Commun. Math. Phys. 114, 147–165 (1988)

Communications in
Mathematical
Physics

The Two-Dimensional, $N=2$ Wess-Zumino Model on a Cylinder*

Arthur Jaffe, Andrzej Lesniewski and Jonathan Weitsman**

Harvard University, Cambridge, MA 02138, USA

Abstract. We construct a family of supersymmetric, two-dimensional quantum field models. We establish the existence of the Hamiltonian H and the supercharge Q as self-adjoint operators. We establish the ultraviolet finiteness of the model, independent of perturbation theory. We develop functional integral representations of the heat kernel which are useful for proving estimates in these models. In a companion paper [1] we establish an index theorem for Q, an infinite dimensional Dirac operator on loop space. This paper and, another related one [2], provide the technical justification for our claim that Q is Fredholm, and for our computation of its index by a homotopy onto quantum mechanics.

Contents

I. Introduction

In this paper we construct a family of $N=2$, Wess-Zumino quantum field models on a cylinder $T^1 \times \mathbb{R}$ [3]. The one-torus (circle) corresponds to periodic boundary conditions in space. We use a mixture of Hamiltonian and Euclidean methods to construct the generator H of time translations.

* Supported in part by the National Science Foundation under Grant DMS/PHY 86-45122
** Hertz Foundation Graduate Fellow

These models have a local, conserved operator, the supercharge Q, related to H by the identity $H=Q^2$. The operators Q and H are integrals over T^1 of densities $Q(x)$ and $H(x)$, respectively. It is a remarkable fact that these models are ultraviolet finite – no infinite renormalizations occur in H. Furthermore, the vacuum energy of H is identically zero; the models are thus infrared finite and have unbroken supersymmetry. We believe that this provides the first complete construction of a nonlinear, supersymmetric quantum field model.

The nonlinear models which we study also have nontrivial vacuum structure. We extend the present analysis in a companion paper [1], where we establish an index theorem for Q and prove nonuniqueness of the vacuum state. The supercharge Q is a function of an analytic polynomial $V(\varphi)$, the superpotential. For a superpotential V of degree n, there are at least $n-1$ linearly-independent, zero-energy eigenstates of H.

We establish the relevant estimates on the Hamiltonian H using heat kernel methods. We represent $\exp(-\beta H)$ by a functional integral which is Gaussian in the fermionic degrees of freedom but which, in general, is non-Gaussian in the bosonic degrees of freedom. We establish a Feynman-Kac representation for the heat kernel which, after evaluation of the Gaussian fermionic integral, yields a Fredholm determinant.

To establish heat kernel representations we first introduce an approximate (ultraviolet regularized) Hamiltonian $H(\kappa)$ and supercharge $Q(\kappa)$, with $H(\kappa)=Q(\kappa)^2$. The representation for $\exp(-H(\kappa))$ involves a Fredholm determinant of $I-K^{(\kappa)}$. We replace the Fredholm determinant $\det(I-K^{(\kappa)})$ with the regularized determinant $\det_3(I-K^{(\kappa)})$, where

$$\det(I-K^{(\kappa)})=\det_3(I-K^{(\kappa)})\exp(-\mathrm{Tr}K^{(\kappa)}-\tfrac{1}{2}\mathrm{Tr}(K^{(\kappa)})^2). \qquad \text{(I.1)}$$

This allows us to remove the ultraviolet regularization κ. The singular terms in the exponential, namely $-\mathrm{Tr}K^{(\kappa)}-\frac{1}{2}\mathrm{Tr}(K^{(\kappa)})^2$ exactly compensate for other singularities which occur in the bosonic action. The result is that no ad hoc renormalization is necessary. (This finiteness does not, however, hold in the corresponding $N=1$ models.)

We use these methods to prove uniform bounds of two types. In the first place we establish a lower bound on the Hamiltonian of the form

$$\zeta N_\tau \leqq H(\kappa)+C. \qquad \text{(I.2)}$$

Here $\tau<1$ and $\zeta=\zeta(\tau)>0$, $C<\infty$ are constants independent of κ. Also N_τ is a quadratic expression interpolating for $\tau\in[0,1]$ between the particle number operator $N=N_0$ and the free Hamiltonian $H_0=N_1$, see Sect. II. This bound is useful to establish elementary properties of H, such as the compactness of its resolvent.

In the second place, we establish norm continuity estimates $\kappa\to(H(\kappa)+I)^{-1}$ and convergence as $\kappa\to\infty$. This allows us to also prove norm continuity and convergence for the supercharge resolvents $\kappa\to(Q(\kappa)\pm i)^{-1}$. The resulting $\kappa=\infty$ operators Q and H are self-adjoint, cutoff independent, and $H=Q^2$. The Feynman-Kac representations also have $\kappa\to\infty$ limits.

The family of models we study have superpotentials

$$V(\varphi)=\tfrac{1}{2}m\varphi^2+P(\varphi), \qquad m>0, \qquad \text{(I.3)}$$

which are analytic polynomials in the complex-valued field φ of degree $n \geqq 2$. We use the mass term $\frac{1}{2}m\varphi^2$ in the covariance of the Gaussian measure, and we consider $P=V-\frac{1}{2}m\varphi^2$ as a perturbation. The bosonic energy density of self interaction is $|\partial V(\varphi)|^2$, a polynomial of degree $2n-2$. The boson-fermionic interaction has the form of a "generalized Yukawa" interaction $\bar{\psi}\Lambda_+\psi\partial^2 V+\bar{\psi}\Lambda_-\psi(\partial^2 V)^*$, where $\Lambda_\pm$ are projections onto chiral subspaces of spinors, see below. The φ^4-Yukawa theory resulting from the choice of cubic V and its renormalizability was studied [4] from a constructive point of view. In case $P=0$, this generalized Yukawa interaction reduces to a free field mass term $m\bar{\psi}\psi$, and the total Hamiltonian reduces to a free, supersymmetric, mass m model.

II. The Model and the Main Results

We review the notation established in [1]. The Hilbert space $\mathscr{H}$ of our model is a tensor product of the bosonic Hilbert space $\mathscr{H}_b$ and the fermionic Hilbert space $\mathscr{H}_f$, namely $\mathscr{H}=\mathscr{H}_b\otimes\mathscr{H}_f$. In both cases we assume that the one particle space is built over the circle (one-torus) T^1 of length ℓ.

II.1. The Bosonic Fock Space

The one particle space of the complex scalar field is

$$W=L_2(T^1)\oplus L_2(T^1)\equiv W_+\oplus W_- .$$

The Fock space $\mathscr{H}_b$ is a symmetric tensor algebra over W with the natural inner product yielding on the n-fold tensor product $\|f\otimes \dots \otimes f\|=\|f\|^n$, $f\in W$. In the Fourier space (momentum representation) we define annihilation operators $a_\pm(p)$ on $W_\pm$ so that $a_\pm\Omega_0^b=0$, $\Omega_0^b=(1,0,\dots,0,\dots)$, and

$$\begin{aligned}&[a_\pm(p),a_\pm(q)]=[a_\pm(p),a_\mp(q)]=[a_\pm(p),a_\mp^*(q)]=0,\\&[a_\pm(p),a_\pm^*(q)]=\delta_{pq},\end{aligned} \tag{II.1}$$

where $p\in\hat{T}^1\equiv\frac{2\pi}{\ell}\mathbb{Z}$ and δ_{pq} is the Kronecker delta. The time zero field is defined by

$$\varphi(x)=(2\ell)^{-1/2}\sum_{p\in\hat{T}^1}\omega(p)^{-1/2}(a_+^*(p)+a_-(-p))e^{-ipx}, \tag{II.2}$$

where $\omega(p)=(p^2+m^2)^{1/2}$, and $m>0$. The canonical momentum is

$$\pi(x)=i(2\ell)^{-1/2}\sum_{p\in\hat{T}^1}\omega(p)^{1/2}(a_-^*(p)-a_+(-p))e^{-ipx}. \tag{II.3}$$

The scalar field satisfies the commutation relations

$$\begin{aligned}&[\varphi(x),\varphi(y)]=[\pi(x),\pi(y)]=[\pi^*(x),\varphi(y)]=0,\\&[\pi(x),\varphi(y)]=-i\delta(x-y),\end{aligned} \tag{II.4}$$

where $\delta(x-y)$ is the Dirac measure.

II.2. The Fermionic Fock Space

The fermionic Fock space $\mathscr{H}_f$ is the anti-symmetric tensor algebra over $L^2(T^1)\oplus L^2(T^1)$. The annihilation operators are $b_\pm(p)$, $p\in\hat{T}^1$ and they satisfy

$$\begin{aligned}&\{b_\pm(p), b_\pm(q)\}=\{b_\pm(p), b_\mp(q)\}=\{b_\pm(p), b^*_\mp(q)\}=0\,,\\&\{b_\pm(p), b^*_\pm(q)\}=\delta_{pq}\,,\end{aligned}\tag{II.5}$$

where $\{\cdot,\cdot\}$ is the anti-commutator. The time zero Fermi fields are defined by

$$\begin{aligned}\psi_1(x)&=(2\ell)^{-1/2}\sum_{p\in\hat{T}^1}\omega(p)^{-1/2}(v(-p)b^*_-(p)+v(p)b_+(-p))e^{-ipx}\,,\\ \psi_2(x)&=(2\ell)^{-1/2}\sum_{p\in\hat{T}^1}\omega(p)^{-1/2}(v(p)b^*_-(p)-v(-p)b_+(-p))e^{-ipx}\,,\end{aligned}\tag{II.6}$$

where $v(p)=(\omega(p)+p)^{1/2}$. Let $\bar{\psi}_1(x)\equiv\psi^*_2(x)$, $\bar{\psi}_2(x)\equiv\psi^*_1(x)$, corresponding to $\bar{\psi}=\psi^*\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Then

$$\begin{aligned}&\{\psi_\mu(x), \psi_\nu(y)\}=0\,,\quad \mu,\nu=1,2\,,\\&\{\bar{\psi}_\mu(x), \psi_\mu(y)\}=0\,,\quad \mu=1,2\,,\\&\{\bar{\psi}_1(x), \psi_2(y)\}=\{\bar{\psi}_2(x), \psi_1(y)\}=\delta(x-y)\,.\end{aligned}\tag{II.7}$$

II.3. The Operators N_τ

For $0\leqq\tau\leqq 1$ we define the operators

$$\begin{aligned}N_{\tau,b}&=\sum_{j=\pm}\sum_{p\in\hat{T}^1}\omega(p)^\tau a^*_j(p)a_j(p)\,,\\ N_{\tau,f}&=\sum_{j=\pm}\sum_{p\in\hat{T}^1}\omega(p)^\tau b^*_j(p)b_j(p)\,,\end{aligned}\tag{II.8}$$

on dense subspaces of $\mathscr{H}_b$ and $\mathscr{H}_f$, respectively. Let

$$N_\tau=N_{\tau,b}\otimes I_{\mathscr{H}_f}+I_{\mathscr{H}_b}\otimes N_{\tau,f}$$

be defined on $\mathscr{H}$. Clearly the number operator is $N=N_0$ and the free field Hamiltonian is $H_0=N_1$. For $0<\tau<1$ these N_τ operators interpolate between N and H_0. It clearly causes no confusion to suppress the tensor products with I.

II.4. The Cutoff Interaction

Let V be given by (I.2). The supercharge Q is defined as a bilinear form on $\mathscr{H}$.

$$Q=\frac{1}{\sqrt{2}}\int_{T_1}(\psi_1(\pi-\partial_x\varphi^*-i\partial V(\varphi))+\psi_2(\pi^*-\partial_x\varphi-i\partial V(\varphi)^*))dx+\text{h.c.}\,,\tag{II.9}$$

where h.c. denotes hermitian conjugate. The domain $\mathscr{D}_0$ of Q we choose consists of Fock states with finite number of particles and $\mathscr{D}(T^1)$-valued wave functions.

We use the following smooth approximation to the periodic Dirac measure based on a cutoff function χ satisfying

(i) $0 \leqq \chi \in \mathscr{S}(\mathbb{R})$,

(ii) $\int_{-\infty}^{\infty} \chi(x)dx = 1$,

(iii) $\chi(-x) = \chi(x)$,

(iv) $\hat{\chi}(p) \geqq 0$,

(v) $\operatorname{supp}\hat{\chi}(p) \subset [-1, 1]$, $\hat{\chi}(p) > 0$ for $|p| \leqq 1/2$.

We set

$$\chi_\kappa(x) = \kappa \sum_{n\in\mathbb{Z}} \chi(\kappa(x - n\ell)), \tag{II.10}$$

where $\kappa > 0$. We define regularized (cutoff) fields by convoluting with χ_κ on T^1,

$$\varphi_\kappa(x) = \chi_\kappa * \varphi(x), \qquad \psi_{\mu,\kappa}(x) = \chi_\kappa * \psi_\mu(x).$$

The regularized supercharge $Q(\kappa)$ is defined as a bilinear form on $\mathscr{H}$,

$$Q(\kappa) = Q_0 + Q_{i,\kappa}, \tag{II.11}$$

where

$$Q_0 = \frac{1}{\sqrt{2}} \int_{T^1} (\psi_1(\pi - \partial_x\varphi^* - im\varphi) + \psi_2(\pi^* - \partial_x\varphi - im\varphi^*))dx + \text{h.c.}, \tag{II.12}$$

and

$$Q_{i,\kappa} = -\frac{i}{\sqrt{2}} \int_{T^1} (\psi_1 \partial P(\varphi_\kappa) + \psi_2 \partial P(\varphi_\kappa)^*)dx + \text{h.c.}, \tag{II.13}$$

where $P(\varphi)$ is defined by (I.3).

Proposition II.1. *The form $Q(\kappa)$ defines a symmetric operator with domain $\mathscr{D}_0$, such that (as a form) its square equals*

$$\begin{aligned} H(\kappa) \equiv Q(\kappa)^2 = H_0 &+ \int_{T^1} (m\varphi^*\partial P(\varphi_\kappa) - (\bar{\psi}_1\psi_1)_\kappa \partial^2 P(\varphi_\kappa) + \text{h.c.})dx \\ &+ \int_{T^1} |\partial P(\varphi_\kappa)|^2 dx. \end{aligned} \tag{II.14}$$

Here $(\bar{\psi}_\mu\psi_\mu)_\kappa \equiv \frac{1}{2}(\bar{\psi}_{\mu,\kappa}\psi_\mu + \bar{\psi}_\mu\psi_{\mu,\kappa})$. Thus $H(\kappa)$ extends uniquely to a symmetric operator with domain $\mathscr{D}_0$.

Proof. See [1].

II.5. The Zero Momentum Limit

Set

$$\varphi_0 = \ell^{-1/2}\hat{\varphi}(0), \qquad \psi_{\mu,0} = \ell^{-1/2}\hat{\psi}_\mu(0), \tag{II.15}$$

where $\hat{\varphi}(p) = \ell^{-1/2} \int_{T^1} dx\varphi(x)e^{ipx}$. Define

$$Q(0) = Q_0 + Q_{i,0}, \tag{II.16}$$

where

$$Q_{i,0} = -\frac{i}{\sqrt{2}} \ell(\psi_{1,0}\partial P(\varphi_0) + \psi_{2,0}\partial P(\varphi_0)^*) + \text{h.c.} \tag{II.17}$$

We also set $H(0) = Q(0)^2$. Here $H(0)$ is the Hamiltonian of a theory where the only interacting mode is the zero mode.

II.6. The Main Results

We first state the results pertaining to the regularity of $Q(\kappa)$ and $H(\kappa)$ and their dependence on regularization.

Theorem II.2. (i) *The operators $Q(\kappa)$ and $H(\kappa)$ are essentially self-adjoint on the domain $\mathscr{D}_0$ for all $0 \leqq \kappa < \infty$.*

(ii) *The resolvents of their closures converge in the operator norm as $\kappa \to \infty$ to the resolvents of self-adjoint operators Q and $H = Q^2$, respectively.*

(iii) *Define $Q(\infty) \equiv Q$, $H(\infty) \equiv H$. The mappings $\kappa \to \text{Resolvent}(Q(\kappa))$ and $\kappa \to \text{Resolvent}(H(\kappa))$ are continuous in the operator norm for $0 \leqq \kappa \leqq \infty$.*

Remark. It is transparent from our proof that the limiting operators Q and H are independent of the choice of the regularizing function χ. Thus the Hamiltonian and supercharge are uniquely determined by the parameters of the superpotential V.

Secondly, we state our integral representations for the index of $Q(\kappa)$. Let $\Gamma = (-I)^{N_{0,f}}$ and let $P_\pm = \frac{1}{2}(I \pm \Gamma)$. Define $Q_+(\kappa) = P_+ Q(\kappa) P_-$, and let $i(Q_+(\kappa))$ denote the index of $Q_+(\kappa)$. In Theorem IV.2 of [1] we established the integral representation

$$i(Q_+(\kappa)) = \int_{\mathscr{D}'(T^2)} \det(1 - K^{(\kappa)}(\Phi)) \exp(-A^{(\kappa)}(\Phi)) d\mu_C(\Phi), \tag{II.18}$$

where $C = C_{\ell,\beta}$ is the Green's function of $-\Delta + m^2$ on the torus, and where $d\mu_C$ is a Gaussian measure on periodic distributions. Also det denotes a Fredholm determinant, and $K^{(\kappa)}$ and $A^{(\kappa)}$ are given in (IV.7–8) of [1]. In [2] we establish a related representation for $\kappa = \infty$. We use the regularized determinant defined by (I.1).

Theorem II.3. *With*

$$\mathscr{A}(\Phi) = \lim_{\kappa \to \infty} [A^{(\kappa)}(\Phi) + \operatorname{Tr} K^{(\kappa)}(\Phi) + \tfrac{1}{2} \operatorname{Tr} K^{(\kappa)}(\Phi)^2],$$

the index has the representation

$$i(Q_+) = \int_{\mathscr{D}'(T^2)} \det{}_3(I - K) \exp(-\mathscr{A}) d\mu_C. \tag{II.19}$$

III. Fundamental a priori Elliptic Estimates

In this section we state the crucial part of our construction, the fundamental a priori estimates. These estimates generalize certain classical elliptic estimates for differential operators on $L_2(\mathbb{R}^M)$ to operators on L_2 of an infinite dimensional (loop) space. The estimates will be proved in [2]. For example, a fundamental a priori estimate in partial differential equations is Gårding's inequality which bounds an elliptic operator from below by a power of the Laplace operator. Our first estimate generalizes Gårding's inequality to an infinite dimensional setting:

Theorem III.1. *Choose $\tau \in [0, 1)$. Then there exist constants $\zeta > 0$ and $C < \infty$ which are independent of κ and for which*

$$\zeta N_\tau \leqq H(\kappa) + C. \tag{III.1}$$

The fact that the bound (III.1) is uniform in κ is characteristic of the a priori bounds established here. We use such estimates to establish the existence of the $\kappa \to \infty$ limit. Such a philosophy is standard in the constructive field theory [5]. The $H(\kappa)$ with $\kappa < \infty$ are operators with a finite number of degrees of freedom (plus an infinite number of uncoupled degrees of freedom). It is important that the constants in our estimates are independent of the number of degrees of freedom $= O(\kappa)$. Thus we develop the theory of infinite dimensional elliptic estimates in terms of finite dimensional, uniform approximations.

We next state the continuity and convergence of the finite dimensional approximations for the semigroups

$$\beta \to \exp(-\beta H(\kappa)), \quad \beta \geqq 0. \tag{III.2}$$

Theorem III.2. *For $\beta > 0$ fixed, the map*

$$\kappa \to \exp(-\beta H(\kappa)) \tag{III.3}$$

is norm-continuous for $0 \leqq \kappa$. Furthermore, the family

$$\{\exp(-\beta H(\kappa))\}$$

is norm-convergent as $\kappa \to \infty$.

We denote the limiting semigroup by $T(\beta)$, $\beta \geqq 0$, namely $\exp(-\beta H(\kappa)) \to T(\beta)$. In order to express $T(\beta)$ in terms of an infinitesimal generator H, we require continuity of $T(\beta)$. The consequence of strong continuity at $\beta = 0$ is the representation $T(\beta) = e^{-\beta H}$, with H a self-adjoint operator on $\mathscr{H}$. The delicate domain question of whether H has a dense domain is more subtle in the infinite dimensional setting than in finite dimensions. For example, no vector in the smooth domain $\mathscr{D}_0$ of C^∞ wave functions with a finite number of particles is in the domain of H. This is the case, even though no renormalizations of H are necessary!

Theorem III.3. *The semigroup $T(\beta)$ is strongly continuous at $\beta = 0$,*

$$\operatorname*{st\,lim}_{\beta \to 0} T(\beta) = I. \tag{III.4}$$

Corollary to Theorems III.2, 3. *The limiting Hamiltonian H satisfies the Gårding estimate* (III.1)

$$\zeta N_\tau \leqq H + C. \tag{III.5}$$

In our examples, the supercharge $Q(\kappa)$ is related to $H(\kappa)$ by $H(\kappa) = Q(\kappa)^2$. We wish to construct a limiting Q as well as a limiting H, and we desire $H = Q^2$. The supercharge is a Dirac operator on loop space, while H is a Laplace operator. We require continuity of $Q(\kappa)$ in κ, as well as convergence of $Q(\kappa)$ in the following manner as $\kappa\to\infty$. Let $\delta Q=(Q(\kappa)-Q(\kappa'))^{-}$, where $^{-}$ denotes the operator closure.

Theorem III.4. *Let* $\beta>0$. *Then* $\mathrm{Range}(e^{-\beta H(\kappa)})\subset\mathrm{Domain}(\delta Q)$ *and*

$$\|e^{-\beta H(\kappa')}\delta Q e^{-\beta H(\kappa)}\|=o(1) \tag{III.6}$$

as $|\kappa-\kappa'|\to 0$, *and as* $\kappa,\kappa'\to\infty$.

IV. The Laplacian H on Loop Space (The Hamiltonian)

In this section we assume that $H(\kappa)$ is essentially self-adjoint on $\mathscr{D}_0$ (as proved in Sect. VI), and we assume the fundamental a priori bounds of Sect. III. We then establish the existence of a self-adjoint $H=\lim_{\kappa\to\infty} H(\kappa)$. The limit exists in the sense of norm convergence of the resolvents. Basically, the existence and self-adjointness of H is a consequence of the a priori bounds.

Theorem IV.1. *The resolvent* $\kappa\to R_\kappa=(H(\kappa)+I)^{-1}$ *is continuous in norm, and the family* R_κ *converges in norm as* $\kappa\to\infty$. *The limiting operator* $R=\lim_\kappa R_\kappa$ *is the resolvent* $(H+I)^{-1}$ *of a self-adjoint operator* H.

We use a standard result in functional analysis: if for one $\beta>0$, $\exp(-\beta H_n)$ is a norm convergent sequence of self-adjoint operators, then the resolvents $(H_n+I)^{-1}$ also converge in norm. Thus resolvent convergence is a consequence of the estimate on the continuity and convergence of the heat kernels, Theorem III.2. The existence of H requires the construction of a dense domain for the infinitesimal generator of $\lim_n \exp(-\beta H_n)$. This follows from the strong continuity of $T(\beta) = \lim_n \exp(-\beta H_n)$, as stated in Theorem III.3.

V. The Dirac Operator Q on Loop Space (The Supercharge)

In this section we establish the properties of $Q=\lim_\kappa Q(\kappa)$. We use the notation

$$S_\kappa=(Q(\kappa)+i)^{-1}, \qquad R_\kappa=(H(\kappa)+I)^{-1}. \tag{V.1}$$

Theorem V.1. *The resolvents* S_κ, S_κ^* *of the supercharge are norm-continuous in* κ *and norm-convergent as* $\kappa\to\infty$. *The limiting operator* $S=\lim_{\kappa\to\infty} S_\kappa$ *is the resolvent of a self-adjoint operator* Q, *and* $H=Q^2$.

Lemma V.2. *The operator* $R_\kappa^{-1/2}S_\kappa$ *is unitary.*

Proof. Note that $S_\kappa S_\kappa^*=R_\kappa=S_\kappa^*S_\kappa$. On the domain $\mathscr{D}_0$,

$$(R_\kappa^{-1/2}S_\kappa)^*(R_\kappa^{-1/2}S_\kappa)=R_\kappa^{-1}S_\kappa^*S_\kappa=I\,,$$

and similarly for the product in the opposite order. These identities extend to $\mathscr{H}$ by continuity. [The operator $R_\kappa^{-1/2}S_\kappa$ is actually a square root of the Cayley transform of $Q(\kappa)$.]

Proof of Theorem V.1. Choose $\varepsilon>0$. We claim that for κ, κ' sufficiently large,

$$\|S_\kappa - S_{\kappa'}\| \leqq 7\varepsilon\,. \tag{V.2}$$

Let $E_\kappa = E_\kappa(\lambda)$ denote the spectral projection onto the subspace $H(\kappa) \leqq \lambda$. We choose $\lambda = \varepsilon^{-2}$. Since $Q(\kappa)$ commutes with $H(\kappa)$, these operators can be simultaneously diagonalized and E_κ commutes with S_κ. We study $\delta S = S_\kappa - S_{\kappa'}$. Then

$$\delta S = E_{\kappa'}\delta S E_\kappa + (I - E_{\kappa'})\delta S E_\kappa + \delta S(I - E_\kappa)\,. \tag{V.3}$$

We claim that for κ, κ' large,

$$\|\delta S(I - E_\kappa)\| \leqq 3\varepsilon\,, \qquad \|(I - E_{\kappa'})\delta S E_\kappa\| \leqq 3\varepsilon\,. \tag{V.4}$$

In fact, using the lemma

$$\|S_\kappa(I - E_\kappa)\| = \|R_\kappa^{-1/2} S_\kappa R_\kappa^{1/2}(I - E_\kappa)\| \leqq \|R_\kappa^{1/2}(I - E_\kappa)\| \leqq (\lambda+1)^{-1/2} \leqq \varepsilon\,. \tag{V.5}$$

Furthermore, for κ, κ', sufficiently large, we infer from Theorem IV.1 that

$$\|R_\kappa^{1/2} - R_{\kappa'}^{1/2}\| \leqq \varepsilon\,. \tag{V.6}$$

Here we use the fact that norm convergence of resolvents implies norm convergence of the square root. Thus

$$\begin{aligned}\|R_{\kappa'}^{1/2}(I - E_\kappa)\| &\leqq \|R_\kappa^{1/2}(I - E_\kappa)\| + \|(R_{\kappa'}^{1/2} - R_\kappa^{1/2})(I - E_\kappa)\| \\ &\leqq \varepsilon + \varepsilon = 2\varepsilon\,,\end{aligned}$$

and by Lemma V.2,

$$\begin{aligned}\|S_{\kappa'}(I - E_\kappa)\| &\leqq \|R_{\kappa'}^{-1/2} S_{\kappa'} R_{\kappa'}^{1/2}(I - E_\kappa)\| \\ &\leqq \|R_{\kappa'}^{1/2}(I - E_\kappa)\| \leqq 2\varepsilon\,.\end{aligned} \tag{V.7}$$

It follows from (V.5), (V.7) that

$$\|\delta S(I - E_\kappa)\| \leqq 3\varepsilon\,,$$

which is the estimate on the last term in (V.3). The estimate on $(I - E_{\kappa'})\delta S E_\kappa$ is similar. Hence

$$\|\delta S\| \leqq 6\varepsilon + \|E_{\kappa'}\delta S E_\kappa\|\,. \tag{V.8}$$

We now use the facts that $\|S_\kappa\| \leqq 1$, and that the resolvent identity

$$\delta S = S_\kappa - S_{\kappa'} = S_{\kappa'}(Q(\kappa') - Q(\kappa))S_\kappa$$

holds as a bilinear form. Thus we also have the form identity

$$E_{\kappa'}\delta S E_\kappa = S_{\kappa'} E_{\kappa'} \delta Q E_\kappa S_\kappa\,, \tag{V.9}$$

where $\delta Q = (Q(\kappa') - Q(\kappa))^-$.

By Theorem III.4, with $\beta>0$, and with κ, κ' sufficiently large,

$$\begin{aligned}\|E_{\kappa'}\delta Q E_\kappa\| &= \|E_{\kappa'} e^{\beta H(\kappa')} e^{-\beta H(\kappa')} \delta Q e^{-\beta H(\kappa)} e^{\beta H(\kappa)} E_\kappa\| \\ &\leqq e^{2\lambda\beta} \|e^{-\beta H(\kappa')} \delta Q e^{-\beta H(\kappa)}\| \leqq e^{2\lambda\beta} o(1) \leqq \varepsilon\,.\end{aligned} \tag{V.10}$$

From (V.8–10), we infer $\|\delta S\| \leqq 7\varepsilon$ as claimed.

This completes the proof of convergence of S_κ as $\kappa \to \infty$. The same type of argument shows that $\{S_\kappa\}$ is continuous in κ for $\kappa < \infty$. Since $\|\delta S\| = \|\delta S^*\|$, the continuity and convergence of S_κ^* follows. This completes the proof of the norm continuity S_κ.

We now proceed to show that $S = \lim_{\kappa \to \infty} S_\kappa$ is the resolvent of a self-adjoint operator Q. The main technical issue is to show that S is invertible, namely that $\operatorname{kernel}(S) = 0$. A similar issue arose in the proof of self-adjointness of H, and it was solved by showing that the semigroup $T(\beta) = \lim_{\kappa \to \infty} \exp(-\beta H(\kappa))$ was strongly continuous at $\beta = 0$, cf. Theorem IV.3. In this case we have no heat kernel representation for Q, but we use the existence of a dense domain for H. In fact,

$$S^*S = \lim_{\kappa \to \infty} S_\kappa^* S_\kappa = \lim_{\kappa \to \infty} (H(\kappa) + I)^{-1} = (H + I)^{-1} . \tag{V.11}$$

Since $0 \leqq H = H^*$, the null space of $(H+I)^{-1}$ is zero. Thus the null space of S is trivial and S is invertible. It then follows by Theorem 4 of [6] that $S = (Q+i)^{-1}$ is the resolvent of a self-adjoint operator Q. Furthermore,

$$S^*S = (Q^2 + I)^{-1} = (H + I)^{-1} ,$$

so $H = Q^2$ and the proof of Theorem V.1 is complete.

VI. The Cutoff Theory

In this section we define well behaved approximations to the Hamiltonian H and to the family of modified Hamiltonians H_τ, used to establish the N_τ bounds of Theorem III.1. The integral representations for the heat kernels of these approximating Hamiltonians $H_\tau(\kappa)$ yield elliptic regularity estimates for $H_\tau(\kappa)$, as well as continuity properties in κ.

The approximating operators $H_\tau(\kappa)$ are unitarily equivalent to operators of the form $h_0 + h_1 + h_2$, where h_2 is a partial differential operator on $L_2(\mathbb{R}^M)$, where $M = O(\kappa)$ is large but finite. The operator h_0 can be diagonalized in closed form [on a Fock space $\mathscr{H} \backslash L_2(\mathbb{R}^M)$ of a system with an infinite number of degrees of freedom]. The operator h_1 is an infinitesimal perturbation of $h_0 + h_2$ (in the sense of Rellich and Kato). Thus $H_\tau(\kappa)$ is an approximation to H_τ whose properties are determined by the action of $H_\tau(\kappa)$ on functions with $O(\kappa)$ degrees of freedom. Our analysis of the $\kappa \to \infty$ limit depends on uniformity of the constants in the elliptic estimates as a function of κ.

In this section we establish the integral representations which we use to establish estimates. In [2] we prove the desired uniform estimates. Throughout these sections we fix $m > 0$ and $\ell < \infty$ (the length of the circle); we do not discuss uniformity of our estimates in these parameters.

VI.1. The Operators $H_\tau(\kappa)$ and $Q(\kappa)$

We begin with the definition of $H_\tau(\kappa)$. Choose $0 \leqq \tau < 1$ and $0 \leqq \zeta < m^{1-\tau}$. Consider the following operators with domain $\mathscr{D}_0$.

$$H_{0,\tau,b} = H_{0,b} - \zeta N_{\tau,b} , \qquad H_{0,\tau,f} = H_{0,f} - \zeta N_{\tau,f} , \tag{VI.1}$$

$$W_1 = \int_{T^1} m[\varphi^* \partial P(\varphi_\kappa) + \varphi \partial P(\varphi_\kappa)^*] dx, \tag{VI.2}$$

$$W_2 = \int_{T^1} |\partial P(\varphi_\kappa)|^2 dx, \tag{VI.3}$$

$$H_{\tau,b} = H_{0,\tau,b} + W_1 + W_2, \tag{VI.4}$$

and

$$H_{b,f} = -\int_{T^1} ((\bar{\psi}_1 \psi_1)_\kappa \partial^2 P(\varphi_\kappa) + (\bar{\psi}_2 \psi_2)_\kappa \partial^2 P(\varphi_\kappa)^*) dx. \tag{VI.5}$$

In terms of these operators

$$H_\tau(\kappa) = H_{\tau,b} + H_{0,\tau,f} + H_{b,f}. \tag{VI.6}$$

Note that for $\omega(p) = (p^2 + m^2)^{1/2}$, there exists $\varepsilon = \varepsilon(\zeta, \tau) > 0$ such that $\omega - \zeta\omega^\tau \geqq \varepsilon\omega$, so $\varepsilon H_0 \leqq H_{0,\tau,b} + H_{0,\tau,f}$. Also note that for $\zeta = 0$, $H_\tau(\kappa) = H(\kappa)$ of (II.14). The N_τ estimate of Theorem III.1 is equivalent to

$$0 \leqq H_\tau(\kappa) + C, \tag{VI.7}$$

where $C = C(\zeta, \tau)$ is a constant independent of κ.

Proposition VI.1. *The operators $H_\tau(\kappa)$ and $Q(\kappa)$ are essentially self-adjoint. Furthermore, for ζ sufficiently small,* (VI.7) *holds, but with a constant C which is not necessarily uniform in κ as $\kappa \to \infty$.*

Let $I_p(\mathscr{H})$, $p \geqq 1$, denote the Banach space of trace class operators on the Hilbert space $\mathscr{H}$ with the norm $\|T\|_p = \{\mathrm{Tr}(T^*T)^{p/2}\}^{1/p}$.

Corollary VI.2. *Let $\beta > 0$ and let $\mathscr{H}$ denote Fock space. Then* $\exp(-\beta H_\tau(\kappa)) \in I_p(\mathscr{H})$ *for all $p \geqq 1$.*

Proof. This follows from the κ-dependent bound (VI.7) and Proposition II.1 of [1].

Lemma VI.3. *The operator $H_{\tau,b}(\kappa)$ is essentially self-adjoint.*

Proof. Decompose the Fock space $\mathscr{H}_b$ as a tensor product

$$\mathscr{H}_b = \mathscr{H}_{\leqq} \otimes \mathscr{H}_{>}, \tag{VI.8}$$

where $\mathscr{H}_{\leqq}$ is spanned by states of the form $R\Omega_0$, where R is any polynomial in the creation operators $a_j^*(p)$ for $|p| \leqq (n-1)\kappa$. Then the operators $H_{\tau,b}$ can be represented as $H_{\tau,b}^{\leqq} \otimes I + I \otimes H_0^{>}$, where $H_0^{>}$ contains no interacting modes. The operator $H_{\tau,b}^{\leqq}$ is equivalent to a Schrödinger operator $-\Delta + v$ on $L_2(\mathbb{R}^M)$, with a polynomial potential v. See [7] for a proof of this representation and the proof of essential self-adjointness of $-\Delta + v$. This completes the proof.

Lemma VI.4. *Given $\varepsilon > 0$, there exists a finite constant $C = C(\varepsilon, \kappa) < \infty$ such that on $\mathscr{D}_0 \times \mathscr{D}_0$*

$$H_{b,f}(\kappa)^2 \leqq \varepsilon W_2 + C. \tag{VI.9}$$

Proof. The perturbation $H_{b,f}$ of (VI.5) can be written as a sum of four terms of the form

$$\sum_{|p| \leqq \kappa} \sum_{|q| \leqq (n-2)\kappa} \bar{\psi}_i^\wedge(-p-q) \hat{\psi}_i(p) \hat{\chi}_\kappa(p) (\partial^2 P(\varphi_\kappa))^\wedge(q), \tag{VI.10}$$

$i=1,2$. Since each $\hat{\psi}_i(p)$ is a bounded operator, for C_κ sufficiently large,

$$\begin{aligned} H_{b,f}^2 &\leqq C_\kappa \sum_{|q|\leqq(n-2)\kappa} |\partial^2 P(\varphi_\kappa)^\wedge(q)|^2 \\ &= C_\kappa \int_{T^1} |\partial^2 P(\varphi_\kappa)|^2 dx . \end{aligned} \tag{VI.11}$$

It follows that for $\varepsilon>0$ there exists $C<\infty$ such that

$$H_{b,f}^2 \leqq \varepsilon \int_{T^1} |\partial P(\varphi_\kappa)|^2 dx + C ,$$

as claimed.

Lemma VI.5. *Assume $n=\deg P\geqq 3$, so $\deg W_2 \geqq 4$. Then there exists $\eta=\eta(\kappa)<1$, $C=C(\eta,\kappa)<\infty$, and $\zeta_0>0$ such that for $\zeta<\zeta_0$,*

$$|W_1| \leqq \eta(H_{0,\tau,b}+W_2)+C . \tag{VI.12}$$

Furthermore, $\eta(\kappa)$ is bounded uniformly away from 1.

Proof. We write $\varphi(x)=\varphi_1(x)+\varphi_2(x)+\varphi^{(>)}(x)$, where $\varphi_1(x)$ denotes the contribution to $\varphi(x)$ from Fourier modes with $|p|\leqq\kappa/2$, and where $\varphi_2(x)$ is the contribution to $\varphi(x)$ from Fourier modes with $\kappa/2<|p|\leqq(n-1)\kappa$. Then for arbitrary $\varepsilon>0$,

$$\begin{aligned} \left| m \int_{T^1} \varphi_1 \partial P(\varphi_\kappa)^* dx \right| &\leqq m \left(\int_{T^1} |\varphi_1|^2 dx \right)^{1/2} \left(\int_{T^1} |\partial P(\varphi_\kappa)|^2 dx \right)^{1/2} \\ &\leqq \varepsilon \int_{T^1} |\partial P(\varphi_\kappa)|^2 dx + O(\varepsilon^{-1}) \int_{T^1} |\varphi_1(x)|^2 dx . \end{aligned} \tag{VI.13}$$

Since $\hat{\varphi}_\kappa(p)=\hat{\varphi}(p)\hat{\chi}(p/\kappa)$, and $\hat{\chi}(p/\kappa)>\mu>0$ for $|p|\leqq\kappa/2$, and $\hat{\chi}(p/\kappa)\equiv 0$ for $|p|\geqq\kappa$, it follows that

$$\int_{T^1} |\varphi_1(x)|^2 dx \leqq \mu^{-2} \int_{T^1} |\varphi_\kappa(x)|^2 dx \leqq \varepsilon \int_{T^1} |\partial P(\varphi_\kappa)|^2 dx + O(\varepsilon^{-1}\mu^{-2}) .$$

Thus, after a new choice of ε,

$$\left| m \int_{T^1} \varphi_1 \partial P(\varphi_\kappa)^* dx \right| \leqq \tfrac{1}{2}\varepsilon \int |\partial P(\varphi_\kappa)|^2 dx + \text{large constant} . \tag{VI.14}$$

Similarly, and using the fact that $\partial P(\varphi_\kappa)^\wedge(p)$ vanishes for $|p|>(n-1)\kappa$, we have

$$\begin{aligned} \left| m \int_{T^1} (\varphi_2+\varphi^{(>)}) \partial P(\varphi_\kappa)^* dx \right| &= \left| m \int_{T^1} \varphi_2 \partial P(\varphi_\kappa)^* dx \right| \\ &\leqq \tfrac{1}{2}\eta \int_{T^1} |\partial P(\varphi_\kappa)|^2 dx + \tfrac{1}{2} m^2 \eta^{-1} \int_{T^1} |\varphi_2|^2 dx . \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{2} m^2 \eta^{-1} \int_{T^1} |\varphi_2|^2 dx &= \frac{1}{2} m^2 \eta^{-1} \sum_{\kappa/2\leqq|p|\leqq(n-1)\kappa} |\hat{\varphi}(p)|^2 \\ &= \frac{1}{2}\eta \sum_{\kappa/2\leqq|p|\leqq(n-1)\kappa} |\hat{\varphi}(p)|^2 m^2 \eta^{-2} \left(1+\frac{p^2}{m^2}\right)\left(1+\frac{p^2}{m^2}\right)^{-1} \\ &\leqq \frac{\alpha}{2}\eta \sum_{|p|\leqq(n-1)\kappa} |\hat{\varphi}(p)|^2 (p^2+m^2) , \end{aligned} \tag{VI.15}$$

where

$$\alpha = \sup_{|p|\geqq \kappa/2} \eta^{-2}(1+p^2/m^2)^{-1} = \eta^{-2}(1+\kappa^2/4m^2)^{-1}. \quad \text{(VI.16)}$$

For fixed κ, m we choose $\eta<1$, but sufficiently close to one that $\alpha<1$. Note that our estimate appears to suggest that $\eta\to 1$ as $\kappa\to 0$. However, our operators are constant for $\kappa<\pi/\ell$, so it is sufficient to establish the estimate for $\kappa>\pi/\ell$, showing that η is bounded away from 1. We are not interested here in the fact that our constants diverge as $\kappa\to\infty$, since estimates uniform in κ are only established in [2].

We now collect together the bounds (VI.14–15), as well as the identical bounds for the complex conjugate term in (VI.2). Thus

$$|W_1| \leqq (\varepsilon+\eta)W_2 + \eta \sum_{|p|\leqq (n-1)\kappa} |\hat{\varphi}(p)|^2(p^2+m^2) + C_1,$$

where C_1 is the constant from (VI.14). Furthermore, with $\omega=(p^2+m^2)^{1/2}$, and

$$H_{0,b}(p) = \sum_{j=\pm} \omega a_j(p)^* a_j(p) = :|\hat{\pi}(p)|^2: + \omega^2 :|\hat{\varphi}(p)|^2:,$$

we have

$$\begin{aligned}\omega^2|\hat{\varphi}(p)|^2 &= \omega^2 :|\hat{\varphi}(p)|^2: + \omega \\ &\leqq :|\hat{\pi}(p)|^2 + \omega^2|\hat{\varphi}(p)|^2: + 2\omega \\ &= H_{0,b}(p) + 2\omega .\end{aligned}$$

Here we use $|\hat{\pi}|^2 = :|\hat{\pi}|^2: + \omega$, $|\hat{\varphi}|^2 = :|\hat{\varphi}|^2: + \omega^{-1}$. Summing over the modes $|p|\leqq (n-1)\kappa$ and increasing the bound by $0\leqq H_{0,b}(p)$ for each remaining mode yields, after a new choice of $\eta<1$ and C, the desired inequality (VI.12) for $\zeta=0$, and it completes the proof for $\zeta=0$. A similar proof holds for $\zeta>0$.

Proof of Proposition VI.1. The operator $H_{\tau,b}+H_{0,f}$ is essentially self-adjoint by Lemma VI.3 and the fact that $H_{\tau,b}$ and $H_{0,f}$ operate on distinct factors of the tensor product $\mathscr{H}=\mathscr{H}_b\otimes\mathscr{H}_f$. The bound of Lemma VI.4 shows that

$$(1-\eta)(H_{0,\tau,b}+W_2) + H_{0,f} \leqq H_{\tau,b} + H_{0,f} + C(\kappa). \quad \text{(VI.17)}$$

Using Lemma VI.4 we conclude that $H_{b,f}$ is an infinitesimal perturbation of W_2. Thus by the Rellich-Kato bound on the Neumann series, Theorem V.4.3 of [8], $H_\tau(\kappa)$ is essentially self-adjoint and (VI.7) holds with small ζ and a κ-dependent constant $C(\kappa)$. The essential self-adjointness of $Q(\kappa)$ follows from (for instance) $H(\kappa)=Q(\kappa)^2$ and the commutator theorem (Theorem 19.4.3 of [5]).

VI.2. The Feynman-Kac Formula

We consider the space $\mathscr{S}'(\mathbb{R}\times T^1)$ of distributions periodic in the x_1 direction. Let C_ℓ denote the Green's function for $-\Delta+m^2$ on the cylinder $\mathbb{R}\times T^1$. It has the integral kernel

$$C_\ell(x-y) = (2\pi\ell)^{-1} \sum_{p_1\in \hat{T}^1} \int \frac{1}{p^2+m^2} e^{-ip(x-y)} dp_0 . \quad \text{(VI.18)}$$

Also, let $d\mu_{C_\ell}(\Phi)$ be a Gaussian measure on $\mathcal{S}'(\mathbb{R}\times T^1)$ with covariance C_ℓ. Let $\Phi_\kappa(x)$ be a regularized approximation to $\Phi(x)$:

$$\Phi_\kappa(x)=\chi_\kappa \star \Phi(x)\equiv \int_{T^1} \chi_\kappa(x_1-x_1')\Phi(x_0,x_1')dx_1', \tag{VI.19}$$

where χ_κ is given by (II.10). We set

$$A_\ell^{(\kappa)}(\Phi)=\int_{[0,\beta]\times T^1}(m\Phi\partial P(\Phi_\kappa)^*+m\Phi^*\partial P(\Phi_\kappa)+|\partial P(\Phi_\kappa)|^2)dx. \tag{VI.20}$$

Similarly, let $S_\ell(x-y)$ denote the Green's function for the Euclidean Dirac operator on the cylinder. Its integral kernel is

$$S_\ell(x-y)=(2\pi\ell)^{-1}\sum_{p_1\in\hat{T}^1}\int\frac{-\not{p}+m}{p^2+m^2}e^{-ip(x-y)}dp_0, \tag{VI.21}$$

where $\not{p}=p_0\gamma_0^E+p_1\gamma_1^E$, and where γ_μ^E are the Euclidean Dirac matrices:

$$\gamma_0^E=\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix},\quad \gamma_1^E=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Let $\mathcal{K}_\alpha(\mathbb{R}\times T^1)$ denote the Hilbert space

$$\mathcal{K}_\alpha(\mathbb{R}\times T^1)=\mathcal{H}_\alpha(\mathbb{R}\times T^1)\oplus\mathcal{H}_\alpha(\mathbb{R}\times T^1), \tag{VI.22}$$

where $\mathcal{H}_\alpha(\mathbb{R}\times T^1)$ is the Sobolev space of order α over $\mathbb{R}\times T^1$. The norm on $\mathcal{H}_\alpha$ is

$$\|f\|_\alpha^2=\sum_{p_1\in\hat{T}^1}\int(p^2+m^2)^\alpha|\hat{f}(p)|^2dp_0. \tag{VI.23}$$

Elsewhere we require $\mathcal{K}_\alpha(T^2)$, the Sobolev space over the torus T^2. It will be clear from the context which space is relevant. Let $K_\ell^{(\kappa)}(\Phi)$ be the operator on $\mathcal{K}_{1/2}$ whose integral kernel is given by

$$\begin{aligned}K_\ell^{(\kappa)}(\Phi)(x,y)=\tfrac{1}{2}\int_{T^1}&([S_\ell(x-z)\partial^2P(\Phi_\kappa(z))\chi_\kappa(z_1-y_1)\\&+S_\ell(x-z)\chi_\kappa(z_1-y_1)\partial^2P(\Phi_\kappa(y))]\Lambda_+\\&+[S_\ell(x-z)\partial^2P(\Phi_\kappa(z))^*\chi_\kappa(z_1-y_1)\\&+S_\ell(x-z)\chi_\kappa(z_1-y_1)\partial^2P(\Phi_\kappa(y))^*]\Lambda_-)dz_1,\end{aligned} \tag{VI.24}$$

where

$$\Lambda_+=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \Lambda_-=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

are the chiral projections, and where $z=(y_0,z_1)$. We remark that $K_\ell^{(\kappa)}(\Phi)\in I_1(\mathcal{K}_{1/2})$ for almost all Φ (with respect to $d\mu_{C_\ell}$). This estimate is a special case of more delicate, related estimates in [2]. However, it can be seen directly by application of the Schwarz inequality in the trace norm. Let $K\equiv K_\ell^{(\kappa)}(\Phi)$ and $\mu=(-d^2/dx_1^2+m^2)^{1/2}$. Then taking I_1 norms on the Hilbert space $\mathcal{K}_{1/2}$,

$$\|K\|_1=\mathrm{Tr}((K^*K)^{1/2})\leqq\|\mu^{-1}\|_2\|\mu(K^*K)^{1/2}\|_2.$$

Since K^*K is self-adjoint on $\mathcal{K}_{1/2}$, and since the adjoints of μ on $\mathcal{K}_0$ and on $\mathcal{K}_1$ agree, we use $\|\mu^{-1}\|_2\leqq$const and

$$\|\mu(K^*K)^{1/2}\|_2\leqq 1+\|\mu(K^*K)^{1/2}\|_2^2=1+\mathrm{Tr}(\mu^2K^*K).$$

Since K has a cutoff,

$$\int \mathrm{Tr}(\mu^2 K^* K) d\mu_C < \infty ,$$

and

$$\int \|K\|_1 d\mu_C < \infty ,$$

as claimed. It follows that the Fredholm determinant $\det(I - K_\ell^{(\kappa)}(\Phi))$ is a random variable.

Consider the following function on $\mathscr{S}'(\mathbb{R} \times T^1)$:

$$F_\ell^{(\kappa)}(\Phi) = \det(I - K_\ell^{(\kappa)}(\Phi)) \exp(-A_\ell^{(\kappa)}(\Phi)) . \tag{VI.25}$$

Proposition VI.6. *There exists $\alpha > 0$ independent of κ such that*

$$F_\ell^{(\kappa)} \in L_p(\mathscr{S}'(\mathbb{R} \times T^1), d\mu_{C_\ell}) , \tag{VI.26}$$

for all κ, $0 \leqq \kappa < \infty$, and for all p satisfying $1 \leqq p \leqq 1 + \alpha$.

Remark. The restriction $p \leqq 1 + \alpha$ in (VI.26) arises because two occurrences of Φ in $A_\ell^{(\kappa)}(\Phi)$ of (VI.20) are not regularized. The proof below shows that the integrability properties of $F_\ell^{(\kappa)}(\Phi)$ improve as $\kappa \to \infty$.

Proof. Since

$$|\det(I - K_\ell^{(\kappa)}(\Phi))|^2 = \det(I + L_\ell^{(\kappa)}(\Phi)) , \tag{VI.27}$$

where

$$L_\ell^{(\kappa)}(\Phi) = -K_\ell^{(\kappa)}(\Phi) - K_\ell^{(\kappa)}(\Phi)^* + K_\ell^{(\kappa)}(\Phi)^* K_\ell^{(\kappa)}(\Phi) . \tag{VI.28}$$

Clearly $L_\ell^{(\kappa)}(\Phi) \geqq -I$, and thus

$$|\det(I - K_\ell^{(\kappa)}(\Phi))| \leqq \exp(\tfrac{1}{2} \mathrm{Tr}\, L_\ell^{(\kappa)}(\Phi)) .$$

We claim that the function

$$Z_\ell^{(\kappa)}(\Phi) = \exp(-A_\ell^{(\kappa)}(\Phi) + \tfrac{1}{2} \mathrm{Tr}\, L_\ell^{(\kappa)}(\Phi)) \tag{VI.29}$$

has the required integrability properties.

We decompose C_ℓ as

$$C_\ell = C_\ell^{(\leqq)} \oplus C_\ell^{(>)} . \tag{VI.30}$$

In the Fourier series for C_ℓ in the spatial variable, this splitting is according to whether $|p_1| \leqq (n-1)\kappa$ or $|p_1| > (n-1)\kappa$. Then we write $d\mu_{C_\ell} = d\mu_{C_\ell^{(\leqq)}} \otimes d\mu_{C_\ell^{(>)}}$. Clearly,

$$\int Z_\ell^{(\kappa)}(\Phi)^p d\mu_{C_\ell}(\Phi) = \int Z_\ell^{(\kappa)}(\Phi^{(\leqq)})^p d\mu_{C_\ell^{(\leqq)}}(\Phi^{(\leqq)}) , \tag{VI.31}$$

where $\Phi^{(\leqq)}$ is the contribution to Φ from the Fourier modes with $|p_1| \leqq (n-1)\kappa$.

To prove that there is $\alpha > 0$ such that the right-hand side of (VI.31) is finite for $1 \leqq p \leqq 1 + \alpha$ we employ the same technique as in the proof of Lemma VI.5. We write $\Phi^{(\leqq)} = \Phi_1 + \Phi_2$, where Φ_1 is the sum of Fourier modes with $|p_1| \leqq \kappa/2$, and correspondingly $C_\ell^{(\leqq)} = C_\ell^{(1)} \oplus C_\ell^{(2)}$. The covariance $C_\ell^{(2)}$ satisfies $(C^{(2)})^{-1} \geqq m^2 + (\kappa/2)^2$. This and the bound

$$\mathrm{Tr}\, L_\ell^{(\kappa)}(\Phi) \leqq C \left(\int_{[0,\beta] \times T^1} |\Phi_\kappa(x)|^{2(n-1)} dx + 1 \right)$$

with $C=C(\kappa)<\infty$ yield the required integrability properties. We leave the details of this argument to the reader.

Let $g_j, h_j \in \mathscr{S}(\mathbb{R}\times T^1)$, $j=1,\ldots,k$. We consider the k-fold exterior product $\bigwedge^k \mathscr{H}_0$ with its natural inner product. Then define

$$F_\ell^{(\kappa)}(\Phi,g,h)=\left(\bigwedge_{j=1}^{k} g_j, \bigwedge_{j=1}^{k}(I-K_\ell^{(\kappa)}(\Phi))^{-1}S_\ell h_j\right)_{\bigwedge^k\mathscr{H}_0} \det(I-K_\ell^{(\kappa)}(\Phi))\exp(-A_\ell^{(\kappa)}(\Phi)). \tag{VI.32}$$

$F_\ell^{(\kappa)}(\Phi,g,h)$ is well defined since for $K\in I_1$ the mapping

$$z\to \bigwedge^k (I-zK)^{-1}\det(I-zK)$$

is an entire, operator-valued function.

Proposition VI.7. *$F_\ell^{(\kappa)}(\Phi,g,h)\in L_p(d\mu_{C_\ell})$, for $0\leqq\kappa<\infty$ and $1\leqq p\leqq 1+\alpha$. The L_p norms are continuous for $g_j, h_j\in\mathscr{H}_{-1/2}$, and $F_\ell^{(\kappa)}$ extends by continuity to this space.*

Proof. Following Seiler [9] we write

$$|F_\ell^{(\kappa)}(\Phi,g,h)|^2=\left|\left(\bigwedge_{j=1}^{k} C_\ell^{1/2}g_j, \bigwedge_{j=1}^{k}(I-K_\ell^{(\kappa)}(\Phi))^{-1}S_\ell h_j\right)_{\bigwedge^k\mathscr{H}_{1/2}}\right|^2 \times\exp(-2A_\ell^{(\kappa)}(\Phi))\det(I+L_\ell^{(\kappa)}(\Phi)). \tag{VI.33}$$

Let $L_\ell^{(\kappa)}(\Phi)_+$ and $L_\ell^{(\kappa)}(\Phi)_-$ be the positive and negative parts of the self-adjoint operator $L_\ell^{(\kappa)}(\Phi)$, $L_\ell^{(\kappa)}(\Phi)=L_\ell^{(\kappa)}(\Phi)_+-L_\ell^{(\kappa)}(\Phi)_-$. It follows that

$$\begin{aligned}|F_\ell^{(\kappa)}(\Phi,g,h)|&\leqq \prod_{j=1}^{k}\|C_\ell^{1/2}g_j\|_{\mathscr{H}_{1/2}}\|S_\ell h_j\|_{\mathscr{H}_{1/2}}\\ &\times\left\|\det(I-L_\ell^{(\kappa)}(\Phi)_-)\bigwedge^k(I-L_\ell^{(\kappa)}(\Phi)_-)^{-1}\right\|^{1/2}\exp\left(\frac{1}{2}\operatorname{Tr}L_\ell^{(\kappa)}(\Phi)_+-A_\ell^{(\kappa)}(\Phi)\right)\\ &\leqq\prod_{j=1}^{k}\|g_j\|_{\mathscr{H}_{-1/2}}\|h_j\|_{\mathscr{H}_{-1/2}}\exp\left(\frac{1}{2}\operatorname{Tr}L_\ell^{(\kappa)}(\Phi)-A_\ell^{(\kappa)}(\Phi)+\frac{1}{2}\right)\\ &=e^{1/2}\prod_{j=1}^{k}\|g_j\|_{\mathscr{H}_{-1/2}}\|h_j\|_{\mathscr{H}_{-1/2}}Z_\ell^{(\kappa)}(\Phi),\end{aligned}$$

with $Z_\ell^{(\kappa)}(\Phi)$ given by (VI.29). This upper bound yields the same function of Φ which occurred in the proof of Proposition VI.6. The extension by continuity follows from the continuity of the estimate in g and h.

Let $u_j\in\mathscr{H}_{-1/2}(T^1)$, $j=1,\ldots,q$, and let $w_j\in\mathscr{H}_{-1}(T^1)$, $j=1,\ldots,p$. Write

$$\begin{aligned}\xi_j&=\psi_{\mu_j}(u_j)\quad\text{or}\quad\bar{\psi}_{\nu_j}(u_j), && j=1,\ldots,q,\\ \xi_{q+j}&=\varphi(w_j)\quad\text{or}\quad\varphi^*(w_j), && j=1,\ldots,p.\end{aligned} \tag{VI.34}$$

For $s\geqq 0$ set

$$\xi_j(s)=e^{-sH_0}\xi_j e^{sH_0},\qquad j=1,\ldots,p+q. \tag{VI.35}$$

Let $0\leqq s_j\leqq\beta$, $j=1,\ldots,p+q$. We define the time ordered product

$$T\left(\prod_{j=1}^{p+q}\xi_j(s_j)\right)=\operatorname{sgn}(\pi)\prod_{j=1}^{q}\xi_{\pi(j)}(s_{\pi(j)})\prod_{j=q+1}^{p+q}\xi_{\varrho(j)}(s_{\varrho(j)}), \tag{VI.36}$$

where π is a permutation of $\{1,\ldots,q\}$ such that (i) $s_{\pi(1)}\leqq\ldots\leqq s_{\pi(q)}$, and (ii) if $s_i=s_j$ and $\xi_i=\psi_{\mu_i}(u_i)$, $\xi_j=\bar{\psi}_{v_j}(u_j)$, then we place ξ_i left to ξ_j. Also, ϱ is a permutation of $\{q+1,\ldots,p+q\}$ which puts the numbers s_j into a nondecreasing order. Let $\alpha_j(t)$, $j=1,\ldots,p+q$ be smooth functions supported in $(0,\beta)$. We define the vector

$$\Omega=\int T\left(\prod_{j=1}^{p+q}\xi_j(s_j)\right)\prod_{j=1}^{p+q}\alpha_j(s_j)\Omega_0 d^{p+q}s. \tag{VI.37}$$

The Feynman-Kac formula gives a path integral representation for the matrix elements $\langle\Omega,\exp(-\beta H(\kappa))\Omega'\rangle$ with Ω and Ω' of the form (VI.37). Let q_1 and q_1' be the number of the ψ fields in Ω and Ω', respectively. Let q_2 and q_2' be the number of the $\bar{\psi}$ fields in Ω and Ω', respectively. It is clear that $\langle\Omega,\exp(-\beta H(\kappa))\Omega'\rangle=0$, unless $q_1+q_2'=q_1'+q_2\equiv k$. Let $i_1<\ldots<i_{q_1}$ (and $i_1'<\ldots<i_{q_1'}'$) be the indices corresponding to the field ψ in Ω (and in Ω', respectively). Let $j_1<\ldots<j_{q_2}$ (and $j_1'<\ldots<j_{q_2'}'$) be the indices corresponding to $\bar{\psi}$ in Ω (and in Ω', respectively). Let $(\theta_\beta\alpha)(t)=\alpha(\beta-t)^*$. We set

$$g_1=w_{j_{q_2}}^*\theta_\beta\alpha_{j_{q_2}},\ldots,g_{q_2}=w_{j_1}^*\theta_\beta\alpha_{j_1},g_{q_2+1}=w_{i_1'}'\alpha_{i_1'}',\ldots,g_k=w_{i_{q_1'}'}'\alpha_{i_{q_1'}'},$$

$$h_1=w_{i_{q_1}}^*\theta_\beta\alpha_{i_{q_1}},\ldots,h_{q_1}=w_{i_1}^*\theta_\beta\alpha_{i_1},\ldots,h_{q_1+1}=w_{j_1'}'\alpha_{j_1'},\ldots,h_k=w_{j_{q_2'}'}'\alpha_{j_{q_2'}'}'$$

and relabel correspondingly the spinor indices. Similarly, we set $f_1=u_p^*\theta_\beta\alpha_p,\ldots,f_p=u_1^*\theta_\beta\alpha_1, f_{p+1}=u_1'\alpha_1',\ldots,f_{2p}=u_p'\alpha_p'$.

Proposition VI.8 (Feynman-Kac formula). *With the above definitions*

$$\langle\Omega,\exp(-\beta H(\kappa))\Omega'\rangle=\varepsilon\int F_\ell^{(\kappa)}(\Phi)\left(\bigwedge_{j=1}^{k}g_j,\bigwedge_{j=1}^{k}(I-K_\ell^{(\kappa)}(\Phi))^{-1}S_\ell h_j\right)_{\bigwedge^k\mathscr{K}_0}$$
$$\times\prod_{j=1}^{2p}\Phi^{\#}(f_j)d\mu_{C_\ell}(\Phi), \tag{VI.38}$$

where $\#$ *means possible complex conjugate and where* $\varepsilon=\pm1$.

The proof follows the lines of [10, 11] and is based on the following well-known bound to establish convergence of a semigroup convergence expansion:

Lemma VI.9. *Let* $H\geqq0$ *be a self-adjoint operator and let* H_1 *be symmetric, and such that* $D(H)\subset D(H_1^2)$ *and*

$$H_1^2\leqq aH+b \tag{VI.39}$$

with $a>0$. *Then* $\exp(-\beta(H+H_1))$ *has a norm convergent perturbation series*

$$\exp(-\beta(H+H_1))=\sum_{n\geqq0}(-1)^n\int_{0\leqq s_1\leqq\ldots\leqq s_n\leqq\beta}\exp(-s_1H)H_1\ldots\exp(-(s_n-s_{n-1})H)$$
$$\times H_1\exp(-(\beta-s_n)H)d^ns. \tag{VI.40}$$

We apply this lemma to $H=H_b(\kappa)+H_{0,f}$ (adding a constant if necessary) and $H_1=H_{b,f}(\kappa)$. The estimates of Sect. VI.1 ensure that (VI.39) is satisfied. Using

$$\langle\Omega_0^f,\psi_{\mu_1}(x_1)e^{-(t_1-s_1)H_{0,f}}\bar{\psi}_{v_1}(y_1)e^{-(s_2-t_1)H_{0,f}}\ldots e^{-(s_n-t_{n-1})H_{0,f}}\bar{\psi}_{v_n}(y_n)\Omega_0^f\rangle$$
$$=\det\{(S_\ell)_{\mu_iv_j}(s_i-t_j;x_i-y_j)\}, \tag{VI.41}$$

valid for $0\leqq s_1\leqq t_1\leqq\ldots\leqq s_n\leqq t_n$ and the Feynman-Kac formula for bosons, we obtain (VI.38).

VI.3. Finite Temperature States

We consider the finite temperature states defined in Sect. IV.1 of [1]. These are just trace states regularized by the heat kernel of H. Let $C_{\ell,\beta}(x-y)$ be the periodic covariance with period β in the x_0 direction, and let $\tilde{S}_{\ell,\beta}(x-y)$ be the fermionic covariance which is antiperiodic in the x_0 direction with period β. Let $\tilde{K}^{(\kappa)}_{\ell,\beta}(\Phi)$ be given by (VI.24) with S_ℓ replaced by $\tilde{S}_{\ell,\beta}$. Below we state a simple Feynman-Kac formula involving such trace states. The final representation differs from the Feynman-Kac formulas in Sect. VI.2 only by replacing the Green's functions C_ℓ and S_ℓ by $C_{\ell,\beta}$ and $\tilde{S}_{\ell,\beta}$, respectively. The validity of these representations is a consequence of the similar representations in the Gaussian case, see Proposition VI.1 of the first paper of [1]. The analytic proof then follows by the convergence of VI.40. Similar trace state representations also hold for states of the form

$$\frac{\mathrm{Tr}(e^{-\beta_1 H}A_1 e^{-\beta_2 H}A_2 \dots e^{-\beta_n H})}{\mathrm{Tr}\, e^{-\beta H}}, \tag{VI.42}$$

where $\beta_j \geqq 0$ and $\sum_j \beta_j = \beta$. We set

$$A^{(\kappa)}_{\ell,\beta}(\Phi) = \int_{[0,\beta]\times T^1} [m\Phi\partial P(\Phi_\kappa)^* + m\Phi^*\partial P(\Phi_\kappa) + |\partial P(\Phi_\kappa)|^2]dx. \tag{VI.43}$$

Also, let

$$\Xi_{\ell,\beta} = \mathrm{Tr}(e^{-\beta H_0}) = \prod_{p\in\hat{T}^1} \coth^2(\beta\omega(p)). \tag{VI.44}$$

Proposition VI.10. *The following identity holds*

$$\mathrm{Tr}(\exp(-\beta H(\kappa)) = \Xi_{\ell,\beta}\int \det(I - \tilde{K}^{(\kappa)}_{\ell,\beta}(\Phi))\exp\{-A^{(\kappa)}_{\ell,\beta}(\Phi)\}d\mu_{C_{\ell,\beta}}(\Phi). \tag{VI.45}$$

VI.4. Path Integral Representation of $\mathrm{Tr}(\exp\{-\beta H_\tau(\kappa)\})$

We use covariance operators studied by Osipov [12]:

$$C_\tau(x-y) = (2\pi)^{-2}\int \frac{\omega_\tau(p_1)}{\omega(p_1)(p_0^2+\omega_\tau(p_1)^2)} e^{-ip(x-y)}dp \tag{VI.46}$$

and

$$\begin{aligned} S_\tau(x-y) = {} & (2\pi)^{-2}\int \frac{(-\not{p}+m)\omega_\tau(p_1)}{(p_0^2+\omega_\tau(p_1)^2)\omega(p_1)} e^{-ip(x-y)}dp \\ & + i\zeta\varepsilon(x_0-y_0)\gamma_0^E(2\pi)^{-2}\int \frac{\omega_\tau(p_1)}{(p_0^2+\omega_\tau(p_1)^2)\omega(p_1)} e^{-ip(x-y)}dp, \end{aligned} \tag{VI.47}$$

where $\omega_\tau(p_1) = \omega(p_1) - \zeta\omega(p_1)^\tau$, and where

$$\varepsilon(x_0) = \begin{cases} 1, & \text{if } x_0 \geqq 0, \\ -1, & \text{if } x_0 < 0. \end{cases}$$

Let $C_{\tau,\ell,\beta}$ be the periodization of C_τ with period ℓ in the x_1 direction and period β in the x_0 direction. Similarly, let $\tilde{S}_{\tau,\ell,\beta}$ be the periodization of S_τ periodic with period ℓ in the x_1 direction and antiperiodic with period β in the x_0 direction. Let

$\tilde{K}^{(\kappa)}_{\tau,\ell,\beta}(\Phi)$ be given by (VI.24) with $\tilde{S}_\ell$ replaced by $\tilde{S}_{\tau,\ell,\beta}$. As in Subsect. VI.2 we show that

$$F^{(\kappa)}_{\tau,\ell,\beta}(\phi) \equiv \det(I - \tilde{K}^{(\kappa)}_{\tau,\ell,\beta}(\Phi)) \exp(-A^{(\kappa)}_{\ell,\beta}(\Phi)) \in L_p(d\mu_{C_{\tau,\ell,\beta}}), \qquad \text{(VI.48)}$$

provided ζ is sufficiently small and p is close to 1.

Proposition VI.11. *With the above definitions,*

$$\mathrm{Tr}(\exp(-\beta H_\tau(\kappa)) = \Xi_{\tau,\ell,\beta} \int \det(I - \tilde{K}^{(\kappa)}_{\tau,\ell,\beta}(\Phi)) \exp(-A^{(\kappa)}_{\ell,\beta}(\Phi)) d\mu_{C_{\tau,\ell,\beta}}(\Phi) \qquad \text{(VI.49)}$$

for $\beta > 0$, *where*

$$\Xi_{\tau,\ell,\beta} = \prod_{p \in \hat{T}^1} \coth^2(\beta\omega_\tau(p)). \qquad \text{(VI.50)}$$

Acknowledgement. We wish to thank Konrad Osterwalder for comments on a draft of this paper.

References

1. Jaffe, A., Lesniewski, A., Weitsman, J.: Index of a family of Dirac operators on loop space. Commun. Math. Phys. **112**, 75–88 (1987)
   Jaffe, A., Lesniewski, A., Lewenstein, M.: Ground state structure in supersymmetric quantum mechanics. Ann. Phys. **178**, 313–329 (1987)
2. Jaffe, A., Lesniewski, A.: A priori estimates for $N=2$ Wess-Zumino models on a cylinder. Commun. Math. Phys. (to appear)
3. Wess, J., Zumino, B.: Nucl. Phys. B **70**, 39 (1974)
   Cecotti, S., Girardello, L.: Stochastic and parastochastic aspects of supersymmetric functional measures: a new nonperturbative approach to supersymmetry. Ann. Phys. **145**, 81–99 (1983)
4. McBryan, O.: Unpublished
5. Glimm, J., Jaffe, A.: Quantum physics. Berlin, Heidelberg, New York: Springer 1987
6. Glimm, J., Jaffe, A.: Singular perturbations of self-adjoint operators. Commun. Pure Appl. Math. **22**, 401–414 (1969)
7. Jaffe, A.: Existence theorems for a cutoff $\lambda\varphi^4$ field theory. In: Mathematical theory of elementary particles. Goodman, R., Segal, I. (eds.). New York: MIT Press 1966
8. Kato, T.: Perturbation theory for linear operators. Berlin, Heidelberg, New York: Springer 1984
9. Seiler, E.: Schwinger functions for the Yukawa model in two dimensions with space-time cutoff. Commun. Math. Phys. **42**, 163–182 (1975)
10. Seiler, E., Simon, B.: Nelson's symmetry and all that in the Yukawa$_2$ and φ^4_3 field theories. Ann. Phys. **97**, 470–518 (1976)
11. Gross, L.: On the formula of Mathews and Salam. J. Funct. Anal. **25**, 162–209 (1977)
12. Osipov, E.P.: The Yukawa$_2$ field theory: linear N_τ bound, locally Fock property. Ann. Inst. H. Poincaré **30**, 470–518 (1976)

Communicated by A. Jaffe

Received June 26, 1987; in revised form September 17, 1987