

# Quantum Riemann Surfaces I. The Unit Disc\*

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**Abstract.** We construct a non-commutative  $\mathbf{C}^*$ -algebra  $C_\mu(\bar{U})$  which is a quantum deformation of the algebra of continuous functions on the closed unit disc  $\bar{U}$ .  $C_\mu(\bar{U})$  is generated by the Toeplitz operators on a suitable Hilbert space of holomorphic functions on  $U$ .

## 1. Introduction

Alain Connes has shown [8] that a substantial part of differential geometry can be extended to a non-commutative setup in which a non-commutative  $*$ -algebra replaces an algebra of functions on a manifold. Clearly, not every non-commutative algebra has an interesting geometry and, while a satisfactory concept of a “non-commutative differentiable manifold” has not been formulated yet, it is desirable to study examples of such structures.

Recently, a growing number of examples of “non-commutative differentiable manifolds” has been studied, see e.g. [16, 18, 22]. These examples form, in a sense, a testing ground for probing general concepts of non-commutative differential geometry and so give us a better insight into the properties of a non-commutative manifold. Also, some important applications of these ideas have been found in other areas of mathematics and physics [3, 7, 9].

In this paper, we begin a program of developing a theory of non-commutative Riemann surfaces. We believe that non-commutative Riemann surfaces should play a distinguished role in non-commutative differential geometry, very much like ordinary Riemann surfaces in the commutative case. Also, there are some speculations [13] that quantum Riemann surfaces might be helpful in studying certain integrable systems arising in string theory.

Conceptually, the simplest method of constructing non-commutative manifolds is the framework of deformation quantization, see e.g. [2, 4, 19, 22]. This is the starting point of our approach.

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Let  $M$  be a Riemann surface and let  $\omega$  be a symplectic form on  $M$ . Then, the  $*$ -algebra  $C^\infty(M)$  of smooth functions on  $M$  comes equipped with a Poisson bracket:

$$\{f, g\} := \omega^{-1}(df, dg), \quad f, g \in C^\infty(M). \tag{I.1}$$

We wish to construct a family of  $\mathbf{C}^*$ -algebras  $\mathcal{A}_\mu, 0 < \mu < 1$ , called quantum Riemann surfaces, together with “quantization maps”  $T^{(\mu)}: C^\infty(M) \rightarrow \mathcal{A}_\mu$  such that:

- (i)  $T^{(\mu)}$  is linear and  $*$ -preserving and (ii)  $\lim_{\mu \rightarrow 0} \left\| \frac{1}{i\mu} [T_f^{(\mu)}, T_g^{(\mu)}] - T_{\{f,g\}} \right\|^{(\mu)} = 0$ . Furthermore, we require that the biholomorphisms of  $M$  act on  $\mathcal{A}_\mu$  and that  $\mathcal{A}_\mu$  is, roughly speaking, the fixed point algebra of the quantum universal covering of  $M$ .

In this paper we construct a quantum deformation of the unit disc ( $\equiv$  universal covering space of hyperbolic Riemann surfaces). Our construction draws some ideas from Berezin’s paper [4] on quantization by covariant symbols. In a subsequent paper [11] we will describe quantum deformations of hyperbolic Riemann surfaces. In another development [12] we study a two-parameter deformation of the unit disc which is closely related to the quantum group  $SU_q(1, 1)$ . We should also mention that quantum Riemann surfaces of genus 0 and 1 have already been studied in the literature, see [16, 20] (quantum sphere) and [8, 18] (quantum torus). Also, quantizations of the unit disk using various symbolic calculi have been studied (see e.g. [14, 21] and references therein).

Recall that the group of biholomorphisms  $SU(1, 1)/\mathbf{Z}_2$  of  $U = \{z \in \mathbf{C} : |z| < 1\}$  consists of fractional transformations

$$z \rightarrow (az + b)(\bar{b}z + \bar{a})^{-1}, \quad |a|^2 - |b|^2 = 1. \tag{I.2}$$

The (essentially unique)  $SU(1, 1)/\mathbf{Z}_2$ -invariant symplectic form on  $U$  is

$$\omega = \frac{i}{2}(1 - |z|^2)^{-2} dz \wedge d\bar{z}, \tag{I.3}$$

with the corresponding Poisson bracket

$$\{f, g\} = i(1 - |z|^2)^2(\partial f \bar{\partial} g - \bar{\partial} f \partial g). \tag{I.4}$$

In Sect. II we give the definition of the quantum unit disc  $C_\mu(\bar{U})$  in terms of generators and construct an  $SU(1, 1)/\mathbf{Z}_2$  action on  $C_\mu(\bar{U})$ . In Sects. III and IV we study the representation theory of  $C_\mu(\bar{U})$  in terms of the algebra of Toeplitz operators on a suitable Hilbert space. This leads us to structural theorems for the quantum unit disc which are closely related to the Brown–Douglas–Fillmore theory [5]. In Sect. V we show that the  $SU(1, 1)/\mathbf{Z}_2$  action can be implemented on this space by a unitary projective representation of  $SU(1, 1)/\mathbf{Z}_2$ . Finally, in Sect. VI we study asymptotic expansions of products of Toeplitz operators and prove that, indeed,  $T^{(\mu)}$  satisfies requirement (ii) formulated above.

## II. Quantum Unit Disc

For  $0 < \mu < 1$  we let  $P_\mu(\bar{U})$  denote the unital  $\mathbf{C}$ -algebra generated by two elements  $z$  and  $\bar{z}$  with the following relation:

$$[z, \bar{z}] = \mu(I - z\bar{z})(I - \bar{z}z). \tag{II.1}$$

We call  $P_\mu(\bar{U})$  the algebra of polynomials on the closed quantum unit disc. The goal of this section is to study some general properties of representations of  $P_\mu(\bar{U})$ . We show that  $P_\mu(\bar{U})$  has a nontrivial universal enveloping  $\mathbf{C}^*$ -algebra. We call this  $\mathbf{C}^*$ -algebra the algebra of continuous functions on the closed quantum unit disc.

Let  $\mathcal{H}$  be a Hilbert space and let  $\pi: P_\mu(\bar{U}) \rightarrow \mathcal{L}(\mathcal{H})$  be a representation of  $P_\mu(\bar{U})$  by bounded linear operators on  $\mathcal{H}$  satisfying the condition  $\pi(z)^* = \pi(\bar{z})$ . By abuse of terminology, we will refer to such representations as  $*$ -representations (strictly speaking,  $P_\mu(\bar{U})$  is not a  $*$ -algebra). Setting  $\pi(z) = e^{i\theta}$ ,  $\pi(\bar{z}) = e^{-i\theta}$ ,  $0 \leq \theta < 2\pi$ , we see that nontrivial  $*$ -representations of  $P_\mu(\bar{U})$  exist. By  $\pi(P_\mu(\bar{U}))^-$  we denote the  $\mathbf{C}^*$ -algebra of operators on  $\mathcal{H}$  defined as the norm closure of  $\pi(P_\mu(\bar{U}))$ .

We will frequently use the following notation:

$$x := z\bar{z}, \quad y := \bar{z}z. \tag{II.2}$$

Applying  $\pi$  and taking the adjoint of (II.1) we find that

$$[\pi(x), \pi(y)] = 0, \tag{II.3}$$

i.e.,  $\pi(x)$  and  $\pi(y)$  generate an abelian subalgebra of  $\pi(P_\mu(\bar{U}))$ .

**Theorem II.1.** *The operators  $\pi(z)$  and  $\pi(\bar{z})$  satisfy*

$$\|\pi(z)\| = \|\pi(\bar{z})\| = 1. \tag{II.4}$$

*Proof.* Let  $\mathbf{C}^*(x, y)$  denote the abelian unital sub- $\mathbf{C}^*$ -algebra of  $\mathcal{L}(\mathcal{H})$  generated by  $\pi(x)$  and  $\pi(y)$ . As a consequence of a Gelfand–Naimark theorem,  $\mathbf{C}^*(x, y)$  is isometrically isomorphic with  $C(\Sigma)$ , the  $\mathbf{C}^*$ -algebra of continuous functions on the spectrum  $\Sigma$  of  $\mathbf{C}^*(x, y)$ . Let  $x(\sigma)$  and  $y(\sigma)$ ,  $\sigma \in \Sigma$ , denote the images of  $\pi(x)$  and  $\pi(y)$  under this isomorphism. Clearly,  $x(\sigma), y(\sigma) \geq 0$ .

From Eq. (II.1) we have

$$y(\sigma)[\mu x(\sigma) + 1 - \mu] = (\mu + 1)x(\sigma) - \mu. \tag{II.5}$$

Since  $\mu x(\sigma) + 1 - \mu > 0$ , we have for all  $\sigma \in \Sigma$ ,

$$y(\sigma) = \frac{(\mu + 1)x(\sigma) - \mu}{\mu x(\sigma) + 1 - \mu}. \tag{II.6}$$

The function  $0 \leq t \rightarrow [(\mu + 1)t - \mu]/[\mu t + 1 - \mu]$  is continuous and monotonically increasing, and thus as a consequence of (II.6),

$$\|\pi(y)\| = \frac{(\mu + 1)\|\pi(x)\| - \mu}{\mu\|\pi(x)\| + 1 - \mu}. \tag{II.7}$$

But from (II.2),  $\|\pi(x)\| = \|\pi(y)\| = \|\pi(z)\|^2$ . Substituting this into (II.7) and solving the resulting algebraic equation yields  $\|\pi(z)\| = 1$ , as claimed.  $\square$

**Lemma II.2.**

$$(i) \quad \text{Ker}(I - \pi(x)) = \text{Ker}(I - \pi(y)), \tag{II.8}$$

$$(ii) \quad \text{Ran}(I - \pi(x)) = \text{Ran}(I - \pi(y)). \tag{II.9}$$

*Proof.* (i) This is an immediate consequence of (II.1) and (II.3).

(ii) If  $\psi = (I - \pi(x))\varphi$ , then from (II.1) and (II.3),  $(I + \mu(I - \pi(y)))\psi = (I - \pi(y))\varphi$ . But  $\|\mu(I - \pi(y))\| < 1$ , and thus  $\psi = (I - \pi(y))(I + \pi(y))^{-1}\varphi \in \text{Ran}(I - \pi(y))$ , proving that  $\text{Ran}(I - \pi(x)) \subset \text{Ran}(I - \pi(y))$ . By a similar argument,  $\text{Ran}(I - \pi(y)) \subset \text{Ran}(I - \pi(x))$ .  $\square$

**Lemma II.3.**

- (i)  $\text{Ker}(I - \pi(x))$  is invariant under  $\pi(P_\mu(\bar{U}))^-$ ,
- (ii)  $\text{Ran}(I - \pi(x))$  is invariant under  $\pi(P_\mu(\bar{U}))$ .

*Proof.* (i) Indeed, if  $\psi \in \text{Ker}(I - \pi(x))$ , then by (II.8),  $(I - \pi(x))\pi(z)\psi = \pi(z)(I - \pi(y))\psi = 0$ , and  $(I - \pi(y))\pi(\bar{z})\psi = \pi(\bar{z})(I - \pi(x))\psi = 0$ , i.e.,  $\pi(z)\psi, \pi(\bar{z})\psi \in \text{Ker}(I - \pi(x))$ . Since  $\text{Ker}(I - \pi(x))$  is closed, the claim follows.

(ii) If  $\psi = (I - \pi(x))\varphi \in \text{Ran}(I - \pi(x))$ , then by (II.9),  $\pi(\bar{z})\psi = (I - \pi(y))\pi(\bar{z})\varphi \in \text{Ran}(I - \pi(y)) = \text{Ran}(I - \pi(x))$ . Furthermore,  $\psi = (I - \pi(y))\varphi'$  and thus  $\pi(z)\psi = (I - \pi(x))\pi(z)\varphi' \in \text{Ran}(I - \pi(x))$ .  $\square$

We now write

$$\mathcal{H} = \text{Ker}(I - \pi(x)) \oplus \mathcal{H}^\circ, \tag{II.10}$$

where  $\mathcal{H}^\circ$  is the orthogonal complement of  $\text{Ker}(I - \pi(x))$ . As a consequence of Lemma II.3 (i) and the fact that  $\pi(P_\mu(\bar{U}))^-$  is a  $*$ -algebra, both direct summands in (II.10) are invariant under  $\pi(P_\mu(\bar{U}))^-$ .

Let now  $\gamma \in SU(1, 1)$ . We will write the inverse of  $\gamma$  as

$$\gamma^{-1} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \quad a, b \in \mathbb{C}, \tag{II.11}$$

where

$$|a|^2 - |b|^2 = 1. \tag{II.12}$$

**Lemma II.4.** *If  $\gamma \in SU(1, 1)$ , then  $(b\pi(z) + a)^{-1} \in \pi(P_\mu(\bar{U}))^-$ .*

*Proof.* Consider the series

$$a^{-1} \sum_{n \geq 0} (-b/a)^n \pi(z)^n. \tag{II.13}$$

As a consequence of (II.12),  $|b/a| < 1$ . Since by Theorem II.1,  $\|\pi(z)\| = 1$ , the series (II.13) converges to an element of  $\mathcal{L}(\mathcal{H})$ . A routine calculation show that its limit is, in fact,  $(b\pi(z) + a)^{-1}$ .  $\square$

We now set for  $\gamma \in SU(1, 1)$ ,

$$\begin{aligned} \rho_\gamma(\pi(z)) &:= (\bar{a}\pi(z) + \bar{b})(b\pi(z) + a)^{-1}, \\ \rho_\gamma(\pi(\bar{z})) &:= (\bar{b}\pi(\bar{z}) + \bar{a})^{-1}(a\pi(\bar{z}) + b). \end{aligned} \tag{II.14}$$

In the theorem below we assert that  $\rho_\gamma$  defines an action of  $SU(1, 1)$  on  $\pi(P_\mu(\bar{U}))^-$ .

**Theorem II.5.** *Let  $\gamma \in SU(1, 1)$ . Then:*

(i)  $\rho_\gamma(\pi(z))^* = \rho_\gamma(\pi(\bar{z}))$ , (II.15)

(ii)  $[\rho_\gamma(\pi(z)), \rho_\gamma(\pi(\bar{z}))] = \mu(I - \rho_\gamma(\pi(z))\rho_\gamma(\pi(\bar{z}))(I - \rho_\gamma(\pi(\bar{z}))\rho_\gamma(\pi(z)))$ . (II.16)

*Proof.* (i) This is clear.

(ii) We verify (II.16) on  $\text{Ker}(I - \pi(x))$  and  $\mathcal{H}$  separately. We set  $z' := \rho_\gamma(\pi(z))$  and  $\bar{z}' := \rho_\gamma(\pi(\bar{z}))$ . If  $\omega \in \text{Ker}(I - \pi(x))$ , then a straightforward calculation shows that

$$z'\bar{z}'\psi = \bar{z}'z'\psi = \psi,$$

which proves that (II.16) holds on  $\text{Ker}(I - \pi(x))$ .

To prove that (II.16) holds on  $\mathcal{H}$  we observe that the inverses  $(I - \pi(x))^{-1}$  and  $(I - \pi(y))^{-1}$  exist as densely defined unbounded operators. Furthermore, as a consequence of Lemma II.3 (ii),  $\text{Dom}((I - \pi(x))^{-1}) = \text{Dom}((I - \pi(y))^{-1}) = \text{Ran}(I - \pi(x))$ .

**Lemma II.6.** *Relation (II.1) restricted to  $\mathcal{H}$  is equivalent to the following equation on  $\text{Ran}(I - \pi(x))$ :*

$$(I - \pi(x))^{-1} - (I - \pi(y))^{-1} = \mu. \tag{II.17}$$

*Proof.* Multiplying (II.17) by  $I - \pi(x)$  on one side and by  $I - \pi(y)$  on the other yields (II.1). Conversely, writing (II.1) as

$$((I - \pi(y)) - (I - \pi(x)))\varphi = \mu(I - \pi(x))(I - \pi(y))\varphi, \quad \varphi \in \mathcal{H},$$

and using (II.3) yields (II.17).  $\square$

We can now conclude the proof of the theorem. As a consequence of (II.12) and Lemma II.4,

$$I - z'\bar{z}' = (b\pi(z) + a)^{-1}(I - \pi(x))(\bar{b}\pi(\bar{z}) + \bar{a})^{-1}, \tag{II.18}$$

and

$$I - \bar{z}'z' = (\bar{b}\pi(\bar{z}) + \bar{a})^{-1}(I - \pi(y))(b\pi(z) + a)^{-1}. \tag{II.19}$$

We claim that  $I - z'\bar{z}'$  and  $I - \bar{z}'z'$  are invertible on  $\text{Ran}(I - \pi(x))$ . Indeed, we set for  $\psi \in \text{Ran}(I - \pi(x))$

$$(I - z'\bar{z}')^{-1}\psi = (\bar{b}\pi(\bar{z}) + \bar{a})(I - \pi(x))^{-1}(b\pi(z) + a)\psi, \tag{II.20}$$

and

$$(I - \bar{z}'z')^{-1}\psi = (b\pi(z) + a)(I - \pi(y))^{-1}(\bar{b}\pi(\bar{z}) + \bar{a})\psi. \tag{II.21}$$

As a consequence of Lemma II.3 (ii) these are well defined. A simple calculation shows that (II.20) and (II.21) are indeed the inverses of  $I - z'\bar{z}'$  and  $I - \bar{z}'z'$ , respectively.

As a consequence of the above considerations we have, as operator equations on  $\text{Ran}(I - \pi(x))$ :

$$\begin{aligned} (I - z'\bar{z}')^{-1} - (I - \bar{z}'z')^{-1} &= (\bar{b}\pi(\bar{z}) + \bar{a})(I - \pi(x))^{-1}(b\pi(z) + a) \\ &\quad - (b\pi(z) + a)(I - \pi(y))^{-1}(\bar{b}\pi(\bar{z}) + \bar{a}) \\ &= |b|^2\{\pi(y)(I - \pi(y))^{-1} - \pi(x)(I - \pi(x))^{-1}\} \\ &\quad + |a|^2\{(I - \pi(x))^{-1} - (I - \pi(y))^{-1}\} \\ &= -|b|^2\{[\pi(z), \pi(\bar{z})](I - \pi(y))^{-1} + \mu\pi(x)\} + \mu|a|^2 \\ &= \mu(|a|^2 - |b|^2) = \mu. \end{aligned}$$

This shows that  $z'$  and  $\bar{z}'$  satisfy (II.17) and the proof of the theorem is complete.  $\square$

We now let for  $a \in P_\mu(\bar{U})$ ,

$$\|a\| := \sup_\pi \|\pi(a)\|, \tag{II.22}$$

where the supremum is taken over all possible  $*$ -representations of  $P_\mu(\bar{U})$ . As a consequence of Theorem II.1,  $\|a\| < \infty$  for all  $a \in P_\mu(\bar{U})$ . Let  $N := \{a \in P_\mu(\bar{U}) : \|a\| = 0\}$  be the null-ideal in  $P_\mu(\bar{U})$ . We define  $C_\mu(\bar{U})$  to be the completion of  $P_\mu(\bar{U})/N$  in the norm induced by  $\|\cdot\|$  and call it the algebra of continuous functions on the closed quantum unit disc. By definition,  $C_\mu(\bar{U})$  is a unital  $\mathbf{C}^*$ -algebra with involution defined by  $z^* := \bar{z}$ . The following theorem is a consequence of the above considerations.

**Theorem II.7.** *The  $\mathbf{C}^*$ -algebra  $C_\mu(\bar{U})$  is generated by two elements  $z$  and  $\bar{z}$  satisfying (II.1) and such that  $\|z\| = 1$ . The mapping*

$$\gamma \rightarrow \rho_\gamma(z) := (\bar{a}z + \bar{b})(bz + a)^{-1}$$

*defines an action of  $SU(1, 1)$  on  $C_\mu(\bar{U})$ .*

### III. Representation Theory

In this section we classify all irreducible  $*$ -representations of the algebra of polynomials on the quantum unit disc  $P_\mu(\bar{U})$ . As in Sect. II, let  $\pi: P_\mu(\bar{U}) \rightarrow \mathcal{L}(\mathcal{H})$  be a representation of  $P_\mu(\bar{U})$  on a Hilbert space  $\mathcal{H}$ .

**Proposition III.1.** *Let  $\lambda_n := n\mu(1 + n\mu)^{-1}$ ,  $n = 0, 1, 2, \dots$ . Then*

$$(i) \quad \text{Spec}(\pi(x)) \subset \{\lambda_n\}_{n \geq 1} \cup \{1\}, \tag{III.1}$$

$$(ii) \quad \text{Spec}(\pi(y)) \subset \{\lambda_n\}_{n \geq 0} \cup \{1\}. \tag{III.2}$$

*Proof.* By Theorem II.1,  $\|\pi(x)\| = \|\pi(y)\| = 1$ . Therefore,  $\text{Spec}(\pi(x)), \text{Spec}(\pi(y)) \subset [0, 1]$ . Furthermore,  $\text{Spec}(\pi(x)) \setminus \{0\} = \text{Spec}(\pi(y)) \setminus \{0\}$ . Let  $I_n, n = 0, 1, 2, \dots$  be a sequence of open subsets of  $[0, 1]$  defined by  $I_n := (\lambda_n, \lambda_{n+1})$ . We claim that  $I_n \cap \text{Spec}(\pi(x)) = \emptyset$  and  $I_n \cap \text{Spec}(\pi(y)) = \emptyset$ , for  $n \geq 0$ . Indeed, as a consequence of (II.1),

$$\pi(y) = \{(\mu + 1)\pi(x) - \mu\} \{\mu\pi(x) + 1 - \mu\}^{-1}, \tag{III.3}$$

and thus  $\pi(x) \geq \lambda_1$ . Therefore,  $I_0 \cap \text{Spec}(\pi(x)) = \emptyset$ . Consequently,  $I_0 \cap \text{Spec}(\pi(y)) = \emptyset$ . We now proceed inductively. Assume that  $I_0, I_1, \dots, I_n$  do not intersect the spectra of  $\pi(x)$  and  $\pi(y)$ . If  $\lambda \in I_{n+1}$  is in the spectrum of  $\pi(x)$ , then by (III.3)  $\bar{\lambda} = \{(\mu + 1)\lambda - \mu\} \{\mu\lambda + 1 - \mu\}^{-1}$  is in the spectrum of  $\pi(y)$ . But  $\bar{\lambda} \in I_n$ , which is a contradiction. Therefore,  $I_{n+1}$  does not intersect the spectrum of  $\pi(x)$  and, consequently, it does not intersect the spectrum of  $\pi(y)$ .  $\square$

**Theorem III.2.** *An irreducible  $*$ -representation  $(\mathcal{H}, \pi)$  of  $P_\mu(\bar{U})$  is unitarily equivalent to one of the two following representations:*

( $\alpha$ ) *a one-dimensional representation  $\pi(z) = e^{-i\theta}, \pi(\bar{z}) = e^{i\theta}$ , where  $0 \leq \theta < 2\pi$ ;*

( $\beta$ ) *an infinite dimensional representation defined as follows: let  $\mathcal{H} = l^2(\mathbf{Z}_+)$  (where  $\mathbf{Z}_+$  is the set of nonnegative integers) and let  $\{f_n\}_{n \geq 0}$  be an orthonormal basis for  $\mathcal{H}$ .*

Then

$$\pi(\bar{z})f_n = \left\{ \frac{(n+1)\mu}{1+(n+1)\mu} \right\}^{1/2} f_{n+1}, \quad n \geq 0, \tag{III.4}$$

$$\pi(z)f_n = \begin{cases} 0 & n = 0, \\ \left\{ \frac{n\mu}{1+n\mu} \right\}^{1/2} f_{n-1}, & n \geq 1. \end{cases} \tag{III.5}$$

*Proof.* We assume that  $(\mathcal{H}, \pi)$  is an irreducible  $*$ -representation of  $P_\mu(\bar{U})$ . We write  $\mathcal{H}$  as in (II.10) and observe that either  $\text{Ker}(I - \pi(x)) = 0$  or  $\mathcal{H} = 0$ . If  $\mathcal{H} = 0$ , then  $\pi(z)\pi(z)^* = \pi(z)^*\pi(z) = I$ , i.e.,  $\pi(z)$  is unitary. As a consequence,  $\mathcal{H}$  is one-dimensional and  $\pi(z) = e^{-i\theta}$ . Let  $\text{Ker}(I - \pi(x)) = 0$  and let  $\mathcal{G}_n, n = 1, 2, \dots$  be the eigenspaces of  $\pi(x)$  corresponding to the eigenvalue  $\lambda_n$ . By the spectral theorem,  $\mathcal{H} = \bigoplus_{n \geq 1} \mathcal{G}_n$ . Observe that, as a consequence of (II.1),  $\pi(\bar{z})$  maps  $\mathcal{G}_n$  into  $\mathcal{G}_{n+1}$ , and  $\pi(z)$  maps  $\mathcal{G}_{n+1}$  into  $\mathcal{G}_n$ . Indeed, if  $f \in \mathcal{G}_n$ , then

$$\begin{aligned} (I + \mu - \mu\pi(\bar{z}z))\pi(z\bar{z})\pi(\bar{z})f &= \mu\pi(\bar{z})f - \mu\pi(\bar{z})\pi(z\bar{z})f \\ &\quad + \pi(\bar{z})\pi(z\bar{z})f = (\mu - \mu\lambda_n + \lambda_n)\pi(\bar{z})f. \end{aligned}$$

Consequently,

$$\begin{aligned} \pi(z\bar{z})\pi(\bar{z})f &= (\mu - \mu\lambda_n + \lambda_n)(I + \mu - \mu\pi(\bar{z}z))^{-1}\pi(\bar{z})f \\ &= (\mu - \mu\lambda_n + \lambda_n)\pi(\bar{z})(I + \mu - \mu\pi(z\bar{z}))^{-1}f \\ &= (\mu - \mu\lambda_n + \lambda_n)(1 + \mu - \mu\lambda_n)^{-1}\pi(\bar{z})f \\ &= \lambda_{n+1}\pi(\bar{z})f, \end{aligned}$$

i.e.,  $\pi(\bar{z})f \in \mathcal{G}_{n+1}$ . In the same way we show that  $\pi(z): \mathcal{G}_{n+1} \rightarrow \mathcal{G}_n$ . Now, let  $i_n$  denote the restriction of  $\pi(\bar{z})$  to  $\mathcal{G}_n$  and let  $r_n$  denote the restriction of  $\pi(z)$  to  $\mathcal{G}_n$ . Then  $i_n r_{n+1} = \lambda_n I$  and  $r_{n+1} i_n = \lambda_n I$ . This shows that the subspaces  $\mathcal{G}_n$  and  $\mathcal{G}_{n+1}$  are isomorphic,  $n = 1, 2, \dots$ . We claim that  $\mathcal{G}_1$  is one-dimensional. Indeed, if  $e_1, e_2 \in \mathcal{G}_1$  are nonzero and mutually orthogonal, then the subspaces generated by  $\{\pi(\bar{z}^m)e_1\}_{m \geq 1}$  and  $\{\pi(\bar{z}^m)e_2\}_{m \geq 1}$  are nonzero, invariant under  $\pi(P_\mu(\bar{U}))$  and mutually orthogonal. As a consequence, all  $\mathcal{G}_n$  are one-dimensional. Let  $f_0 \in \mathcal{G}_1$  with  $\|f_0\| = 1$ , and let  $f_n, n \geq 1$ , be defined by

$$f_n := \|\pi(\bar{z}^n)f_0\|^{-1}\pi(\bar{z}^n)f_0 \in \mathcal{G}_{n+1}. \tag{III.6}$$

Then  $\{f_n\}_{n \geq 0}$  forms an orthonormal basis for  $\mathcal{H}$ . A simple computation shows that

$$\begin{aligned} \pi(\bar{z})f_n &= \sqrt{\lambda_{n+1}}f_{n+1}, \quad n \geq 0, \\ \pi(z)f_n &= \begin{cases} 0 & n = 0 \\ \sqrt{\lambda_n}f_{n-1}, & n \geq 1. \end{cases} \quad \square \end{aligned}$$

We now describe a representation of  $P_\mu(\bar{U})$  on a Hilbert space of holomorphic functions on the unit disc  $U \subset \mathbb{C}$ . This space was previously studied by Berezin [4] in the context of quantization of covariant symbols.

We set

$$r := \frac{1}{\mu} + 1, \tag{III.7}$$

and observe that  $r > 2$ . Let  $\mathcal{H}_U$  be the Hilbert space of holomorphic functions on  $U$  equipped with the Petersson scalar product

$$(\varphi, \psi) := \frac{i}{2} \int_U \varphi(\zeta) \overline{\psi(\zeta)} (1 - |\zeta|^2)^{r-2} d\zeta \wedge d\bar{\zeta}. \tag{III.8}$$

We set

$$d\mu_r(\zeta) = \frac{i}{2} (1 - |\zeta|^2)^{r-2} d\zeta \wedge d\bar{\zeta}. \tag{III.9}$$

It is well known that the functions

$$\varphi_n(\zeta) = \left\{ \frac{1}{\pi n! \Gamma(r-1)} \right\}^{1/2} \zeta^n \tag{III.10}$$

form an orthonormal basis for  $\mathcal{H}_U$ . This leads us to the following expression for the corresponding Bergman kernel:

$$K(\zeta, \eta) = \pi^{-1} (r-1) (1 - \zeta\bar{\eta})^{-r}. \tag{III.11}$$

Recall that  $K(\zeta, \eta)$  is the integral kernel of the orthogonal projection  $P$  of  $L^2(U, d\mu_r)$  onto  $\mathcal{H}_U$ . For each continuous function  $f$  on  $\bar{U}$  we consider an operator  $T_f$  on  $\mathcal{H}_U$  defined by

$$T_f \varphi(\zeta) := (PM_f \varphi)(\zeta) = \int_U K(\zeta, \eta) f(\eta) \varphi(\eta) d\mu_r(\eta). \tag{III.12}$$

Here  $M_f$  denotes the multiplication operator on  $L^2(U, d\mu_r)$ . Observe that  $T_f \in \mathcal{L}(\mathcal{H}_U)$ . An operator of the form  $T_f$  is called a Toeplitz operator with symbol  $f$  (see e.g. [10]). Note that  $T_{\bar{f}} = T_f^*$ . Let  $\mathcal{T}_\mu(\bar{U})$  denote the  $\mathbb{C}^*$ -algebra generated by the Toeplitz operators.

We define the following Toeplitz operators:

$$\begin{aligned} (\pi(\bar{z})\varphi)(\zeta) &:= \int_U K(\zeta, \eta) \eta \varphi(\eta) d\mu_r(\eta) = \zeta \varphi(\zeta), \\ (\pi(z)\varphi)(\zeta) &:= \int_U K(\zeta, \eta) \bar{\eta} \varphi(\eta) d\mu_r(\eta) = (\pi(\bar{z})^* \varphi)(\zeta). \end{aligned} \tag{III.13}$$

**Lemma III.3.** *We have*

$$(\pi(z)\varphi)(\zeta) = \zeta^{-r} \int_0^\zeta \eta^{r-1} \varphi'(\eta) d\eta. \tag{III.14}$$

*Proof.* We set  $\psi(\zeta) := (\pi(z)\varphi)(\zeta)$ . Then using (III.11),

$$\begin{aligned} \psi(\zeta) &= \pi^{-1} (r-1) \int_U \bar{\eta} (1 - \zeta\bar{\eta})^{-r} \varphi(\eta) d\mu_r(\eta) \\ &= \pi^{-1} \frac{d}{d\zeta} \int_U (1 - \zeta\bar{\eta})^{-r+1} \varphi(\eta) d\mu_r(\eta) \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{r-1} \frac{d}{d\zeta} \int_U (1 - \zeta\bar{\eta}) K(\zeta, \eta) \varphi(\eta) d\mu_r(\eta) \\
 &= \frac{1}{r-1} \left[ \varphi'(\zeta) - \frac{d}{d\zeta} (\zeta\psi(\zeta)) \right].
 \end{aligned}$$

As a consequence, we obtain a differential equation on  $\psi$ ,

$$\zeta\psi' + r\psi = \varphi'. \tag{III.15}$$

Solving this equation yields

$$\psi(\zeta) = \zeta^{-r} \int_0^\zeta \eta^{r-1} \varphi'(\eta) d\eta,$$

as claimed.  $\square$

**Theorem III.4.**  $(\mathcal{H}_U, \pi)$  defines a  $*$ -representation of  $P_\mu(\bar{U})$ . This representation is unitarily equivalent with representation  $(\beta)$  of Theorem III.2.

*Proof.* From (III.10) and (III.13),

$$(\pi(\bar{z})\varphi_n)(\zeta) = \zeta\varphi_n(\zeta) = \left\{ \frac{n+1}{n+r} \right\}^{1/2} \varphi_{n+1}(\zeta),$$

which proves (III.4). Likewise, from (III.14),

$$(\pi(z)\varphi_n)(\zeta) = \zeta^{-r} \int_0^\zeta \eta^{r-1} \varphi_n'(\eta) d\eta = \left\{ \frac{n}{n-1+r} \right\}^{1/2} \varphi_{n-1}(\zeta),$$

for  $n \geq 1$  and  $(\pi(z)\varphi_0)(\zeta) = 0$ . This proves (III.5).  $\square$

In other words,  $\pi$  is a homomorphism of  $P_\mu(\bar{U})$  into  $\mathcal{F}_\mu(\bar{U})$ . Our next result states that, in fact,  $C_\mu(\bar{U})$  is isomorphic with  $\mathcal{F}_\mu(\bar{U})$ . Therefore, the concrete  $\mathbf{C}^*$ -algebra  $\mathcal{F}_\mu(\bar{U})$  may serve as a universal representation of the abstract  $\mathbf{C}^*$ -algebra  $C_\mu(\bar{U})$ .

**Theorem III.5.**  $\pi: P_\mu(\bar{U}) \rightarrow \mathcal{F}_\mu(\bar{U})$  induces an isomorphism of the  $\mathbf{C}^*$ -algebras  $C_\mu(\bar{U})$  and  $\mathcal{F}_\mu(\bar{U})$ .

We will prove this theorem in Sect. IV.

#### IV. Toeplitz Operators

Our goal in this section is to prove Theorem III.5. The technique to achieve this is a detailed analysis of the structure of the  $\mathbf{C}^*$ -algebra  $\mathcal{F}_\mu(\bar{U})$ . Our analysis follows the standard methods (in particular those of [10, 1 and 6]). For the reader's convenience we include the details of the proofs referring to the literature at appropriate places. Throughout this section,  $f$  denotes a continuous function on  $\bar{U}$  and  $\|f\|_\infty$  denotes its sup-norm.

**Lemma IV.1.** (i)  $\|T_f\| \leq \|f\|_\infty$ ,  
 (ii)  $T_f = 0$ , if and only if  $f = 0$ .

*Proof.* Part (i) is clear. To prove part (ii) we observe that  $T_f = 0$  if and only if  $(\varphi_m, T_f \varphi_n) = 0, m, n = 0, 1, 2, \dots$ , where  $\varphi_n$  is defined by (III.10). Since

$$(\zeta^m, T_f \zeta^n) = \int_U \zeta^m \bar{\zeta}^n f(\zeta) d\mu_r(\zeta),$$

and since the monomials  $\zeta^m \bar{\zeta}^n$  generate a dense subspace in  $L^2(U, d\mu_r)$  this is possible if and only if  $f = 0$ .  $\square$

Let  $\mathcal{K}(\mathcal{H}_U)$  denote the  $\mathbb{C}^*$ -algebra of compact operators on  $\mathcal{H}_U$ .

**Lemma IV.2.**  $T_f \in \mathcal{K}(\mathcal{H}_U)$  if and only if  $f \upharpoonright_{\partial U} = 0$ .

*Proof.* We follow [1] and [6]. Suppose that  $f \upharpoonright_{\partial U} = 0$ . There is a sequence  $\{f_n\}$  of continuous functions such that  $\text{supp } f_n \subset U$  and  $\|f - f_n\|_\infty \rightarrow 0$ . Let  $\{\varphi_m\}$  be a sequence in  $\mathcal{H}_U$  such that  $\varphi_m \rightarrow 0$  weakly (recall that a weakly convergent sequence is bounded). We claim that  $T_f \varphi_m \rightarrow 0$  in norm which means that  $T_f$  is compact. Indeed, since  $\varphi_m(\zeta) \rightarrow 0$  pointwise, it follows that, for a fixed  $n, \varphi_m(\zeta) \rightarrow 0$  uniformly on  $\text{supp } f_n$ . Therefore,

$$\begin{aligned} \|T_f \varphi_m\| &\leq \|T_{f_n} \varphi_m\| + \|T_{f-f_n} \varphi_m\| \\ &\leq C \left( \sup_{\zeta \in \text{supp } f_n} |\varphi_m(\zeta)| \|f_n\| + \|f - f_n\|_\infty \|\varphi_m\| \right), \end{aligned}$$

where  $C = \left\{ \int_U d\mu_r(\zeta) \right\}^{1/2} < \infty$ . Choosing  $n$  sufficiently large and using the fact that  $\|\varphi_m\| \leq K$ , we see that the right-hand side of this inequality is arbitrarily small.

To prove the converse we consider the sequence of elements of  $\mathcal{H}_U$

$$\varphi_n(\zeta) := K(\zeta, \eta_n) / K(\eta_n, \eta_n)^{1/2},$$

where  $\eta_n \in U, \eta_n \rightarrow \eta \in \partial U$ . Then  $\|\varphi_n\| = 1$  and  $\varphi_n(\zeta) \rightarrow 0$  uniformly outside a neighborhood  $V$  of  $\eta$ . Passing to a subsequence, if necessary, we may assume that  $\varphi_n \rightarrow 0$  weakly. Since

$$\begin{aligned} \|(T_f - f(\eta))\varphi_n\|^2 &\leq \int_U |f(\zeta) - f(\eta)|^2 |\varphi_n(\zeta)|^2 d\mu_r(\zeta) \\ &\leq \left\{ \sup_{\zeta \in V} |f(\zeta) - f(\eta)| \right\}^2 + \left\{ \sup_{\zeta \in U \setminus V} |\varphi_n(\zeta)| \right\}^2 \|f - f(\eta)\|^2, \end{aligned}$$

we see that  $\|(T_f - f(\eta))\varphi_n\| \rightarrow 0$ . As a consequence,

$$|f(\eta)| = \|f(\eta)\varphi_n\| \leq \|T_f \varphi_n\| + \|(T_f - f(\eta))\varphi_n\| \rightarrow 0,$$

since  $T_f$  is compact. Therefore,  $f(\eta) = 0$ .  $\square$

**Lemma IV.3.**  $T_f T_g - T_{f_g}, [T_f, T_g] \in \mathcal{K}(\mathcal{H}_U)$ .

*Proof.* We follow [1]. Obviously, the second claim is a consequence of the first, so we need prove that  $T_f T_g - T_{f_g} \in \mathcal{K}(\mathcal{H}_U)$ . Clearly  $T_{f_g} - T_f T_g = P M_f (1 - P) M_g$ . Let  $\mathcal{H}_U^\perp$  denote the orthogonal complement of  $\mathcal{H}_U$  in  $L^2(U, d\mu_r)$  and let  $H_f := (1 - P) M_f P: \mathcal{H}_U \rightarrow \mathcal{H}_U^\perp$ . Then

$$T_{f_g} - T_f T_g = H_f^* H_g. \tag{IV.1}$$

Furthermore,

$$H_{f_g} = S_f H_g + H_f T_g, \tag{IV.2}$$

where  $S_f: \mathcal{H}_U^\perp \rightarrow \mathcal{H}_U^\perp$  is defined by  $S_f := (1 - P)M_f$ . Let now  $\mathcal{B} := \{f \in C(\bar{U}): H_f \text{ is compact}\}$ . As a consequence of (IV.2),  $\mathcal{B}$  is a closed subalgebra of  $C(\bar{U})$ . Also,  $H_1 = 0$ ,  $H_\zeta = 0$  and by (IV.1),

$$H_\zeta^* H_{\bar{\zeta}} = T_{\zeta\bar{\zeta}} - T_\zeta T_{\bar{\zeta}} = [\pi(z), \pi(\bar{z})] \in \mathcal{K}(\mathcal{H}_U).$$

So  $1, \zeta, \bar{\zeta} \in \mathcal{B}$ , and thus by the Stone–Weierstrass theorem  $\mathcal{B} = C(\bar{U})$ .  $\square$

Let  $J \subset \mathcal{T}_\mu(\bar{U})$  be the commutator ideal of  $\mathcal{T}_\mu(\bar{U})$ , i.e., the smallest norm-closed, two sided ideal in  $\mathcal{T}_\mu(\bar{U})$  containing all the commutators.

**Lemma IV.4.**  $J = \mathcal{K}(\mathcal{H}_U)$ .

*Proof.* We follow [1]. To show that  $\mathcal{K}(\mathcal{H}_U) \subset \mathcal{T}_\mu(\mathcal{H}_U)$  we use Theorem 5.39 in [10]. Firstly,  $\mathcal{K}(\mathcal{H}_U) \cap \mathcal{T}_\mu(\mathcal{H}_U) \neq \emptyset$ , as  $[T_\zeta, T_{\bar{\zeta}}] \in \mathcal{K}(\mathcal{H}_U)$ . Secondly, we claim that  $\mathcal{T}_\mu(\bar{U})$  is irreducible. Indeed, let  $Q$  be an orthogonal projection in  $\mathcal{H}_U$  which commutes with  $\mathcal{T}_\mu(\bar{U})$ . Then  $(Q\varphi)(\zeta) = QT_{\varphi(\bar{\zeta})}1 = \varphi(\zeta)(Q1)(\zeta)$  for all polynomials  $\varphi \in \mathcal{H}_U$ . From  $Q^2 = Q$  we have  $Q1 = 1$  and the claim follows. Therefore, by the quoted theorem,  $\mathcal{K}(\mathcal{H}_U) \subset \mathcal{T}_\mu(\bar{U})$ . As a consequence of Lemma IV.3,  $J \subset \mathcal{K}(\mathcal{H}_U)$ . Since  $\mathcal{K}(\mathcal{H}_U)$  is simple [10],  $\mathcal{K}(\mathcal{H}_U) = J$ .  $\square$

**Theorem IV.5.** (i) *There is a short exact sequence of  $\mathbf{C}^*$ -algebra (the Brown–Douglas–Fillmore sequence)*

$$0 \rightarrow \mathcal{K}(\mathcal{H}_U) \rightarrow \mathcal{T}_\mu(\bar{U}) \rightarrow C(\partial U) \rightarrow 0. \tag{IV.3}$$

(ii) *Every element  $A \in \mathcal{T}_\mu(\bar{U})$  can be written as*

$$A = T_f + K, \tag{IV.4}$$

where  $K \in \mathcal{K}(\mathcal{H}_U)$ . Furthermore,

$$\|T_f + K\| \geq \sup_{|\zeta|=1} |f(\zeta)|. \tag{IV.5}$$

*Proof.* (i) We claim that

$$\mathcal{T}_\mu(\bar{U})/\mathcal{K}(\mathcal{H}_U) \cong C(\partial U), \tag{IV.6}$$

as  $\mathbf{C}^*$ -algebras. Indeed, let  $\sigma: C(\bar{U}) \rightarrow \mathcal{T}_\mu(\bar{U})/\mathcal{K}(\mathcal{H}_U)$  be the map defined by  $\sigma(f) := T_f \pmod{\mathcal{K}(\mathcal{H}_U)}$ . By Lemma IV.3,  $\sigma$  is a  $*$ -homomorphism whose range is dense. By Lemma IV.2,  $\text{Ker}(\sigma) = C^{(0)}(\bar{U}) := \{f \in C(\bar{U}): f \upharpoonright_{\partial U} = 0\}$ . So  $\sigma$  induces an injective  $*$ -homomorphism

$$\bar{\sigma}: C(\bar{U})/C^{(0)}(\bar{U}) \cong C(\partial U) \rightarrow \mathcal{T}_\mu(\bar{U})/\mathcal{K}(\mathcal{H}_U).$$

By Proposition 4.67 in [10],  $\bar{\sigma}$  is an isometry, so  $\text{Ran}(\sigma) = \text{Ran}(\bar{\sigma}) = \mathcal{T}_\mu(\bar{U})/\mathcal{K}(\mathcal{H}_U)$ , and  $\bar{\sigma}$  is an isomorphism of  $\mathbf{C}^*$ -algebras.

(ii) As a consequence of (i), every element in  $\mathcal{T}_\mu(\bar{U})$  is of the form (IV.4). Since  $j: \mathcal{T}_\mu(\bar{U}) \rightarrow C(\partial U)$ ,  $j(T_f + K) := f \upharpoonright_{\partial U}$  is a  $*$ -monomorphism of  $\mathbf{C}^*$ -algebras, it follows from Proposition 4.67 in [10] that  $\sup_{\zeta \in \partial U} |f(\zeta)| \leq \|T_f + K\|$ .  $\square$

Now we are in a position to prove Theorem III.5.

*Proof of Theorem III.5.* Let

$$a = \sum_{\substack{k_1, \dots, k_n \\ l_1, \dots, l_n}} \alpha_{kl} z^{k_1} \bar{z}^{l_1} \dots z^{k_n} \bar{z}^{l_n} \in P_\mu(\bar{U}), \tag{IV.7}$$

where  $\alpha_{kl} \in \mathbb{C}$ . By (II.22) and Theorems III.2 and II.4,

$$\|a\| = \max \left\{ \left\| \pi \left( \sum_{k,l} \alpha_{kl} z^{k_1} \bar{z}^{l_1} \dots z^{k_n} \bar{z}^{l_n} \right) \right\|, \sup_{|\zeta|=1} \left| \sum_{k,l} \alpha_{kl} \zeta^{k_1 + \dots + k_n} \bar{\zeta}^{l_1 + \dots + l_n} \right| \right\}.$$

By (IV.5), the maximum is achieved at  $\left\| \pi \left( \sum_{k,l} \alpha_{kl} z^{k_1} \bar{z}^{l_1} \dots z^{k_n} \bar{z}^{l_n} \right) \right\|$ . Therefore  $\|a\| = \|\pi(a)\|$ . The claim follows.  $\square$

As a consequence of Theorems III.5 and IV.5 we obtain:

**Corollary IV.6.** *The  $\mathbb{C}^*$ -algebra  $C_\mu(\bar{U})$  is a GCR algebra.*

Let now  $\mathcal{H} = l^2(\mathbb{Z}_+)$  and let  $\{f_n\}_{n \geq 0}$  be an orthonormal basis for  $\mathcal{H}$ . The operator  $S$  defined by  $Sf_n := f_{n+1}$ ,  $n \geq 0$ , is called the unilateral shift [10]. Let  $\mathbb{C}^*(S)$  be the unital  $\mathbb{C}^*$ -algebra generated by  $S$ .

**Theorem IV.7.**  *$C_\mu(\bar{U})$  is isomorphic with  $\mathbb{C}^*(S)$ .*

*Proof.* Let  $E_{m,n} \in \mathcal{L}(\mathcal{H})$  be defined by  $E_{m,n} f_p := \delta_{np} f_m$ . Then  $S = \sum_{n \geq 0} E_{n+1,n}$  and  $\pi(\bar{z}) = \sum_{n \geq 0} \{(n+1)\mu / (1+(n+1)\mu)\}^{1/2} E_{n+1,n}$ . It is clear that  $S - \pi(\bar{z}) \in \mathcal{K}(\mathcal{H})$ . Consequently,  $\mathbb{C}^*(S) \subset \pi(C_\mu(\bar{U})) \cong C_\mu(\bar{U})$ . On the other hand, since  $S + K \in \mathbb{C}^*(S)$  (see [10]), it follows that  $\pi(\bar{z}) \in \mathbb{C}^*(S)$ , i.e.,  $\pi(C_\mu(\bar{U})) \subset \mathbb{C}^*(S)$ .  $\square$

### V. The Action of $SU(1, 1)$

In Sect. II we constructed a group action  $SU(1, 1) \rightarrow \text{Aut}(C_\mu(\bar{U}))$ . Under the identification  $C_\mu(\bar{U}) \cong \mathcal{F}_\mu(\bar{U})$ , this action can be implemented by a projective unitary representation of  $SU(1, 1)$  on  $\mathcal{H}_U$ .

For  $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$  we set  $\log \lambda := \log |\lambda| + i \arg \lambda$ , where  $-\pi < \arg \lambda \leq \pi$ . If  $\gamma \in SU(1, 1)$ , we let  $a_\gamma$  and  $b_\gamma$  denote the matrix elements of  $\gamma^{-1}$  as in (II.11). The function [14]

$$\frac{1}{2\pi i} \{ \log(\bar{b}_{\gamma_2} \gamma_1^{-1} \zeta + \bar{a}_{\gamma_2}) - \log(\bar{b}_{\gamma_1 \gamma_2} \zeta + \bar{a}_{\gamma_1 \gamma_2}) + \log(\bar{b}_{\gamma_1} \zeta + \bar{a}_{\gamma_1}) \}, \tag{V.1}$$

where  $\gamma_1, \gamma_2 \in SU(1, 1)$ ,  $\zeta \in U$ , is independent of  $\zeta$  (to see this, differentiate with respect to  $\zeta$ ). We denote the value of (V.1) by  $\lambda(\gamma_1, \gamma_2)$ . Furthermore, setting  $\zeta = 0$  it is easy to see that  $\lambda(\gamma_1, \gamma_2) \in \{-1, 0, 1\}$ . In fact, the function  $\lambda: SU(1, 1) \times SU(1, 1) \rightarrow \mathbb{Z}$  defines a cohomology class in  $H^2(SU(1, 1), \mathbb{Z})$ , i.e.,

$$\lambda(\gamma_2, \gamma_3) - \lambda(\gamma_1 \gamma_2, \gamma_3) + \lambda(\gamma_1, \gamma_2 \gamma_3) - \lambda(\gamma_1, \gamma_2) = 0 \tag{V.2}$$

(to see this, use the fact that (V.1) is independent of  $\zeta$ ). We set

$$\sigma(\gamma_1, \gamma_2) := \exp \{ 2\pi i r \lambda(\gamma_1, \gamma_2) \}. \tag{V.3}$$

For  $\gamma \in SU(1, 1)$  and  $\varphi \in \mathcal{H}_U$  we define

$$(U_\gamma \varphi)(\zeta) := (\bar{b}_\gamma \zeta + \bar{a}_\gamma)^{-r} \varphi(\gamma^{-1} \zeta), \tag{V.4}$$

where  $(\bar{b}_\gamma \zeta + \bar{a}_\gamma)^{-r} := \exp\{-r \log(\bar{b}_\gamma \zeta + \bar{a}_\gamma)\}$ . Then  $U_\gamma$  is a unitary operator on  $\mathcal{H}_U$ , and

$$U_{\gamma_1 \gamma_2} = \sigma(\gamma_1, \gamma_2) U_{\gamma_1} U_{\gamma_2}. \tag{V.5}$$

Therefore, as a consequence of (V.2),  $\gamma \rightarrow U_\gamma$  is a projective unitary representation of  $SU(1, 1)$ .

**Theorem V.1.** *The  $SU(1, 1)$  action  $\gamma \rightarrow \rho_\gamma$  on  $C_\mu(\bar{U})$  is implemented by  $\gamma \rightarrow U_\gamma$ .*

*Proof.* We need verify that

$$\rho_\gamma(\bar{z}) = U_\gamma \bar{z} U_{\gamma^{-1}}. \tag{V.6}$$

Indeed, from (III.11),

$$\begin{aligned} (U_\gamma \bar{z} U_{\gamma^{-1}} \varphi)(\zeta) &= (\bar{b}\zeta + \bar{a})^{-r} (\bar{z} U_{\gamma^{-1}} \varphi)(\gamma^{-1} \zeta) = (\bar{b}\zeta + \bar{a})^{-r} \frac{a\zeta + b}{\bar{b}\zeta + \bar{a}} (U_{\gamma^{-1}} \varphi)(\gamma^{-1} \zeta) \\ &= (\bar{b}\zeta + \bar{a})^{-r} \frac{a\zeta + b}{\bar{b}\zeta + \bar{a}} (-\bar{b}\gamma^{-1}\zeta + a)^{-r} \varphi(\zeta) = \frac{a\zeta + b}{\bar{b}\zeta + \bar{a}} \varphi(\zeta) = \rho_\gamma(\bar{z}) \varphi(\zeta), \end{aligned}$$

as claimed.  $\square$

## VI. Asymptotics of Products of Toeplitz Operators

In this section we show that, in a suitable sense,  $C_\mu(U)$  is a quantum deformation of the unit disc. To be more specific, we prove that for smooth  $f$  and  $g$ ,  $\frac{1}{\mu} [T_f, T_g]$  approaches its classical value, as  $\mu \rightarrow 0$ .

Let  $C^\infty(\bar{U})$  denote the Fréchet space of smooth functions on  $U$  whose derivatives extend to continuous functions on  $\bar{U}$ . For  $f, g \in C^\infty(\bar{U})$  we define

$$\{f, g\}(\zeta) := i(1 - |\zeta|^2)^2 [\partial f(\zeta) \bar{\partial} g(\zeta) - \bar{\partial} f(\zeta) \partial g(\zeta)], \tag{VI.1}$$

the Poisson bracket of  $f$  and  $g$  corresponding to the Poincaré symplectic form on  $U$ ,  $\omega := \frac{i}{2} (1 - |\zeta|^2)^{-2} d\zeta \wedge d\bar{\zeta}$ . Obviously,  $\{f, g\} \in C^\infty(\bar{U})$ . The goal of this section is to prove the following theorems.

**Theorem VI.1.** *Let  $f \in C(\bar{U})$ . Then*

$$\|T_f\| \leq \|f\|_\infty \leq \|T_f\| + o(1), \tag{VI.2}$$

as  $r \rightarrow \infty$ . In particular,  $\lim_{r \rightarrow \infty} \|T_f\| = \|f\|_\infty$ .

**Theorem VI.2.** *Let  $f, g \in C^\infty(\bar{U})$  and let  $r$  be sufficiently large. Then:*

(i) *There exists a constant  $C$  (depending on  $f$  and  $g$ ) such that*

$$\|r(T_f T_g - T_{fg}) + T_{(1 - |\zeta|^2)^2 \partial f \bar{\partial} g}\| \leq Cr^{-1/2}. \tag{VI.3}$$

(ii) *There is a constant C (depending on f and g) such that*

$$\left\| \frac{r}{i} [T_f, T_g] - T_{\{f,g\}} \right\| \leq Cr^{-1/2}. \tag{VI.4}$$

*Remark.* It should be remembered that the operators and the operator norms in (VI.2–4) depend on r even though, for the sake of notational simplicity, this is not explicit in our formulas. Estimate (VI.2) says that  $\lim_{\mu \rightarrow 0} \|T_f\| = \|f\|$ , while estimate

(IV.4) says that  $\lim_{\mu \rightarrow 0} \left\| \frac{1}{i\mu} [T_f, T_g] - T_{\{f,g\}} \right\| = 0$ . This shows that  $C_\mu(U)$  is a quantum deformation of the unit disc in a sense close to Rieffel’s “strict deformation quantization” [17].

*Proof of Theorem VI.1.* The first inequality in (VI.2) is clear. To prove the second one, we define  $\gamma_\zeta \in SU(1, 1)$ ,  $\zeta \in U$ , by

$$\gamma_\zeta = (1 - |\zeta|^2)^{-1/2} \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & 1 \end{pmatrix}. \tag{VI.5}$$

Observe that  $\gamma_\zeta(0) = \zeta$ , and

$$|\gamma_\zeta(\eta) - \gamma_\zeta(0)| \leq \frac{|\eta|}{1 - |\eta|}.$$

In particular, if  $|\eta| < \delta \leq 1/2$ , then

$$|\gamma_\zeta(\eta) - \gamma_\zeta(0)| \leq 2\delta. \tag{VI.6}$$

**Lemma VI.3.** *If  $f \in C(\bar{U})$ , then*

$$\sup_{\zeta \in \bar{U}} \left| f(\zeta) - (r - 1) \int_U f(\gamma_\zeta(\eta)) d\mu_r(\eta) \right| = o(1), \tag{VI.7}$$

as  $r \rightarrow \infty$ .

We shall prove this lemma after completing the main line of the argument. Using the reproducing property of the Bergman kernel, we write

$$\begin{aligned} f(\zeta) &= (r - 1) \int_U \gamma_\zeta^* f(\eta) d\mu_r(\eta) + \left( f(\zeta) - (r - 1) \int_U \gamma_\zeta^* f(\eta) d\mu_r(\eta) \right) \\ &= (r - 1) \int_{U^2} K(\theta, \eta) \gamma_\zeta^* f(\eta) d\mu_r(\theta) + \left( f(\zeta) - (r - 1) \int_U \gamma_\zeta^* f(\eta) d\mu_r(\eta) \right) \\ &= (T_{\gamma_\zeta^* f} \varphi_0, \varphi_0) + \left( f(\zeta) - (r - 1) \int_U \gamma_\zeta^* f(\eta) d\mu_r(\eta) \right), \end{aligned}$$

where  $\gamma_\zeta^* f(\eta) := f(\gamma_\zeta(\eta))$ , and where  $\varphi_0$  is given by (III.10). As a consequence of Lemma VI.3,

$$\sup_{\zeta \in \bar{U}} |f(\zeta)| \leq \sup_{\zeta \in \bar{U}} \|T_{\gamma_\zeta^* f}\| + o(1).$$

But,  $T_{\gamma_\zeta^* f} = U_{\gamma_\zeta} T_f U_{\gamma_\zeta}^{-1}$ , with  $U_{\gamma_\zeta}$  unitary, and so  $\|f\|_\infty \leq \|T_f\| + o(1)$ , as claimed.

*Proof of Lemma VI.3.* Since  $f$  is continuous on a compact set, it is uniformly continuous. Given  $\varepsilon > 0$ , we take  $\delta \leq 1/2$  so that  $|f(\eta) - f(\eta')| < \varepsilon/2$ , whenever  $|\eta - \eta'| < 2\delta$ . Then,

$$\begin{aligned} f(\eta) - (r-1) \int_U f(\gamma_\zeta(\eta)) d\mu_r(\eta) &= (r-1) \int_U (f(\gamma_\zeta(0)) - f(\gamma_\zeta(\eta))) d\mu_r(\eta) \\ &= \int_{|\eta| < \delta} \dots + \int_{\delta < |\eta| < 1} \dots := I_1 + I_2. \end{aligned}$$

Now, using (VI.6),

$$|I_1| \leq \sup_{|\eta| < \delta} |f(\gamma_\zeta(0)) - f(\gamma_\zeta(\eta))| < \varepsilon/2.$$

Furthermore,

$$|I_2| \leq 2 \|f\|_\infty \int_{\delta < |\eta| < 1} d\mu_r(\eta) = 2 \|f\|_\infty (1 - \delta^2)^{r-1},$$

which is less than  $\varepsilon/2$ , for  $r$  sufficiently large.

*Proof of Theorem VI.2.* Obviously, (ii) is a consequence of (i) and (VI.1). To prove (i) we write

$$(T_f T_g \varphi, \psi) = \int_{U^2} K(\zeta, \eta) f(\zeta) g(\eta) \varphi(\eta) \overline{\psi(\zeta)} d\mu_r(\zeta) d\mu_r(\eta), \quad (\text{VI.8})$$

where  $\varphi, \psi \in \mathcal{H}_U$ . In the  $\eta$ -integral we substitute  $\eta = \gamma_\zeta \theta$ , where  $\gamma_\zeta \in SU(1, 1)$  is defined by (VI.5). Then

$$(T_f T_g \varphi, \psi) = \pi^{-1} (r-1) \int_{U^2} f(\zeta) g(\gamma_\zeta \theta) (U_{\gamma_\zeta^{-1}} \varphi)(\theta) \overline{\psi(\zeta)} (1 - |\zeta|^2)^{-r/2} d\mu_r(\zeta) d\mu_r(\theta). \quad (\text{VI.9})$$

This form of (VI.8) is suitable for asymptotic analysis. From Taylor's theorem,

$$\begin{aligned} g(\gamma_\zeta \theta) &= g(\zeta) + (1 - |\zeta|^2) \partial g(\zeta) \theta + (1 - |\zeta|^2) \bar{\partial} g(\zeta) \bar{\theta} \\ &\quad + (1 - |\zeta|^2) [-\bar{\zeta} \partial g(\zeta) + \frac{1}{2} (1 - |\zeta|^2) \partial^2 g(\zeta)] \theta^2 \\ &\quad + (1 - |\zeta|^2) [-\zeta \bar{\partial} g(\zeta) + \frac{1}{2} (1 - |\zeta|^2) \bar{\partial}^2 g(\zeta)] \bar{\theta}^2 \\ &\quad + (1 - |\zeta|^2)^2 \partial \bar{\partial} g(\zeta) \theta \bar{\theta} + G(\theta, \zeta), \end{aligned} \quad (\text{VI.10})$$

where  $G(\theta, \zeta)$  is the second order remainder. Let us assume for a moment that the following fact is true.

**Lemma IV.4.** *There is a constant  $C$  (depending on  $f$  and  $g$ ) such that*

$$\begin{aligned} &|\pi^{-1} (r-1) \int_{U^2} f(\zeta) G(\theta, \zeta) (U_{\gamma_\zeta^{-1}} \varphi)(\theta) \overline{\psi(\zeta)} (1 - |\zeta|^2)^{-r/2} d\mu_r(\zeta) d\mu_r(\theta)| \\ &\leq C r^{-3/2} \|\varphi\| \|\psi\|. \end{aligned} \quad (\text{VI.11})$$

Substituting (VI.10) into (VI.9) and using the fact that  $\theta \rightarrow (U_{\gamma_\zeta^{-1}} \varphi)(\theta)$  is a holomorphic function we find:

$$\begin{aligned} (T_f T_g \varphi, \psi) &= \int_U f(\zeta) g(\zeta) \varphi(\zeta) \overline{\psi(\zeta)} d\mu_r(\zeta) + \pi^{-1} (r-1) \\ &\quad \cdot \int_{U^2} f(\zeta) \bar{\partial} g(\zeta) \frac{\partial}{\partial \theta} (U_{\gamma_\zeta^{-1}} \varphi)(0) \overline{\psi(\zeta)} (1 - |\zeta|^2)^{-r/2+1} |\theta|^2 d\mu_r(\zeta) d\mu_r(\theta) \end{aligned}$$

$$\begin{aligned}
& + \pi^{-1}(r-1) \int_{U^2} f(\zeta)(1-|\zeta|^2)^2 \partial \bar{\partial} g(\zeta) \varphi(\zeta) \overline{\psi(\zeta)} |\theta|^2 d\mu_r(\zeta) d\mu_r(\theta) \\
& + \pi^{-1}(r-1) \int_{U^2} f(\zeta) [-\zeta \bar{\partial} g(\zeta) + \frac{1}{2}(1-|\zeta|^2) \bar{\partial}^2 g(\zeta)] \frac{\partial^2}{\partial \theta^2} (U_{\gamma_{\zeta}^{-1}} \varphi)(0) \\
& \cdot \overline{\psi(\zeta)} (1-|\zeta|^2)^{-r/2+1} |\theta|^4 d\mu_r(\zeta) d\mu_r(\theta) + R,
\end{aligned} \tag{VI.12}$$

where  $R$  denotes the remainder term which obeys the estimate (VI.11). Observe that the first term on the right-hand side is equal to  $(T_{f_g} \varphi, \psi)$ . Furthermore, using

$$\pi^{-1}(r-1) \int_U |\theta|^2 d\mu_r(\theta) = r^{-1},$$

and

$$(1-|\zeta|^2)^{-r/2} \frac{\partial}{\partial \theta} (U_{\gamma_{\zeta}^{-1}} \varphi)(\theta) = (1-|\zeta|^2)^{-r+1} (1+\bar{\zeta}\theta)^{-1} \frac{\partial}{\partial \zeta} \{(1-|\zeta|^2)^{r/2} (U_{\gamma_{\zeta}^{-1}} \varphi)(\theta)\}, \tag{VI.13}$$

we write the second and third terms in (VI.12) as

$$\begin{aligned}
& \frac{1}{r} \int_U f(\zeta) \bar{\partial} g(\zeta) \overline{\psi(\zeta)} (1-|\zeta|^2)^{-r+2} \frac{\partial}{\partial \zeta} \{(1-|\zeta|^2)^r \varphi(\zeta)\} d\mu_r(\zeta) \\
& + \frac{1}{r} \int_U f(\zeta) (1-|\zeta|^2)^2 \partial \bar{\partial} g(\zeta) \varphi(\zeta) \overline{\psi(\zeta)} d\mu_r(\zeta).
\end{aligned}$$

Integrating by parts we see that this sum is equal to

$$-\frac{1}{r} \int (1-|\zeta|^2)^2 \partial f(\zeta) \bar{\partial} g(\zeta) \varphi(\zeta) \overline{\psi(\zeta)} d\mu_r(\zeta) = -\frac{1}{r} (T_{(1-|\zeta|^2)^2 \partial f \bar{\partial} g} \varphi, \psi).$$

We claim that the fourth term in (VI.12) can be estimated by  $Cr^{-2} \|\varphi\| \|\psi\|$ , with  $C$  independent of  $r$ . We proceed as follows. Using (VI.13) twice, we find that

$$\begin{aligned}
(1-|\zeta|^2)^{-r/2} \frac{\partial^2}{\partial \theta^2} (U_{\gamma_{\zeta}^{-1}} \varphi)(0) & = -(1-|\zeta|^2)^{-r+1} \bar{\zeta} \frac{\partial}{\partial \zeta} \{(1-|\zeta|^2)^r \varphi(\zeta)\} \\
& + (1-|\zeta|^2)^{-r+1} \frac{\partial}{\partial \zeta} \left\{ (1-|\zeta|^2) \frac{\partial}{\partial \zeta} [(1-|\zeta|^2)^r \varphi(\zeta)] \right\}.
\end{aligned}$$

We substitute this into (VI.12) and use the fact that

$$\pi^{-1}(r-1) \int_U |\theta|^4 d\mu_r(\theta) = \frac{2}{r(r+1)}. \tag{VI.14}$$

Integrating by parts we write the integral as

$$\frac{1}{r(r+1)} \int_U F(\zeta) \varphi(\zeta) \overline{\psi(\zeta)} d\mu_r(\zeta), \tag{VI.15}$$

with  $F(\zeta)$  continuous and bounded on  $U$ . Finally, we estimate (VI.15) by  $r^{-2} \sup_{\zeta \in U} |F(\zeta)| \|\varphi\| \|\psi\|$ . This concludes the proof of (VI.3) up to the proof of

Lemma VI.4.  $\square$



*Proof of Lemma VI.4.* We claim that  $G(\theta, \zeta)$  can be written as

$$G(\theta, \zeta) = \sum_{0 \leq j \leq 3} g_j(\theta, \zeta) \bar{\theta}^j \bar{\theta}^{3-j}, \tag{VI.16}$$

where  $g_j, 0 \leq j \leq 3$ , are smooth functions on  $U \times U$  with the following properties:

$$(i) \quad |g_j(\theta, \zeta)| \leq O(1)(1 - |\zeta|^2)(1 - |\theta|)^{-6}, \tag{VI.17}$$

$$(ii) \quad \left| \frac{\partial}{\partial \zeta} \{ (1 - |\zeta|^2)^{-1} g_j(\theta, \zeta) \} \right| + \left| \frac{\partial}{\partial \bar{\zeta}} \{ (1 - |\zeta|^2)^{-1} g_j(\theta, \zeta) \} \right| \leq O(1)(1 - |\theta|)^{-8}, \tag{VI.18}$$

$$(iii) \quad \left| \frac{\partial}{\partial \theta} g_j(\theta, \zeta) \right| + \left| \frac{\partial}{\partial \bar{\theta}} g_j(\theta, \zeta) \right| \leq O(1)(1 - |\zeta|^2)(1 - |\theta|)^{-8}. \tag{VI.19}$$

Indeed, we write

$$\begin{aligned} G(\theta, \zeta) &= \frac{1}{2} \int_0^1 (1-s)^2 \frac{d^3}{ds^3} \gamma_\zeta^* g(s\theta) ds \\ &= \frac{1}{2} \sum_{0 \leq j \leq 3} \binom{3}{j} \int_0^1 (1-s)^2 \partial_{s\bar{\theta}}^j \bar{\partial}_{s\theta}^{3-j} \gamma_\zeta^* g(s\theta) ds \theta^j \bar{\theta}^{3-j}, \end{aligned}$$

and define

$$g_j(\theta, \zeta) := \frac{1}{2} \binom{3}{j} \int_0^1 (1-s)^2 \partial_{s\bar{\theta}}^j \bar{\partial}_{s\theta}^{3-j} \gamma_\zeta^* g(s\theta) ds. \tag{VI.20}$$

To verify that  $g_j(\theta, \zeta)$  obeys (VI.17)–(VI.19) we carry out the differentiations in (VI.20) according to the chain rule. Using the fact that for  $k \geq 1$

$$\frac{\partial^k}{\partial \theta^k} \gamma_\zeta(\theta) = (-1)^{k-1} k! (1 - |\zeta|^2) \bar{\zeta}^{k-1} (1 + \bar{\zeta}\theta)^{-k-1}, \tag{VI.21}$$

as well as the inequality

$$|1 + s\bar{\zeta}\theta|^{-1} \leq (1 - s|\theta|)^{-1} \leq (1 - |\theta|)^{-1}, \quad 0 \leq s \leq 1, \tag{VI.22}$$

we easily obtain the bound

$$|g_j(\theta, \zeta)| \leq O(1) \left\{ \sum_{1 \leq \alpha + \beta \leq 3} \|\partial^\alpha \bar{\partial}^\beta g\|_\infty \right\} (1 - |\zeta|^2)(1 - |\theta|)^{-6},$$

which proves (VI.17). Inequalities (VI.18) and (VI.19) follow from an analogous argument for the derivatives of (VI.20).

Let us now estimate the contribution that each term in (VI.16) gives to (VI.11). Let  $j = 0$ . Using

$$\bar{\theta}(1 - |\theta|^2)^{r-2} = \frac{1}{r-1} \frac{\partial}{\partial \theta} (1 - |\theta|^2)^{r-1}, \tag{VI.23}$$

and integrating by parts with respect to  $\theta$  we find that

$$\pi^{-1}(r-1) \int_{\bar{U}^2} f(\zeta) g_0(\theta, \zeta) \bar{\theta}^3 (U_{\gamma_\zeta^{-1} \varphi})(\theta) \overline{\psi(\zeta)} (1 - |\zeta|^2)^{-r/2} d\mu_r(\zeta) d\mu_r(\theta)$$

$$\begin{aligned}
&= -\pi^{-1} \int_{\bar{U}^2} f(\zeta)(1-|\theta|^2) \frac{\partial}{\partial \theta} g_0(\theta, \zeta) \bar{\theta}^2 (U_{\gamma_{\zeta}^{-1}} \varphi)(\theta) \overline{\psi(\zeta)} (1-|\zeta|^2)^{-r/2} d\mu_r(\zeta) d\mu_r(\theta) \\
&\quad - \pi^{-1} \int_{\bar{U}^2} f(\zeta)(1-|\theta|^2) g_0(\theta, \zeta) \bar{\theta}^2 \frac{\partial}{\partial \theta} (U_{\gamma_{\zeta}^{-1}} \varphi)(\theta) \overline{\psi(\zeta)} (1-|\zeta|^2)^{-r/2} d\mu_r(\zeta) d\mu_r(\theta).
\end{aligned} \tag{VI.24}$$

The first term on the right-hand side can be estimated by

$$\begin{aligned}
&2\pi^{-1} \sup_{\zeta} |f(\zeta)| \sup_{\zeta, \theta} \left| (1-|\zeta|^2)^{-1} (1-|\theta|)^8 \frac{\partial}{\partial \theta} g_0(\theta, \zeta) \right| \\
&\quad \cdot \left\{ \int_{\bar{U}} |\theta|^4 (1-|\theta|)^{-14} d\mu_r(\theta) \right\}^{1/2} \left\{ \int_{\bar{U}} (1-|\zeta|^2)^{-r} d\mu_r(\zeta) \right\}^{1/2} \|\varphi\| \|\psi\| \\
&\leq Cr^{-3/2} \|\varphi\| \|\psi\|,
\end{aligned}$$

where we have used that

$$\int_{\bar{U}} |\theta|^4 (1-|\theta|)^{-14} d\mu_r(\theta) \leq O(1)r^{-3}, \tag{VI.25}$$

and

$$\pi^{-1} \int_{\bar{U}} (1-|\zeta|^2)^{-r} d\mu_r(\zeta) = 1. \tag{VI.26}$$

To estimate the second term in (VI.24) we first use (VI.13) and integrate by parts with respect to  $\zeta$ . The result is

$$\begin{aligned}
&\pi^{-1} \int_{\bar{U}^2} \partial f(\zeta)(1-|\theta|^2) g_0(\theta, \zeta) \bar{\theta}^2 (U_{\gamma_{\zeta}^{-1}} \varphi)(\theta) \overline{\psi(\zeta)} (1-|\zeta|^2)^{-r/2+1} (1+\bar{\zeta}\theta)^{-1} d\mu_r(\zeta) d\mu_r(\theta) \\
&\quad + \pi^{-1} \int_{\bar{U}^2} f(\zeta)(1-|\theta|^2) \frac{\partial}{\partial \zeta} \{ (1-|\zeta|^2)^{-1} g_0(\theta, \zeta) \} \\
&\quad \cdot \bar{\theta}^2 (U_{\gamma_{\zeta}^{-1}} \varphi)(\theta) \overline{\psi(\zeta)} (1-|\zeta|^2)^{-r/2+2} (1+\bar{\zeta}\theta)^{-1} d\mu_r(\zeta) d\mu_r(\theta).
\end{aligned}$$

Using (VI.22) (with  $s=1$ ) and properties (i)–(iii) above, we estimate each of these terms by

$$\begin{aligned}
&C \|\varphi\| \|\psi\| \left\{ \int_{\bar{U}} |\theta|^4 (1-|\theta|)^{-16} d\mu_r(\theta) \right\}^{1/2} \left\{ \int_{\bar{U}} (1-|\zeta|^2)^{-r+2} d\mu_r(\zeta) \right\}^{1/2} \\
&\leq Cr^{-3/2} \|\varphi\| \|\psi\|.
\end{aligned}$$

This completes the proof for  $j=0$ .

Let  $j=1$ . Using (VI.23) and integrating by parts with respect to  $\theta$  we find that

$$\begin{aligned}
&\pi^{-1} (r-1) \int_{\bar{U}^2} f(\zeta) g_1(\theta, \zeta) \bar{\theta}^2 (U_{\gamma_{\zeta}^{-1}} \varphi)(\theta) \overline{\psi(\zeta)} (1-|\zeta|^2)^{-r/2} d\mu_r(\zeta) d\mu_r(\theta) \\
&= -\pi^{-1} \int_{\bar{U}^2} f(\zeta)(1-|\theta|^2) \frac{\partial}{\partial \theta} g_1(\theta, \zeta) |\theta|^2 (U_{\gamma_{\zeta}^{-1}} \varphi)(\theta) \overline{\psi(\zeta)} (1-|\zeta|^2)^{-r/2} d\mu_r(\zeta) d\mu_r(\theta) \\
&\quad - \pi^{-1} \int_{\bar{U}^2} f(\zeta)(1-|\theta|^2) g_1(\theta, \zeta) |\theta|^2 \frac{\partial}{\partial \theta} (U_{\gamma_{\zeta}^{-1}} \varphi)(\theta) \overline{\psi(\zeta)} (1-|\zeta|^2)^{-r/2} d\mu_r(\zeta) d\mu_r(\theta)
\end{aligned}$$

$$-\pi^{-1} \int_{U^2} f(\zeta)(1 - |\theta|^2)g_1(\theta, \zeta)\bar{\theta}(U_{\gamma_\zeta^{-1}}\varphi)(\theta)\overline{\psi(\zeta)}(1 - |\zeta|^2)^{-r/2}d\mu_r(\zeta)d\mu_r(\theta).$$

The first two terms on the right-hand side of this equation can be estimated in the same way as the terms in (VI.24) in the case of  $j = 0$ . To estimate the third term we use (VI.23) (with  $r$  replaced by  $r + 1$ ), integrate by parts with respect to  $\theta$  (by doing so we produce the missing factor of  $O(r^{-1})$ ), and use the Schwarz inequality as in the analysis of  $j = 0$ . The resulting bound is  $Cr^{-3/2} \|\varphi\| \|\psi\|$ .

Let  $j = 2$ . We use

$$\theta(1 - |\theta|^2)^{r-2} = \frac{1}{r-1} \frac{\partial}{\partial \bar{\theta}}(1 - |\theta|^2)^{r-1}, \tag{VI.27}$$

and integrate by parts with respect to  $\theta$  to obtain

$$\begin{aligned} &\pi^{-1}(r-1) \int_{U^2} f(\zeta)g_2(\theta, \zeta)\theta^2\bar{\theta}(U_{\gamma_\zeta^{-1}}\varphi)(\theta)\overline{\psi(\zeta)}(1 - |\zeta|^2)^{-r/2}d\mu_r(\zeta)d\mu_r(\theta) \\ &= -\pi^{-1} \int_{U^2} f(\zeta)(1 - |\theta|^2) \frac{\partial}{\partial \bar{\theta}}g_2(\theta, \zeta)|\theta|^2(U_{\gamma_\zeta^{-1}}\varphi)(\theta)\overline{\psi(\zeta)}(1 - |\zeta|^2)^{-r/2}d\mu_r(\zeta)d\mu_r(\theta) \\ &\quad - \pi^{-1} \int_{U^2} f(\zeta)(1 - |\theta|^2)g_2(\theta, \zeta)\theta(U_{\gamma_\zeta^{-1}}\varphi)(\theta)\overline{\psi(\zeta)}(1 - |\zeta|^2)^{-r/2}d\mu_r(\zeta)d\mu_r(\theta). \end{aligned}$$

Both terms on the right-hand side of this equation have a familiar structure and, by the methods explained above, they can be estimated by  $Cr^{-3/2} \|\varphi\| \|\psi\|$ .

Finally, let  $j = 3$ . Using (VI.27) and integrating by parts with respect to  $\theta$  we obtain

$$\begin{aligned} &\pi^{-1}(r-1) \int_{U^2} f(\zeta)g_3(\theta, \zeta)\theta^3(U_{\gamma_\zeta^{-1}}\varphi)(\theta)\overline{\psi(\zeta)}(1 - |\zeta|^2)^{-r/2}d\mu_r(\zeta)d\mu_r(\theta) \\ &= -\pi^{-1} \int_{U^2} f(\zeta)(1 - |\theta|^2) \frac{\partial}{\partial \bar{\theta}}g_3(\theta, \zeta)\theta^2(U_{\gamma_\zeta^{-1}}\varphi)(\theta)\overline{\psi(\zeta)}(1 - |\zeta|^2)^{-r/2}d\mu_r(\zeta)d\mu_r(\theta). \end{aligned}$$

Again, as in the case of  $j = 0$ , this can be estimated by  $Cr^{-3/2} \|\varphi\| \|\psi\|$ .  $\square$

In conclusion, we notice that formula (VI.9) provides a systematic basis for generating an asymptotic expansion of  $T_f T_g$  in powers of  $r^{-1}$ . The methods we have used to estimate the second order remainder in this expansion can presumably be used to estimate the remainder of arbitrary order.

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