Effective techniques for risk management of mortgage portfolios

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Overview

- Libor Market Model
- Time To Event Model
- Risk management
  - Duration risk (Delta)
  - Convexity risk (Gamma)
  - Volatility risk (Vega)
  - Prepayment risk

Disclaimer

The views expressed in this talk are mine and do not necessarily reflect the views of Ellington Management Group.

Term structure modeling

The purpose of a term structure model is to generate *future* rates scenarios in a manner consistent with *current* markets.

A term structure model should be constructed so that it is:

- Arbitrage free
- Realistic
- Tractable

A good choice is the LIBOR Market Model (LMM).

Dynamics of the LMM

We consider a sequence of approximately equally spaced dates (*standard tenors*)

$$0 \leq T_0 < T_1 < \ldots < T_N,$$

A standard Libor forward rate

$$L^j, \quad j = 0, 1, \ldots, N-1,$$

is associated with a forward rate agreement which starts on T_j and ends on T_{j+1}. Usually, we assume $N = 120$, and the L^j's are 3 month Libor forward rates.

Dynamics of LMM

We model L^j as a continuous time stochastic process $L^j(t)$, $0 \leq t \leq T_{j-1}$ (killed at $t = T_{j-1}$!). The dynamics of the forward process is driven by an N-dimensional, correlated Wiener process $W_0(t), \ldots, W_{N-1}(t)$. The probability measure associated with this Wiener process is denoted by P.

We let ρ_{jk} denote the instantaneous correlation between $W_j(t)$ and $W_k(t)$, i.e.

$$\mathsf{E}^{\mathsf{P}}\left[dW_j(t)\, dW_k(t)\right] = \rho_{jk} dt\,.$$

Dynamics of LMM

The dynamics of LMM is given by a system of stochastic differential equations:

$$dL^j(t) = \Delta^j(L(t), t)\, dt + C^j(L(t), t)\, dW_j(t)\,.$$

The first term on the right hand side is called the *drift term*, and the second term is called the *diffusion term*. The no arbitrage requirement forces a relationship between the drift and the diffusion terms. The form of the drift term depends on the choice of *numeraire* (asset in units of which all prices are expressed).

Dynamics of LMM

If the numeraire is chosen to be the zero coupon bond $P_k\left(t\right)$ maturing at T_k, one finds that:

$$dL^j\left(t\right) = C^j\left(L^j\left(t\right), t\right) \times \begin{cases} -\displaystyle\sum_{j+1 \le i \le k} \frac{\rho_{ji}\delta_i C^i\left(L^i\left(t\right), t\right)}{1+\delta_i L^i\left(t\right)} dt + dW_j\left(t\right), & \text{if } j < k, \\ \displaystyle\sum_{k+1 \le i \le j} \frac{\rho_{ji}\delta_i C^i\left(L^i\left(t\right), t\right)}{1+\delta_i L^i\left(t\right)} dt + dW_j\left(t\right), & \text{if } j > k. \end{cases}$$

Dynamics of LMM

The functions $C^j\left(L^j\left(t\right), t\right)$ are the *local volatilities* defining the volatility structure of the model.

These equations are supplied with initial values for the Libor forwards:

$$L^j\left(0\right) = L_0^j,$$

where L_0^j is the current forward.

Calibrating LMM

To calibrate the model:

- Choose the initial condition to match the current curve
- Choose the local volatilities to match the swaption and cap vols
- Choose a realistic correlation matrix ρ

Uses of LMM

Can do:

- Generate future scenarios consistent with current markets
- Price interest rate sensitive instruments
- Quantify interest rate and volatility risk
- Help identify mispricings in the markets

Cannot do:

- Handle *event risk* (such as prepayments & defaults) embedded in MBSs or credit sensitive instruments

Securities with event risk

Event risk is largely exogenous to the rates process. In order to model it we assume that:

- Interest rates dynamics is modeled by LMM
- Event dynamics is modeled by a random time T
- There exists *intensity process* $\lambda(t) \geq 0$, so that

$$\text{Prob}\left(\{T > t\} \left| \mathcal{F}_t\right.\right) = \exp\left(-\int_0^t \lambda(s)\, ds\right)$$

In the context of prepayment, $\lambda(t)$ is the SMM.

Securities with event risk

The intensity $\lambda(t)$ is itself stochastic. The stochastic process

$$S(t) = \exp\left(-\int_0^t \lambda(s)\, ds\right)$$

is called the *survival probability*, while

$$F(t) = 1 - S(t)$$

is the *event probability*.

Securities with event risk

Valuation of securities with event risk is done by means of Monte Carlo simulations. Each MC scenario produces the value:

$$\sum_j \Big(c_j P(t_j)\, S(t_j) + r_j P(t_j) \left[F_j(t) - F(t_{j-1}) \right] \Big),$$

where c_j are the known cash amounts, and r_j are the recovery rates in case an event occurs. The intensity process defining the survival probability is modeled outside of LMM. The price of the security is calculated by taking the average of the values computed above.

Modeling prepayments

Factors affecting $S(t)$:

- Interest rates
- Unobserved heterogeneity characteristics of borrowers
- Characteristics of loans
- Macroeconomic factors

Modeling prepayments

Decomposition of $S(t)$ into *competing risks*:

- Turnover
- Refi
- Total payoff
- ...

$$S(t) = \mathcal{C}(S_1(t), \ldots, S_r(t)),$$

where $S_j(t)$ are the survival probabilities corresponding to the *latent* risks.

Modeling prepayments

A convenient framework for modeling prepayment is the Cox model.

- Conceptually easy
- Works for non-path dependent risks such as turnover

Does not capture the path dependent burnout phenomenon of refi risk.

Modeling heterogeneity

Probability distribution $G(x)$, $x \in X$, of borrowers

$$\Downarrow$$

Each borrower has intensity $\lambda(t, x)$

$$\Downarrow$$

Each borrower's survival probability is

$$S(t, x) = \exp\left(-\int_0^t \lambda(s, x)\, ds\right)$$

Modeling heterogeneity

The average (population) intensity is

$$\lambda_b(t) = \frac{\int_X \lambda(t,x)\, S(t,x)\, dG(x)}{\int_X S(t,x)\, dG(x)}$$

$\Rightarrow$ the population survival probability is

$$S_b(t) = \exp\left(-\int_0^t \lambda_b(s)\, ds\right)$$

Needed: choice of $\lambda(t,x)$.

Modeling heterogeneity

Two dimensional heterogeneity space $x = (a, c)$.

- $a =$ *alertness*, $c =$ *cost* threshold
- Borrower's intensity process

$$\lambda(t, x) = a\, \mathbf{1}_{C(t) \geq c}\, \lambda_0(t)$$

Incentive process $C(t)$

$$\Uparrow$$

Interest rates, loan characteristics, ...

Modeling prepayments

The full intensity:

$$\lambda(t) = \lambda_b(t)\ \exp\left(\sum \beta_j V_j(t)\right)$$

$V_j(t) =$ stochastic factors affecting prepayments

Factors $V_j(t)$

$\Uparrow$

Macroeconomic factors, loan characteristics, ...

Calibrating prepayments

To do:

- Given historical SMMs for thousands of pools of mortgages or single loans
- Find the model parameters (collectively denoted by θ) that give the best fit
- Do it efficiently

Calibrating prepayments

Kullback-Leibler divergence between the probability distributions: $\{p_i\}$ (observed) and $\{p_i(\theta)\}$ (theoretical):

$$D(p\|p(\theta)) = \sum_i p_i \log\left(\frac{p_i}{p_i(\theta)}\right)$$

Then

- $D(p\|p(\theta)) \geq 0$
- "Best" value of $\theta \Rightarrow$ minimum KL divergence.

Duration risk (delta)

Traditional measures of the portfolio's market risk are: duration and partial durations. Problems:

- Inflexible
- Calculations incompatible with modern curve construction methodologies
- Do not capture well complicated curve moves

A more flexible approach is based on perturbing the forward curve.

Duration risk (delta)

- Choose a *hedging portfolio* consisting of swaps, Eurodollar futures, etc.:

$$\Pi_{\text{hedge}} = \{B_1, \ldots, B_n\}$$

- Let $\mathcal{C}_0$ denote the current forward curve, the *base scenario*. Choose a number of new micro scenarios

$$\mathcal{C}_1, \ldots, \mathcal{C}_p$$

by perturbing non-overlapping segments of $\mathcal{C}_0$

Duration risk (delta)

- The vector $\delta\Pi$ of portfolio's *sensitivities* is

$$\delta_i \Pi = \Pi\left(\mathcal{C}_i\right) - \Pi\left(\mathcal{C}_0\right), \qquad i = 1, \ldots, p,$$

  where by $\Pi\left(\mathcal{C}_i\right)$ we denote the value of the portfolio given the forward curve $\mathcal{C}_i$.

- The matrix δB of sensitivities of the hedging instruments is

$$\delta_i B_j = B_j\left(\mathcal{C}_i\right) - B_j\left(\mathcal{C}_0\right)$$

  Always use more scenario than hedging instruments!

Duration risk (delta)

- The vector Δ of *hedge ratios* is calculated by minimizing

$$\mathcal{L}(\Delta) = \frac{1}{2}\left\|\delta B\,\Delta - \delta\Pi\right\|^2 + \frac{1}{2}\lambda\|Q\,\Delta\|^2$$

  Here, λ is an appropriately chosen small smoothness parameter, and Q is the smoothing operator. Explicitly,

$$\Delta = \left((\delta B)^t\,\delta B + \lambda Q^t\,Q\right)^{-1}(\delta B)^t\,\delta\Pi$$

This methodology is called *ridge regression*.

Convexity risk (gamma)

The *gamma* of a portfolio is sometimes calculated as its global convexity characteristic. This is a rather crude measure, as portfolios typically exhibit complex convexity behaviors. A better way is to construct the portfolio gamma as the change in its delta under specified macro scenarios:

$$\Xi_0, \Xi_1, \ldots, \Xi_r,$$

with Ξ_0 base scenario (no change in rates).

Convexity risk (gamma)

For example:

Ξ_{+50}	All rates up 50 basis points.
Ξ_{+25}	All rates up 25 basis points.
Ξ_{-25}	All rates down 25 basis points.
Ξ_{-50}	All rates down 50 basis points.

For each of the macro scenarios, we calculate the deltas

$$\Delta_1, \ldots, \Delta_r.$$

Convexity risk (gamma)

The quantities:

$$\Gamma_1 = \Delta_1 - \Delta_0,$$
$$\vdots$$
$$\Gamma_r = \Delta_r - \Delta_0,$$

are the portfolio gammas under the corresponding scenarios. For intermediate market moves, the portfolio gamma can be calculated by linearly interpolating the macro scenarios.

Volatility risk (vega)

Traditional way of calculation the vega risk of a portfolio is to perturb vol inputs: shift selected swaption and/or cap volatilities. Problems:

- This may be incompatible with the term structure model
- Does not capture well *shearing* moves in the volatility surface (e.g. large upward move of short dated vol accompanied by no move in longer dated vol)

Volatility risk (vega)

Instead one may perturb the internal vol parameters of the term structure model. LMM builds its internal “volatility surface” $\mathfrak{S}$. We construct volatility micro scenarios by accessing $\mathfrak{S}$ and shifting selected non-overlapping segments. Let us call these scenarios

$$\mathfrak{S}_0, \mathfrak{S}_1, \ldots, \mathfrak{S}_q,$$

with $\mathfrak{S}_0 = \mathfrak{S}$ being the base scenario.

Volatility risk (vega)

- We choose a hedging portfolio Π_{hedge} which may consist of liquid instruments such as swaptions, caps and floors, Eurodollar options, ...
- The rest is a *verbatim* repeat of the delta story. We calculate the sensitivities of the portfolio to the volatility scenarios. We calculate the sensitivities of the hedging portfolio to the volatility scenarios. Finally, we use ridge regression to find the hedge ratios.

Prepayment risk

Quantify the prepayment risk by perturbing the parameters of the prepayment model such as:

- Mortgage rate
- Loadings of the stochastic factors
- ...