitivities ("greeks").

Hedging under SABR

Modified SABR greeks below do a better justice to the model dynamics. Since the processes for σ and F are correlated, whenever F changes, on average σ changes as well. This change is proportional to the correlation coefficient ρ between the Brownian motions driving F and σ. It is easy to see that a scenario consistent with the dynamics is of the form

$$\begin{aligned} F_0 &\to F_0 + \Delta F_0, \\ \sigma_0 &\to \sigma_0 + \delta_F \sigma_0. \end{aligned}$$

Here

$$\delta_F \sigma_0 = \frac{\rho\alpha}{F_0^{\beta}} \, \Delta F_0$$

is the average change in σ_0 caused by the change in the underlying forward. The new delta risk is given by

$$\Delta = \frac{\partial V}{\partial F_0} + \frac{\partial V}{\partial \sigma} \left(\frac{\partial \sigma}{\partial F_0} + \frac{\partial \sigma}{\partial \sigma_0} \, \frac{\rho\alpha}{F_0^{\beta}} \right).$$

This risk incorporates the average change in volatility caused by the change in the underlying.

Hedging under SABR

Similarly, the vega risk should be calculated from the scenario:

$$F_0 \to F_0 + \delta_\sigma F_0,$$
$$\sigma_0 \to \sigma_0 + \Delta\sigma_0,$$

where

$$\delta_\sigma F_0 = \frac{\rho F_0^\beta}{\alpha}\,\Delta\sigma_0$$

is the average change in F_0 caused by the change in the beta vol. This leads to the modified vega risk

$$\Lambda = \frac{\partial V}{\partial \sigma}\frac{\partial \sigma}{\partial \sigma_0} + \Big(\frac{\partial V}{\partial \sigma}\,\frac{\partial \sigma}{\partial F_0} + \frac{\partial V}{\partial F_0}\Big)\frac{\rho F_0^\beta}{\alpha}\;.$$

The first term on the right hand side of the formula above is the classic SABR vega, while the second term accounts for the change in volatility caused by the move in the underlying forward rate.

Hedging under SABR

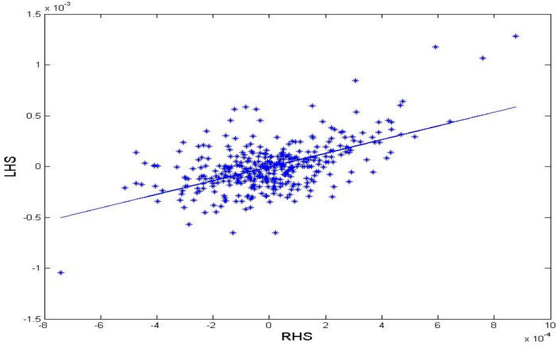


Figure: Regression of $\delta_F \sigma_0$ against $\rho\alpha / F^{\beta}$ for the 1Y into 10Y swaption $\beta = 0.5$.

Hedging under SABR

The graph below shows the historical values of the calibrated parameter ρ.

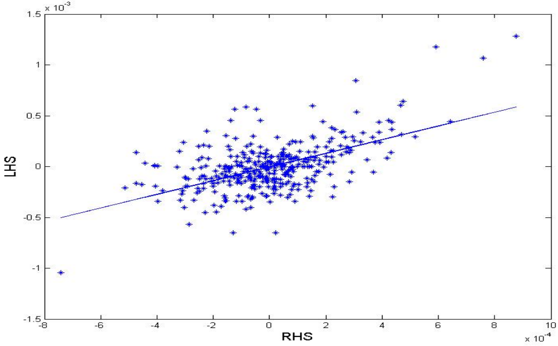


Figure: Regression of $\delta_F \sigma_0$ against $\rho\alpha/F^\beta$ for the 5Y into 5Y swaption $\beta = 0.75$.

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